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ON STRONG COMPARATIVE LOGIC

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1. *Introduction*

Comparative logic is not at all a new area of research: on the contrary, it dates back to Aristotle's *Topics* and was subsequently cultivated by a number of authors throughout the historical development of ancient and medieval logic. However, a clear revival of interest for the subject has taken place in Italy over the last 15 years, mostly due to the work of Ettore Casari (e.g. 1987; 1989; 1997), who introduced a series of logical calculi of increasing complexity which serve as a broad and articulated frame for the formal reconstruction of a theory of comparison. The reader is referred to Casari (1997) for a clear and detailed account of the work done so far.

By "(propositional) strong comparative logic" (henceforth SCL) we mean the propositional fragment of the calculus named "comparative logic of rank 1" in Casari (1987), endowed with an axiom of linearity. It is a first degree system, whereby no nesting of the connective $\leq$ is permitted. Relatively to it, Casari proved a completeness theorem using algebraic techniques —he singled out the class of bifrontal chains as an adequate class of models. Later, Pierluigi Minari (1988) introduced both a relational semantics and a different algebraic semantics for comparative logic of rank 1, i.e. SCL without linearity.

The present paper, although indebted to the basic ideas of Minari's relational approach, takes a third standpoint: it basically consists in an Epstein-style semantics (cp. Epstein, 1990). We shall show that SCL may be semantically characterized by grading the "truth intensities" of zero degree formulae by means of an appropriate relation. Furthermore, some additional properties of SCL will be discussed.

2. *Syntax*

Let P be a standard propositional alphabet, enriched with the new connective $\leq$. The set ZDF of *zero degree formulae* is defined as the smallest set containing the variables and closed under $\neg$, &, $\vee$, $\rightarrow$. The set FDF of *first degree formulae* is inductively defined as follows:

1) if $A \in$ ZDF, then $A \in$ FDF;
2) if $A, B \in$ ZDF, then $A \leq B \in$ FDF;
3) if $A, B \in$ FDF, then $\neg A, A \,\&\, B, A \vee B, A \rightarrow B \in$ FDF;
4) nothing else belongs to FDF.

Members of ZDF, resp. FDF, will be sometimes referred to as "zdfs" and "fdfs". $A \leftrightarrow B$ and $A \geq\leq B$ will be used as abbreviations, respectively, for $(A \rightarrow B) \,\&\, (B \rightarrow A)$, $A \leq B \,\&\, B \leq A$.

A possible intuitive reading for $A \leq B$ is "A is at most as true as B".

Let us now define a formal system, SCL, formulated in the language £ thus obtained (we draw this system from Casari, 1987; by the way, we adopt the same conventions about auxiliary symbols. Axioms C5 and C6.2 below are not independent but are nonetheless included for easier reference).

Axioms

Group A. A standard set of axiom schemata for classical propositional logic, where metavariables stand for elements of FDF (most axioms may be "coded" by replacing the letter "A" by "C" in the corresponding axioms of Group C below).

Group C. For $A, B, C \in$ ZDF, the following formulae are axioms:

C1. $A \leq B \,\&\, B \leq C \rightarrow A \leq C$
C2.1. $(A \,\&\, B) \leq A$
C2.2. $(A \,\&\, B) \leq B$
C2.3. $A \leq B \,\&\, A \leq C \rightarrow A \leq (B \,\&\, C)$
C3.1. $A \leq (A \vee B)$
C3.2. $B \leq (A \vee B)$
C3.3. $A \leq C \,\&\, B \leq C \rightarrow (A \vee B) \leq C$
C4.1. $A \leq \neg\neg A$
C4.2. $\neg\neg A \leq A$
C4.3. $A \leq B \rightarrow \neg B \leq \neg A$
C5. $(A \vee B) \,\&\, (A \vee C) \leq (A \vee (B \,\&\, C))$
C6.1. $A \leq B \rightarrow (A \rightarrow B)$
C6.2. $\neg(A \leq B) \rightarrow (B \rightarrow A)$
C7.1. $\neg A \leq (A \rightarrow B)$
C7.2. $B \leq (A \rightarrow B)$
C7.3. $(A \rightarrow B) \leq (\neg A \vee B)$
L. $A \leq B \vee B \leq A$

Rule of inference

MP. $A, A \rightarrow B \Rightarrow B$

The notions of provability and derivability are as in classical logic. We draw from Casari (1987) the following list of theorems of SCL:

T1. $A \leq A$; T2. $A \geq\leq A$; T3. $A \leq B \,\&\, B \leq A \rightarrow A \geq\leq B$; T4. $A \geq\leq B \rightarrow B \geq\leq A$; T5. $A \geq\leq B \,\&\, B \geq\leq C \rightarrow A \geq\leq C$; T6. $A \geq\leq B \rightarrow \neg A \geq\leq \neg B$; T7. $A \geq\leq B \,\&\, C \geq\leq D \rightarrow (A \,\&\, C) \geq\leq (B \,\&\, D) \,\&\, (A \vee C) \geq\leq (B \vee D) \,\&\, (A \rightarrow C) \geq\leq (B \rightarrow D)$; T8. $A \geq\leq \neg\neg A$; T9. $A \leq B \leftrightarrow \neg B \leq \neg A$; T10. $(A \,\&\, A) \geq\leq A$; T11. $(A \vee A) \geq\leq A$; T12. $(A \,\&\, B) \geq\leq (B \,\&\, A)$; T13. $(A \vee B) \geq\leq (B \vee A)$; T14. $(A \,\&\, (B \,\&\, C)) \geq\leq ((A \,\&\, B) \,\&\, C)$; T15. $(A \vee (B \vee C)) \geq\leq ((A \vee B) \vee C)$; T16. $(A \,\&\, (A \vee B)) \geq\leq A$; T17. $(A \vee (A \,\&\, B)) \geq\leq A$; T18. $(A \,\&\, (B \vee C)) \geq\leq ((A \,\&\, B) \vee (A \,\&\, C))$; T19. $(A \vee (B \,\&\, C)) \geq\leq ((A \vee B) \,\&\, (A \vee C))$; T20. $(\neg(A \,\&\, B)) \geq\leq (\neg A \vee \neg B)$; T21. $(\neg(A \vee B)) \geq\leq (\neg A \,\&\, \neg B)$; T22. $A \leq B \,\&\, A \leq C \leftrightarrow A \leq (B \,\&\, C)$; T23. $A \leq B \vee A \leq C \leftrightarrow A \leq (B \vee C)$; T24. $A \leq C \,\&\, B \leq C \leftrightarrow (A \vee B) \leq C$; T25. $A \leq C \vee B \leq C \leftrightarrow (A \,\&\, B) \leq C$; T26. $\neg A \leq A \leftrightarrow A$; T27. $A \leq \neg A \leftrightarrow \neg A$.

Now, let us extend somewhat the preceding list. If $I(A, B)$ is a shorthand for $(A \vee \neg A) \leq (B \vee \neg B)$ (or, for that matter, for $(B \,\&\, \neg B) \leq (A \,\&\, \neg A)$), for $A, B, C \in$ ZDF the following schemata are provable in SCL as well (T29–T31, indeed, are special cases of T1, C1, and L respectively):

T28. $A, B \Rightarrow A \,\&\, B$
T29. $I(A, A)$
T30. $I(A, B) \,\&\, I(B, C) \rightarrow I(A, C)$
T31. $I(A, B) \vee I(B, A)$
T32. $I(A, \neg B) \leftrightarrow I(A, B)$
$I(\neg A, B) \leftrightarrow I(A, B)$
T33. $I(A, B \rightarrow C) \leftrightarrow I(A, \neg B \vee C)$
T34. $A \,\&\, B \rightarrow (I(C, A \,\&\, B) \leftrightarrow I(C, A) \,\&\, I(C, B))$
$A \,\&\, B \rightarrow (I(C, A \vee B) \leftrightarrow I(C, A) \vee I(C, B))$
$A \,\&\, B \rightarrow (I(A \,\&\, B, C) \leftrightarrow I(A, C) \vee I(B, C))$
$A \,\&\, B \rightarrow (I(A \vee B, C) \leftrightarrow I(A, C) \,\&\, I(B, C))$
T35. $\neg A \,\&\, \neg B \rightarrow (I(C, A \,\&\, B) \leftrightarrow I(C, A) \vee I(C, B))$
$\neg A \,\&\, \neg B \rightarrow (I(C, A \vee B) \leftrightarrow I(C, A) \,\&\, I(C, B))$
$\neg A \,\&\, \neg B \rightarrow (I(A \,\&\, B, C) \leftrightarrow I(A, C) \,\&\, I(B, C))$
$\neg A \,\&\, \neg B \rightarrow (I(A \vee B, C) \leftrightarrow I(A, C) \vee I(B, C))$
T36. $A \,\&\, \neg B \rightarrow (I(C, A \,\&\, B) \leftrightarrow I(C, B))$
$A \,\&\, \neg B \rightarrow (I(C, A \vee B) \leftrightarrow I(C, A))$

$$A \,\&\, \neg B \to (I(A \,\&\, B, C) \leftrightarrow I(B, C))$$
$$A \,\&\, \neg B \to (I(A \vee B, C) \leftrightarrow I(A, C))$$

T37. $\neg A \,\&\, B \to (I(C, A \,\&\, B) \leftrightarrow I(C, A))$
$$\neg A \,\&\, B \to (I(C, A \vee B) \leftrightarrow I(C, B))$$
$$\neg A \,\&\, B \to (I(A \,\&\, B, C) \leftrightarrow I(A, C))$$
$$\neg A \,\&\, B \to (I(A \vee B, C) \leftrightarrow I(B, C))$$

T38. $A \leq B \leftrightarrow (A \to B) \,\&\, (A \,\&\, B \to I(A, B)) \,\&\, (\neg A \,\&\, \neg B \to I(B, A))$

Proofs of T28–T37 are left as an exercise. We shall instead sketch, in an informal natural deduction style, the proof of the very important T38, leaving up to the reader the task of recasting this argument into a more accurate formal one. Remember that a standard deduction theorem is available in SCL (cp. Casari, 1987).

Proof sketch. Left to right. Assume $A \leq B$. Then (C6.1) $A \to B$. Now, assume $A \,\&\, B$, whence (A2.1, A2.2) you get A, B and (T26) $\neg A \leq A$, $\neg B \leq B$. By transitivity from $\neg A \leq A$ and $A \leq B$, you have $\neg A \leq B$, whence, by T28, $\neg A \leq B \,\&\, A \leq B$. Thus, by A3.1, $(\neg A \leq B \,\&\, A \leq B) \vee (A \leq \neg B \,\&\, \neg A \leq \neg B)$, i.e. (T22–T25) $(A \vee \neg A) \leq B \vee (A \vee \neg A) \leq \neg B$, i.e. $(A \vee \neg A) \leq (B \vee \neg B)$. By the deduction theorem, then, $A \,\&\, B \to I(A, B)$ is established under the assumption $A \leq B$. $\neg A \,\&\, \neg B \to I(B, A)$ is established similarly. An application of T28 and a further recourse to the deduction theorem suffice for the left-to-right part of T38.

Right to left. Suppose $\neg(A \leq B)$, $A \to B$, $A \,\&\, B \to I(A, B)$, $\neg A \,\&\, \neg B \to I(B, A)$. Since (C6.2) $\neg(A \leq B) \to (B \to A)$, by T28 we get $A \leftrightarrow B$. Now, we argue following a dylemmatic pattern (notice, also in the following, that SCL has in its classical fragment the appropriate syntactic apparatus to match our "semantical" reasoning), and assume A, whence B. By T26, $\neg A \leq A$, $\neg B \leq B$. Moreover (T28), $A \,\&\, B$, whence $I(A, B)$, i.e. $(A \vee \neg A) \leq (B \vee \neg B)$. Rephrasing it via T22–T25, we get $(A \leq B \,\&\, \neg A \leq B) \vee (A \leq \neg B \,\&\, \neg A \leq \neg B)$. In the first case, by A2.1, $A \leq B$ (contradiction); in the second case, by $A \leq \neg B$, $\neg B \leq B$, via A2.1, T28, C1, we get $A \leq B$ (contradiction). Assuming now $\neg A$, argue as above to get the same contradiction. By T28 and the deduction theorem, then, we obtain our conclusion.

Let A, B, C range over FDF and let the expression "$A[B/C]$" denote the result of the replacement operation, i.e. the result of replacing in A zero or more occurrences of B by C. Moreover, let CF = FDF/ZDF (in this expression, of course, "/" denotes set-theoretical difference). Given the way our set FDF was built up, the replacement operation is not necessarily stable, i.e., not always does it turn well-formed formulae (of either ZDF or FDF) into well-formed formulae. However, it is easily seen that:

1) If $A, B \in$ ZDF and $C \in$ ZDF (CF), then $A[B/C] \in$ ZDF (FDF);
2) If $A, C \in$ FDF and $B \in$ CF, then $A[B/C] \in$ FDF;
3) If $B, C \in$ ZDF and $A \in$ FDF, then $A[B/C] \in$ FDF;
4) If $A \in$ FDF, $B \in$ ZDF and $C \in$ CF, then $A[B/C] \in$ FDF provided that none of the replaced occurrences of B lies in the scope of any occurrence of "$\leq$" in A.

Casari (1987) proved that i) if $A, B, C \in$ ZDF and $\vdash A \geq\leq B$, then $\vdash C \geq\leq C[A/B]$; ii) if $A, B \in$ ZDF, $C \in$ FDF and $\vdash A \geq\leq B$, then $\vdash C \leftrightarrow C[A/B]$. Remark that, in such cases, the morphological condition according to which the replacement operation must transform well-formed formulae into well-formed formulae is satisfied by 1) and 3) above. Hence, in SCL we have replaceability of *comparatively* equivalent zdfs (i.e., zero-degree formulae A and B s.t. $\vdash A \geq\leq B$). On the contrary, we do not generally have replaceability of *materially* equivalent zdfs or fdfs (i.e., zero- or first-degree formulae A and B s.t. $\vdash A \leftrightarrow B$), even when replacement is morphologically permitted.

However, as long as a certain subclass of FDF is concerned, we are able to demonstrate this property. Define a *comparative formula* as a first degree formula having the form $A \leq B$, and a *C-uncomparable formula* ($C \in$ FDF) as a first degree formula which is not a proper subformula of any comparative subformula of C. Remark that every formula containing a comparative formula is a C-uncomparable formula for every $C \in$ FDF. We prove:

Theorem 1. Let A, B, C $\in$ FDF. If $\vdash A \leftrightarrow B$ and A is a C-uncomparable formula, then $\vdash C \leftrightarrow C[A/B]$.

Proof. Induction on the length of C. If $C \in$ ZDF, the theorem boils down to the classical replacement theorem, since SCL has in its classical fragment the required equivalences. Now, suppose that C has the form $D \leq E$ ($D, E \in$ ZDF). Since A is a C-uncomparable formula, it cannot be a proper subformula of $D \leq E$. Thus, either A does not occur in $D \leq E$ (in this case, $\vdash D \leq E \leftrightarrow D \leq E$), or A coincides with $D \leq E$ (and then $\vdash A \leftrightarrow B$ by our hypothesis). The remainder of the proof is, like the first part, an induction on the construction of FDF.

3. *Semantics*

In all comparative logics, hence in SCL too, propositions are conceived of as not merely being true or false, but also having a *degree of truth* (or falsity). In order to provide an Epstein-style semantics for SCL, we reshape this concept splitting it up in the following 5 points:

1) Zdfs can be viewed as having not only a Boolean truth value, but also an *intensity*, which can be numerically measured. Hence, intensities can be compared to one another. Suppose you have an intuitive idea of what a "degree of truth (falsity)" is, and of how to express it by a positive, resp. negative numerical value. Then think of the intensity of a zdf as the *absolute value* of its degree of truth.

2) The only properties of a zdf which matter to SCL are its logical form, its (Boolean) truth value and its intensity.

3) The truth value and the intensity of a zdf are independent of each other as far as *atomic* formulae are at issue.

4) The connective $\leq$ is a "truth-and-intensity" function: it depends both on the truth value and on the intensity of the two formulae under comparison.

5) Formulae containing at least an occurrence of the connective $\leq$ have a truth value, but no intensity. Comparisons of comparisons are not allowed.

Let us now shift from these "conversational" hints to a more accurate formal semantics. Let VAR be the set of variables of £. Moreover, let $\mathfrak{I}$ be a linear preordering of VAR, i.e. a reflexive, transitive, and connected relation on VAR. We extend $\mathfrak{I}$ to the whole of ZDF through I1–I9, listed below:

I1. $\mathfrak{I}(A, A)$
I2. $\mathfrak{I}(A, B)$ and $\mathfrak{I}(B, C)$ imply $\mathfrak{I}(A, C)$
I3. $\mathfrak{I}(A, B)$ or $\mathfrak{I}(B, A)$
I4. $\mathfrak{I}(A, \neg B)$ iff $\mathfrak{I}(A, B)$ iff $\mathfrak{I}(\neg A, B)$
I5. $\mathfrak{I}(A, B \rightarrow C)$ iff $\mathfrak{I}(A, \neg B \vee C)$

Furthermore, for every *classical* valuation v: ZDF $\rightarrow$ {T, F}:

I6. If $v(A) = v(B) = \mathrm{T}$,
$\mathfrak{I}(C, A \mathbin{\&/\vee} B)$ iff $\mathfrak{I}(C, A)$ and/or $\mathfrak{I}(C, B)$
$\mathfrak{I}(A \mathbin{\&/\vee} B, C)$ iff $\mathfrak{I}(A, C)$ or/and $\mathfrak{I}(B, C)$
I7. If $v(A) = v(B) = \mathrm{F}$,
$\mathfrak{I}(C, A \mathbin{\&/\vee} B)$ iff $\mathfrak{I}(C, A)$ or/and $\mathfrak{I}(C, B)$
$\mathfrak{I}(A \mathbin{\&/\vee} B, C)$ iff $\mathfrak{I}(A, C)$ and/or $\mathfrak{I}(B, C)$
I8. If $v(A) = \mathrm{T}$, $v(B) = \mathrm{F}$,

$\Im(C, A \& B)$ iff $\Im(C, B)$
$\Im(C, A \vee B)$ iff $\Im(C, A)$
$\Im(A \& B, C)$ iff $\Im(B, C)$
$\Im(A \vee B, C)$ iff $\Im(A, C)$

I9. If $v(A) = \mathrm{F}$, $v(B) = \mathrm{T}$,
$\Im(C, A \& B)$ iff $\Im(C, A)$
$\Im(C, A \vee B)$ iff $\Im(C, B)$
$\Im(A \& B, C)$ iff $\Im(A, C)$
$\Im(A \vee B, C)$ iff $\Im(B, C)$

Any binary relation on ZDF satisfying the previous conditions is called an *intensity preordering* of ZDF. Intuitively, read $\Im(A, B)$ as: "*A* is at most as intense as *B*". Of course, every intensity preordering $\Im^*$ of VAR, together with a classical valuation *v* on the same set, determines a unique intensity preordering $\Im$ of ZDF. In such a case, we may speak of *the* intensity preordering $\Im$ *co-determined* by *v* and $\Im^*$.

Now, if v: VAR $\rightarrow$ {T, F}, $\Im^*$ is an intensity preordering of VAR and $\Im$ is the intensity preordering of ZDF co-determined by *v* and $\Im^*$, a *valuation* $v_{\Im}$ on FDF is inductively defined as follows:

$v_{\Im}(p) = v(p)$;
$v_{\Im}(\neg A)$, $v_{\Im}(A \& B)$, $v_{\Im}(A \vee B)$, $v_{\Im}(A \rightarrow B)$ are calculated by means of classical truth tables;
$v_{\Im}(A \leq B)$ is calculated by means of the following truth table ("$\Im(A, B) = \mathrm{T}$ (F)" counts as an abbreviation for "$\Im(A, B)$ holds (does not hold)"):

A	B	$\Im(A, B)$	$\Im(B, A)$	$A \leq B$
T	T	T	T	T
T	T	T	F	T
T	T	F	T	F
T	F	any values		F
F	T	any values		T
F	F	T	T	T
F	F	T	F	F
F	F	F	T	T

Notice that, given the linearity of $\Im(A, B)$ (I3), the preceding table has no entry corresponding to $\Im(A, B) = \Im(B, A) = \mathrm{F}$.

We define, for every *A* in FDF:

$v\mathfrak{I} \models A$ (*A true* in $v\mathfrak{I}$) iff $v\mathfrak{I}(A) = \mathrm{T}$;
$\models A$ (*A logically valid*) iff $v\mathfrak{I} \models A$ for every valuation $v\mathfrak{I}$.

A valuation $v\mathfrak{I}$ s.t. A is true in $v\mathfrak{I}$ is also called a *TI-model* for A. If $\Gamma \subseteq$ FDF, then $v\mathfrak{I}$ is said to be a TI-model for Γ iff is a TI-model for every B in Γ. Lastly, A is a *logical consequence* of Γ ($\Gamma \models A$) iff every TI-model for Γ is also a TI-model for A.

Theorem 2. $\Gamma \vdash A$ *iff* $\Gamma \models A$.

Proof. Left to right. Induction on the length of derivations. A tip to save time and paper: since a tabular 2-valued semantics is at hand, standard Beth-Hintikka-style analytic tableaux are feasible. Thus, e.g., to check any of the axioms, suppose it is false and try to derive a contradiction relatively to either any of its atomic subformulae (group A, group C) or an item of the form $\mathfrak{I}(A, B)$ (group C: as you will see, the tableaux will split up several times).

Right to left. We need some more definitions and a couple of intermediate lemmata.

$\Gamma \subseteq$ FDF is a *C-theory* iff 1) contains every axiom of SCL; 2) is closed under modus ponens. Γ is a *complete C-theory* iff, for every A in FDF, at least one of A, $\neg A$ is in Γ; is a *consistent C-theory* iff, for every A in FDF, at most one of A, $\neg A$ is in Γ; is a *prime C-theory* iff, for every A, B in FDF, $A \vee B$ is in Γ iff either A or B is in Γ. Notice that, as for classical logic, a consistent complete C-theory is always prime.

Lemma 1. Γ *is a consistent complete C-theory iff there is a TI-model for* Γ.

Proof. The "if" part is left to the reader. For the other direction, suppose that Γ is a consistent complete C-theory. For every p in VAR, set $c(p) = \mathrm{T}$ if p belongs to Γ, $c(p) = \mathrm{F}$ otherwise. By induction it is easily seen that c is a valuation and that $c(A) = \mathrm{T}$ iff $A \in \Gamma$ for $A \in$ ZDF. Moreover, for p, q in VAR, let $\mathfrak{I}^*(p, q)$ hold iff $I(p, q)$ belongs to Γ. Now, take the intensity preordering $\mathfrak{I}$ co-determined by c and $\mathfrak{I}^*$. By double induction on the complexity of A, B we prove the following

Claim. $\mathfrak{I}(A, B)$ *holds iff* $I(A, B) \in \Gamma$.

Proof. As an example, we prove the inductive step $\mathfrak{I}(A, B \,\&\, C)$ iff $I(A, B \,\&\, C) \in \Gamma$. Case 1: $c(B) = c(C) = \mathrm{T}$. We have that $B, C \in \Gamma$ (def.), whence $B \,\&\, C \in \Gamma$ (T28); thus, by T34, $I(A, B \,\&\, C) \leftrightarrow I(A, B) \,\&\, I(A, C) \in \Gamma$. As a consequence, $\mathfrak{I}(A, B \,\&\, C)$ iff $\mathfrak{I}(A, B)$ and $\mathfrak{I}(A, C)$ (since $\mathfrak{I}$ is co-determined by c and $\mathfrak{I}^*$) iff $I(A, B), I(A, C) \in \Gamma$ (by induction) iff $I(A, B) \,\&\, I(A,$

$C) \in \Gamma$ (T28, A2.1, A2.2) iff $I(A, B \& C) \in \Gamma$ (by MP from the formula above). Cases 2–4, corresponding to different c-values, are handled similarly. Notice that throughout the proof theorems T29–T37 are needed.

Now, consider the valuation $c\mathfrak{J}$ accordingly defined, as on p. 277. Let us show that, for every fdf A, $c\mathfrak{J} \models A$ iff A is in Γ.

If p is a variable, such an equivalence is true by definition.

If A is $\neg B$, then $\neg B \in \Gamma$ iff $B \notin \Gamma$ (consistency and completeness) iff $c\mathfrak{J}(B) = \mathrm{F}$ (ind. hyp.) iff $c\mathfrak{J}(\neg B) = \mathrm{T}$ (def.).

If A is $B \& C$, then $B \& C \in \Gamma$ iff $B, C \in \Gamma$ (A2.1, A2.2, T28) iff $c\mathfrak{J}(B) = c\mathfrak{J}(C) = \mathrm{T}$ (ind. hyp.) iff $c\mathfrak{J}(B \& C) = \mathrm{T}$ (def.).

If A is $B \vee C$, then $B \vee C \in \Gamma$ iff $B \in \Gamma$ or $C \in \Gamma$ (A3.1, A3.2, primality) iff $c\mathfrak{J}(B) = \mathrm{T}$ or $c\mathfrak{J}(C) = \mathrm{T}$ (ind. hyp.) iff $c\mathfrak{J}(B \vee C) = \mathrm{T}$ (def.). The case $A \to B$ obviously reduces to the previous ones.

Our claim is then inductively proved for ZDF. Let us now move to FDF. Suppose A is $B \leq C$.

If $c\mathfrak{J}(B \leq C) = \mathrm{T}$, then $c\mathfrak{J}(B) = \mathrm{F}$ or $c\mathfrak{J}(B) = \mathrm{T}$ i.e. (ind. hyp.) $B \notin \Gamma$ or $C \in \Gamma$ i.e. (completeness) $\neg B \in \Gamma$ or $C \in \Gamma$ i.e. (A3.1 or A3.2) $\neg B \vee C \in \Gamma$ i.e. (classical thesis available in SCL) $B \to C \in \Gamma$. Now, suppose $I(B, C) \notin \Gamma$, i.e. $\mathfrak{J}(B, C)$ does not hold. Since $c\mathfrak{J}(B \leq C) = \mathrm{T}$, this entails either $c\mathfrak{J}(B) = \mathrm{F}$ or $c\mathfrak{J}(C) = \mathrm{F}$ i.e. (ind. hyp. and completeness) $\neg B \in \Gamma$ or $\neg C \in \Gamma$, whence (A3.1 or A3.2) $\neg B \vee \neg C \in \Gamma$, which by a classical thesis available in SCL reduces to $\neg(B \& C) \in \Gamma$. Thus, either $\neg(B \& C) \in \Gamma$ or $I(B, C) \in \Gamma$. By either A3.1 or A3.2, $\neg(B \& C) \vee I(B, C) \in \Gamma$, whence $(B \& C) \to I(B, C) \in \Gamma$. Similarly, we obtain $(\neg B \& \neg C) \to I(C, B) \in \Gamma$. Summing up by T28, $(B \to C) \& (B \& C \to I(B, C)) \& (\neg B \& \neg C \to I(C, B)) \in \Gamma$, which by T38 entails $B \leq C \in \Gamma$.

Conversely, suppose $B \leq C \in \Gamma$. Then (T38) $(B \to C) \& (B \& C \to I(B, C)) \& (\neg B \& \neg C \to I(C, B)) \in \Gamma$. We dismember this conjunction via A2.1–A2.2. From $B \to C$, using our consistency and primality hypotheses, we get $B \notin \Gamma$ or $C \in \Gamma$, whence, by induction, $c\mathfrak{J}(B) = \mathrm{F}$ or $c\mathfrak{J}(B) = \mathrm{T}$. Now, suppose $B \& C \in \Gamma$. Then $I(B, C) \in \Gamma$, i.e. $\mathfrak{J}(B, C)$ holds. Suppose on the contrary $B \& C \notin \Gamma$. Then, using our consistency, completeness, and primality assumptions, as well as classical negation theses, $B \notin \Gamma$ or $C \notin \Gamma$, whence by induction $c\mathfrak{J}(B) = \mathrm{F}$ or $c\mathfrak{J}(C) = \mathrm{F}$. Thus, if $c\mathfrak{J}(B) = c\mathfrak{J}(C) = \mathrm{T}$, then $\mathfrak{J}(B, C)$ holds. Carrying out the same reasoning with respect to $\neg B \& \neg C$, we reconstruct exactly the truth-and-intensity conditions needed for $B \leq C$ to be true.

The induction on FDF is then completed in a straightforward manner. This ends the proof of Lemma 1.

Lemma 2. If it is not the case that $\Gamma \vdash A$, then there is a consistent complete C-theory Σ s.t. $\Gamma \cup \{\neg A\} \subseteq \Sigma$.

Proof. If we want to avoid recourse to the full power of Zorn's Lemma, we may stipulate that VAR be a denumerable set. With this minimal ontological assumption, we can resort to a standard Lindenbaum-Henkin-style saturation procedure, which is by no means affected by our additional stipulations. Remark that, since a full deduction theorem is provable for SCL, we need not introduce an auxiliary notion of &-consistency (unlike in most modal and relatedness contexts).

Now, suppose it is not the case that $\Gamma \vdash A$. Then, by Lemma 2, there is a consistent complete C-theory Σ s.t. $\Gamma \cup \{\neg A\} \subseteq \Sigma$. By Lemma 1, there is a TI-model for Σ. Then, $\Gamma \not\models A$. This ends the proof of Theorem 2.

3.1. *Constructive completeness*

The completeness proof traced above is obviously nonconstructive. However, a more constructive completeness proof is likely to be feasible moving along the lines of the one carried out in Paoli (1996) for relatedness logic. We shall not adapt the whole procedure to the present case, confining ourselves to sketch the proof of a normal form theorem which would serve as a basis for it.

Theorem 3. If A is in FDF, then there exists A^*, containing exactly the same variables as A, say $p_1, \ldots, p_n$, s.t. 1) $\vdash A \leftrightarrow A^*$, 2) A^* is a generalized disjunction $A_1 \vee \ldots \vee A_m$, where for $i \leq m$ $A_i = B_1 \,\&\, \ldots \,\&\, B_q$ and B_j ($j \leq q$) is either a variable p_k ($k \leq n$), or the negation of such, or a formula having the form $I(p_k, p_l)$, ($k, l \leq n$), or the negation of such.

Proof (heuristic hints). In the sequel, steps justified by "replacement of comparative equivalents" are countersigned by "RCE", whereas steps justified by "replacement of material equivalents" are countersigned by "RME". In the latter case, we rest on the fact that the replaced formulae are A-uncomparable formulae for every A in FDF (cp. Theorem 1).

Perform one after another the following steps:

1. Replace every instance of $A \rightarrow B$ by $\neg A \vee B$ ($(A \rightarrow B) \geq\leq (\neg A \vee B)$, RCE).
2. In comparative formulae (see above), put A and B in disjunctive normal form (T8, T10–T21, RCE).
3. Replace every instance of $A \leq (B \vee C)$ by $A \leq B \vee A \leq C$, and every instance of $(A \vee B) \leq C$ by $A \leq C \,\&\, B \leq C$ (T23, T24, RME).

4. Replace every instance of $A \leq (B \ \& \ C)$ by $A \leq B \ \& \ A \leq C$, and every instance of $(A \ \& \ B) \leq C$ by $A \leq C \vee B \leq C$ (T22, T25, RME).

5. Replace every instance of $A \leq B$ by $(A \to B) \ \& \ (A \ \& \ B \to I(A, B)) \ \&$ $(\neg A \ \& \ \neg B \to I(B, A))$ (T38, RME).

6. Replace every instance of $I(A, \neg B)$ and $I(\neg A, B)$ by $I(A, B)$ (T32, RME).

7. Repeat steps 1. and 2., putting the whole formula in disjunctive normal form.

4. *SCL as an "intermediate" logic*

With respect to comparative logic —much in the same way as other logics containing a linearity axiom analogous to L above, such as RM with respect to relevant logic and Dummett's LC with respect to intuitionistic logic— SCL represents a long step towards a classical analysis of implication (cp. Anderson & Belnap, 1975, pp. 397–399). We shall hereafter prove two striking features of SCL which are indeed shared by RM. Failure of interpolation (Theorem 5), however, was proved by Minari (1988) for comparative logic without linearity.

Theorem 4. If $\vdash A \leq B$, *then either A and B share a variable, or else* both $\vdash \neg A$ and $\vdash B$.

Proof. First, we prove that, if A and B share no variable, there is an intensity preordering $\mathfrak{I}$ s.t. $\mathfrak{I}(A, B)$ fails. In fact, let $p_1, \ldots, p_n$ be the variables in A, and $q_1, \ldots, q_m$ the variables in B. Consider the sequence $< q_1, \ldots, q_m, p_1, \ldots, p_n, r_1, r_2, \ldots >$, where $r_1, r_2, \ldots$ are exactly the variables which occur neither in A nor in B. Let $\mathfrak{I}(p, q)$ hold iff p precedes, or coincides with, q in the above sequence. $\mathfrak{I}$ is then an intensity preordering of VAR s.t. for every i, j $\mathfrak{I}(p_i, q_j)$ fails. It is readily seen, by induction on the length of formulae, that $\mathfrak{I}(A, B)$ fails given any classical valuation v.

Now, let A and B be formulae with no variable in common, s.t. $\vdash A \leq B$ but either not $\vdash \neg A$ or not $\vdash B$. By Theorem 2, then, we can construct a classical valuation v^* s.t. *a*) not both $v^*(A) = \mathrm{T}$ and $v^*(B) = \mathrm{F}$; b) either $v^*(A) = \mathrm{T}$ or $v^*(B) = \mathrm{F}$. This leaves two cases open: 1) $v^*(A) = v^*(B) = \mathrm{T}$; 2) $v^*(A) = v^*(B) = \mathrm{F}$.

Case 1): let $v^*\mathfrak{I}$ be such that for every variable r $v^*\mathfrak{I}(r) = v^*(r)$ and such that $\mathfrak{I}(A, B)$ fails as above. Then $v^*\mathfrak{I}(A \leq B) = \mathrm{F}$, i.e. $\not\models A \leq B$, i.e. (Theorem 2) not $\vdash A \leq B$, against our hypothesis.

Case 2): similar (take $\mathfrak{I}$ s.t. $\mathfrak{I}(B, A)$ fails).

Theorem 5. There are A, B belonging to ZDF s.t. 1) $\vdash A \leq B$*; 2) it is not the case that* $\vdash \neg A$*; 3) it is not the case that* $\vdash B$*; 4) there is no C in ZDF s.t.* $\vdash A \leq C$, $\vdash C \leq B$ *and the variables in C are among the variables shared by A and B.*

Proof. We need two lemmata.

Lemma 3. $\vdash (A \,\&\, \neg A) \leq (B \vee \neg B)$
$\vdash A \leq B \Rightarrow\, \vdash A \leq (B \vee C)$
$\vdash A \leq B \Rightarrow\, \vdash (A \,\&\, C) \leq B$

Proof. Exercise.

Lemma 4. Let $\Im^*$ be an intensity preordering of VAR s.t. $\Im^*(p_1, q)$ fails and ... and $\Im^*(p_n, q)$ fails. Consider the intensity preordering $\Im$ of ZDF co-determined by $\Im^*$ and any classical valuation v. Then $\Im(A, q)$ fails if the variables in A are among $\{p_1, ..., p_n\}$.

Proof. Induction on the complexity of A. If A is a variable, $A = p_i$ for $i \leq n$, and we're done. If $A = \neg B$, $\Im(\neg B, q)$ holds iff $\Im(B, q)$ does, and the latter fails by IH. If $A = B \,\&\, C$, $\Im(B \,\&\, C, q)$ iff $\Im(B, q)$ and/or $\Im(C, q)$ (depending on v); either way, it fails by IH. The case for disjunction is similar.

Let us now return to our theorem. Let $A \leq B$ be Meyer's formula $(s \vee (p \,\&\, q \,\&\, \neg q)) \leq ((s \vee p) \,\&\, (s \vee r \vee \neg r))$, which is known to admit of no interpolants in RM. We prove that A and B meet all of the conditions 1)–4) above.

Ad 1). By C3.1, $\vdash s \leq (s \vee p)$ and $\vdash s \leq (s \vee r \vee \neg r)$, whence by T28, C2.3 $\vdash s \leq ((s \vee p) \,\&\, (s \vee r \vee \neg r))$. By Lemma 3, $\vdash (p \,\&\, q \,\&\, \neg q) \leq (s \vee r \vee \neg r)$. By C1, C2.1, C3.2 $\vdash (p \,\&\, q \,\&\, \neg q) \leq (s \vee p)$. Finally, by C2.3, C3.3, $\vdash A \leq B$.

Ad 2), 3). Use classical truth tables and soundness of the system.

Ad 4). Suppose there were C containing no variable outside $\{p, s\}$ and s.t. $\vdash A \leq C$, $\vdash C \leq B$. Consider v s.t. $v(p) = v(q) = v(r) = \mathrm{T}$, $v(s) = \mathrm{F}$; moreover, consider the following intensity preordering $\Im^*$ of VAR:

$$\begin{array}{c} \cdots \\ | \\ p, s \\ | \\ q, r \end{array}$$

(q and r precede p and s which in turn precede all the other variables of £). Let us take the intensity preordering $\Im$ co-determined by $\Im^*$ and v, and the valuation $v\Im$ accordingly defined. Clearly, $v\Im(A) = F$ and $v\Im(B) = T$. If $v\Im(C) = T$, $v\Im(C \leq ((s \vee p) \& (s \vee r \vee \neg r))) = T$ iff $\Im(C, (s \vee p) \& (s \vee r \vee \neg r))$, iff $\Im(C, s \vee p)$ and $\Im(C, s \vee r \vee \neg r)$, iff $\Im(C, p)$ and $\Im(C, r)$. But, since the latter fails by Lemma 4, $v\Im(C \leq B) = F$ and thus not $\vdash C \leq B$, against our hypothesis. Similarly, if $v\Im(C) = F$, $v\Im((s \vee (p \& q \& \neg q)) \leq C) = T$ iff $\Im(C, s \vee (p \& q \& \neg q))$, iff $\Im(C, s)$ and $\Im(C, p \& q \& \neg q)$, iff $\Im(C, s)$ and $\Im(C, \neg q)$, iff $\Im(C, s)$ and $\Im(C, q)$. But the latter fails by Lemma 4 —and once again we contradict one of our hypotheses, namely that $\vdash A \leq C$.

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