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A THEORY OF LOGICAL RELEVANCE

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Usually relevant logic is characterized as a branch of modern symbolic logic, or more precisely, as a unit of modern non-classical logic. This brings into consideration only a set of systems (*E, R, NR, T, RM, EM etc*) free of known paradoxes of logical entailment and implication. According to their view one can consider them as sorts of deductive systems. But in essence they are purely formal (logical) constructions, since they have only formal semantics, where fundamental concepts of the deduction theory — logical entailment, implication, logical law and even a sense of formulae truth evaluation — are not cleared up. The semantics of these systems are often built *ad hoc*, with formulae truth evaluation conditions that have been specially selected to justify theorems of some formally predetermined logical calculus. It is therefore not surprising that no concept of relevance arises here. Moreover, it is a wide-spread opinion that it is impossible to introduce a natural concept of relevance and thus everybody can define their own version of the concept. As a result we can find systems which are "deeply" relevant, or with "deep relevance", and so on (by the way these systems usually exclude laws which are very natural from the common sense point of view). Thus neither the practical application of such systems nor their theoretical significance are certain. We believe that introducing a concept of relevance, one should at least try to show its benefits.

It is possible, however, to interpret relevant logic as a new stage of the development of modern symbolic logic. "Development" here means improvement, or, if you want, "relevantisation" of individual systems of classical, intuitionistic and modal logics.

Generally speaking, this process of reformation involves first the substitution of the paradoxical and therefore inexact concept of entailment by a more definite concept free of negative features, namely, by relevant entailment. This concept should arise naturally as a result of the uncovering of the sources of paradoxes and their elimination. Secondly, there is the natural extension of a language by the introduction of implication as a linguistic

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analogy of entailment. Finally, the concept of entailment should be generalised to cover the case of all formulae of the extended language. We shall show below that the well-known relevant system *E (of entailment)* is the result of applying the above process of transformation to classical propositional logic (*CPL*); as a simple consequence of this one can come to the conclusion that similar transformations of the classical predicate logic lead to the system *EQ (entailment with quantification)* — the quantified extension of the system *E*.

When defining the relevant entailment for *CPL* we follow Ackermann [1] who originally proposed the idea of elaborating the concept of logical entailment "*A entails B*" by interpreting it as the "logical content of *B* is a part of a logical content of *A*". This view of entailment was adopted by Ackerman for his systems of so called "strong implication" [1], the modification of which has led to the system E (of entailment) of A. Anderson and N. Belnap [2]. But the further development of this idea of relevance faced difficulties arised mostly due to the uncertainty of the concept of "logical content of a proposition". We will consider this last concept as a "semantical information of a proposition" represented by its logical form. We believe the theoretical concept of semantical information in our presentation has been essentially refined.

Thus, we want to consider the concept of logical entailment as an informational relation between propositions. It is very natural to interpret this relation as a relevant entailment and to consider that it simultaneously introduces the concept of relevance. Intensional implication "$\rightarrow$" appears as a linguistic analogy of relevant entailment.

In conclusion we define the concept of entailment and intensional implication for the language of the system *E*.[1] Essentially we are building "informational semantics" adequate to *E* (with proofs of the soundness and completeness theorems).

Nevertheless, having informational concepts of relevant entailment and intensional implication and therefore the concept of relevance, we can establish that the system *E* (as well as *EQ*) correctly formalises only some fragments of these concepts. Hence it is possible to extend *E* while not breaking borders of relevance. Note that we simultaneously account for the principle distinction between the informational semantics we proposed and formal semantics (which is aimed, as we have already mentioned, to justify presupposed theorems and to reject everything which does not represent theorems).

Together with the introduction of the intensional implication the new logical constant appears, which corresponds to "if ...then" of our natural lan-

[1] Axiomatics for the system *E* is attached in the appendix.

guage expression used in science for the representation of laws of nature and necessary connections in general. This improvement of deductive apparatus of classical logic and expressiveness of artificial language has considerable methodological significance. Logical entailment plays an important role in the specification of different concepts and procedures of the knowledge. At the same time defects of entailment relation and expressiveness of classical logic language have been found (as a result exactly of the absence of facilities to express necessary connections) as for instance in the case of "ontological necessity", "scientific law", "counterfactual propositions", "disposition predicates", "scientific explanation" and other fundamental concepts of methodology of science.

This reformation of classical logic is also significant for another, philosophical, reason. Appearance of new laws of the types $A \rightarrow B$ and $\neg(A \rightarrow B)$ leads to more precise understanding of the concepts of analytical knowledge and logical necessity (it is important here to keep all laws of the classical logic, accurately considering material implication $A \supset B$ as a counterfactual proposition $\neg A \vee B$).[2] The conception on the nature of logical knowledge is changed due to the revelation of the informativeness of logical laws. This is a form of the development of logical knowledge analogous with the cases of Euclidean and non-Euclidean geometry or classical (Newtonian) and relativitstic mechanics. In these cases, which are governed by the so called "correspondence principle", an old theory — classical logic in our case— being replaced by the new theory, keeps its significance for some certain circumstances, namely, when we are interested in the correlation between truth values of propositions (but not between their contents as it is specified in the formal language of relevant logic).

Thus our task is to develop informational semantics adequate to E based on the analysis of CPL. This semantics should include definitions of relevant entailment and intensional implication. Finally, we are going to show possible extensions of E based on the generalisation of the previously mentioned concepts of entailment and implication. The formation of the prospective informational semantics has two steps:

- Step 1. An analysis of the sources of the paradoxes of entailment in *CPL* and a development of the concept of relevant entailment for formulae of classical logic as a result of the elimination of these sources.

[2] The so called paradoxes of material implication have their source exactly in an incorrect interpretation of material implication "$\supset$" as "if...then". In the best case this approach can be approved for only the main occurence of $\supset$ in some valid formula $A \supset B$ and only with consideration that $\supset$ notes only the connection between truth values of A and B but not between the situations which these propositions indicate.

- Step 2. Establishing a definition of the concept of relevant entailment and intensional implication for the full system E.

In the first part, a semantics of generalized state-descriptions, the concept of informativeness of propositions is essentially used; in the second step, a semantics of informational weakenings deals with the very information of propositions represented by their logical structures.

Using above concepts of relevant entailment, relevant implication and relevance itself we will try to understand in what sense a logical system can be considered as a relevant system.

Notation

- We use $\wedge$, $\vee$, $\neg$, $\supset$ for the connectives of classical logic
- We use bullets to distinguish logical constants (as the expressions of the logical language) from metalanguage expressions. Thus, $\dot\forall\alpha\ (P(\alpha)\ \dot\wedge\ Q(\alpha))$ means metalanguage expression "... for all state-descriptions α it is true that $P(\alpha)$ and $Q(\alpha)$..."
- We use "$\rightarrow$" for the relevant implication, "$\vdash$" in the context "$A \vdash B$" means "B is derivable from A" while in the context "$\vdash A$" means that A is a theorem. "$\models$ " in the context "$A \models B$" means "A entails B", while in the context "$\models A$" means that A is a law.

The meaning of other expressions will be introduced by definitions or explained in the text.

1. *The Semantics of generalised state-descriptions* (s.d.)

1.1. *Relevant entailment for* CPL *formulae*

The suggested CPL language consists of the connectives $\wedge$, $\vee$, $\neg$ and countable set of propositional variables $p_1, p_2, ... p_n$,.... The logical connectives have usual definitions but based upon the concept of *s.d.* ("state-description" — a description of a possible world). Following R. Carnap, we suggest that *s.d.* is a set, every basic element p_i or $\neg p_i$ of which satisfies the following two conditions:

- *(a)* $\dot\forall\alpha\ \dot\forall i\ (p_i \in \alpha\ \dot\vee\ \neg p_i \in \alpha)$ — for any *s.d.* α and any variable p_i this variable p_i or its negation $\neg p_i$ are elements of α and
- *(b)* $\dot\forall\alpha\ \dot\forall i\ \dot\neg(p_i \in \alpha\ \dot\wedge\ \neg p_i \in \alpha)$ — no variable belongs to *s.d.* together with its negation.

Truth evaluations in a world α are defined analogously with the tableau, however *true* and *false* are defined independently.

Dfl

1.	Tp_i/α	iff	$p_i \in \alpha$	1'.	Fp_i/α	iff	$\neg p_i \in \alpha$
2.	$T(A \wedge B)/\alpha$	iff	$TA/\alpha \dot{\wedge} TB\alpha$	2'.	$F(A \wedge B)/\alpha$	iff	$FA/\alpha \dot{\vee}$ $\dot{\vee} FB/\alpha$
3.	$T(A \vee B)/\alpha$	iff	$TA/\alpha \dot{\vee} TB\alpha$	3'.	$F(A \vee B)/\alpha$	iff	$FA/\alpha \dot{\wedge}$ $\dot{\wedge} FB/\alpha$
4.	$T(\neg A)/\alpha$	iff	FA/α	4'.	$F(\neg A)/\alpha$	iff	TA/α

According to the definition of logical entailment in classical logic, this relation $A \models B$ has two main features:

- It depends only on the logical content of the propositions involved, which means

  $$A_0 \models B_0 \text{ iff } A \models B$$

  where A, B are some particular propositions and A_0, B_0 are their logical forms, i.e. formulae of an appropriate language with interpreted logical constants;
- Any case when A is true in relation to some world α (for some interpretation of the propositional variables in A) determines B to be true under the same interpretation.

But what determines this second property? It is clear that "$A \models B$" means that the "informativeness of B — $i(B)$ — is a part of the informativeness of A — $i(A)$". Let us introduce the following notation for this relation:

$$i(B) \preccurlyeq i(A)$$

where $\preccurlyeq$ abbreviates the relation "...is a part of..." In the theory of semantical information, being dependened on the measurement system, in the case of binary system, $i(A)$ is defined as "$-log_2 P(A)$" where $P(A)$ is a probability of A. But this definition is not suitable for our purpose. It is not only difficult to compute this probability, but what is the most important, the definition above does not express the relativity of the informativeness of A, that it depends on some initial set of possibilities N. Taking this dependence into consideration it would be natural to define $i(A, N)$ (the informativeness of A in relation to N) as an index showing how an admission of A restricts the initial set of possibilities N. We can express $i(A, N)$ as a pair $\langle N_A, N \rangle$ (where N_A means a subset of possibilities N where A is true) and the relation $i(B, N) \leq i(A, N)$ as $N_A \subseteq N_B$. Note that the proposed comparison of $i(A, N)$ and $i(B, N)$ differs from the one in the case of the digital interpretation of informativeness. More precisely it deals with the following cases of homogeneity of information of A and B:

- This information is defined in relation to the same set N;
- $N_A \subseteq N_B$ or $N_B \subseteq N_A$. Only in these cases "$i(B, N) \leq i(A, N)$" is equal to "$I(B) \preccurlyeq I(A)$".[3]

In our work, as we are dealing with the information concentrated in the logical form of a proposition, this set N is a set of all *s.d.* possible in the suggested language, and N_C — for any formula C — is $\{\alpha : TC/\alpha\}$. According to well known correlation, $N_A \subseteq N_B$ in this case is equal to $\forall\alpha_{\in N}(TA/\alpha \dot{\rightarrow} TB/\alpha)$.

In some particular cases of defining the informativeness of A one can consider a set of *s.d.* constructed only with the variables of A. However the informativeness of A is not changed by the addition of new variables, which were absent in N. This allows us to consider N as the whole set of states of afairs which provides significant theoretical benefits. In particular, since N in all cases is the same, we can use $i(A)$ instead of $i(A, N)$. However as we will see, taking into account the relativity of informativeness is important in searching for the sources of the paradoxes of the classical entailment.

Nevertheless even this more precise concept of information is still paradoxical: $I(A)$ for any logical law A is a pair $<N, N>$ and for its negation — $<\emptyset, N>$. Therefore for any B we have: $B \models A$ and $\neg A \models B$. The reason for it can be found in some specific consideration of classical *s.d.* or, more precisely, in the concept of semantical information. Defining $I(A)$ in relation to the set N of classical *s.d.* — $I(A, N)$ — we use some knowledge about possible worlds. This is definite knowledge about the conditions *(a)* and *(b)* mentioned above. It is clear now that in reality we define not the informativeness of a proposition A itself, but something that A adds to the informativeness of *(a)* and *(b)*. If the information of A is already included in this knowledge about possible worlds, then $I(A)$ does not express any information, while $I(\neg A)$ is infinite ($\neg A$ expresses all information implied by all formulae of a language). All laws are, according to *Df1*, informationally covered by *(a)* and *(b)* as they can be established for example using truth tables only with the definiton of logical constants based on these principles.

To define the "pure" informativeness of a proposition itself in relation to the set of possible worlds, one has to abstract from any knowledge about

[3]For example if we interpret the informativeness of propositions in digits then the informativeness of the statement "A card taken out of the pack is of a diamond" is less than the informativeness of the statement "A card taken out of the pack is the ace of hearts". But it is clear that informativeness of the first statement completely failed to be a part of informativeness of the second one. A probabilistic theory of information does not consider the information of a proposition but rather the information of accidental choices —an object of the statistical theory of information.

correlation between its elements p_i and $\neg p_i$. This knowledge is only necessary and sufficient for a comparison of informativeness of extensional propositions. Thus, we have to abstract from the principles *(a)* and *(b)* which is equivalent to the abstraction from the correlation between p_i and $\neg p_i$ and therefore to the abstraction from the correlation between truth values of some formula in some world. Hence, *s.d.* α should be understood as a set with elements p_i or $\neg p_i$. Among all possible worlds we allow *s.d.* where for some *i* there are no occurrences of p_i or $\neg p_i$ and we even allow empty *s.d.* Another possible *s.d.* is one, which contains p_i and $\neg p_i$ simultaneously. Let us call these "non-standard" *s.d.* as *generalised s.d.* A generalised *s.d.* α is any subset of a set $\{p_1, p_2, ..., p_n, \neg p_1, \neg p_2, ..., \neg p_n\}$. Obviously, contradictory and uncertain possible worlds (in relation to some p_i) are admitted as abstract possibilities due to the abstraction from *(a)* and *(b)* but only when concerned with the definition of informativeness for formulae.

These *s.d.* are abstractly possible *s.d.*, more precisely, they are descriptions of abstractly possible worlds. Subsets of generalised *s.d.*, which satisfy the principles *(a)* and *(b)*, might be termed classical, or normal, while *s.d.* corresponding to them are actually possible. However, strictly speaking, we can consider only the principle of consistency as a condition for actual possibility.

Worlds where the principle *(a)* fails are actually possible, for example *(a)* fails for a proposition expresses some accidental event in the future. As for the contradictory worlds, they are at least intuitively possible. It also seems that intuition is the only reason to prohibit such worlds, i.e. to accept the ontological law of inconsistency. Let us note that the famous law $\neg(p \wedge \neg p)$, which is always regarded as prohibiting contradictions, according to Df1 is equivalent to $p \vee \neg p$ and thus does not tell us anything about contradictions.

Now if in Df1 and in all definitions above with "$\models$" we consider α as a generalised *s.d.* and replace a set *N* of classical worlds by the set *M* of generalised *s.d.* we will have the following definitions of *relevant entailment* for the *CPL* formulae:

Df2 $A \models B$ iff $I(B, M) \leqslant I(A, M)$ iff $i(B, M) \leqslant i(A, M)$ iff $M_A \subseteq M_B$ iff $\forall \alpha_{\in M}(TA/\alpha \dot{\rightarrow} TB/\alpha)$.

Df2a $\Gamma \models B$ iff $\dot{\exists} A_1, A_2, ... A_n \in \Gamma (A_1 \wedge A_2 \wedge ... \wedge A_n \models B)$ (for any non-empty set of propositions Γ and for $n \geq 1$ and some combination of brackets in conjunction).

With the proposed generalisation of *s.d.* we keep the strictness of the definition of *T* and *F* for any *A* and any *s.d.* from *M*. It is clear that in addition to the classical cases $\langle TA/\alpha, \dot{\neg}FA/\alpha\rangle$, and $\langle FA/\alpha, \dot{\neg}TA/\alpha\rangle$ we have now new cases of $\langle TA/\alpha, FA/\alpha\rangle$ and $\langle \dot{\neg}TA/\alpha, \dot{\neg}FA/\alpha\rangle$.[4] The concept of relevant entailment is not paradoxical, since there is no A, for which M_A is universal (or empty). Logical laws now express some specific information.

Note that the introduction of the generalized concept of information $I(A, M)/\Gamma$ — information of A in relation to a set M with the knowledge Γ about the correlation of p_i and $\neg p_i$ in M worlds— leads to the generalized concept of logical entailment. Several versions of this entailment can be developed for the same language:

- with empty Γ we have relevant entailment;
- $\Gamma = (a), (b)$ leads to the classical entailment;
- $\Gamma = (a)$ yields entailment of the Hao Wang system;
- $\Gamma = (b)$ establishes entailment in the system which is dual to the Hao Wang system;
- $\Gamma = \dot{\forall}\alpha\ (\dot{\exists}\ i\ (p_i \in \alpha\ \dot{\wedge}\ \neg p_i \in \alpha\ )\ \dot{\rightarrow}\ \dot{\forall} i\ (p_i \in \alpha\ \dot{\vee}\ \neg p_i \in \alpha)$ yields the definition of the entailment in the first-degree fragment of Lukasievich logic.

We then extend *CPL* by the introduction of the intensional implication $\rightarrow$ as a new logical constant resulting in appearence of the formulae of the new type. Thus we can express laws $A \rightarrow B$ or $\neg(A \rightarrow B)$. This will constitute the general framework for the second part of the paper (semantics of informational weakenings of propositions).

Since we intend to consider "$\rightarrow$" as a linguistic analogy of the metalanguage entailment, we have:

Df3 $\models_E ((A_1 \rightarrow B_1) \wedge ... \wedge (A_k \rightarrow B_k$ iff $\dot{\forall} i \leq k\ (A_i \models\ \ B_i)$ meaning "B_i is a relevant consequence of A_i", here $\models_E A$ means that A is a law in E.

Laws of the type $(A \rightarrow B)$ for A and B of the *CPL* language are derived according to Df2, their conjunction are obtained according to Df2a. And as Df3 claims, this conjunction is also a law.

The definition Df3 represents the general form of the main laws of E which imply all other laws via the weakening principles which are discussed below.

[4]The existence of these four cases is often interpreted as implying the existence of many truth values (4 semantical values) for the system E. However the truth value of a proposition of some system depends on the sense of this proposition. In our case we kept the sense of the propositions of *CPL* but abstract from known correlations between truth and false.

Until now we have established entailment relations $A \models B$ and laws $A \to B$ for the extensional A and B based on the direct comparison of their informativeness (according to the last part of Df2 — $\forall\alpha_{\in M}(TA/\alpha \to TB/\alpha)$. It is natural to consider as extensional any formula without $\to$, i.e. any formula of classical logic which has only Boolean connectives. However, we can generalise this concept considering any formula A with $\to$ also as extensional if we abstract from the meanings of its subformulae of the type $E \to F$. Thus, considering these formulae as elementary, we can substitute them by new variables or their negations (which have not been occurred in the given proposition). By doing it we extend the set of laws $A \to B$ generated by Df2 and also the set of laws generated by Df3 when expanding definitions Df1, Df2 and Df3 to all formulae of the language of the system E, which are considered as extensional. In this case the former concept of *s.d.* has not been formally changed. Now these *s.d.* describe only extensional aspects of the related worlds, sets of state-descriptions of the factual character (expressible in *CPL*). Note, however, that in the world, together with these situations of the factual character, there are various correlations between them, described by the formulae of the intensional character (i.e. those with occurrences of formulae $E \to F$ where meaning of such implications is essential).

Besides, on the basis of extensional description of a world it is possible to define truth conditions of $A \to B$ with extensional A and B. But first of all we have to consider some problems concerning the relation between the relevant entailment and the intensional implication.

Intensional implication $\to$ is usually charactarised as a language analogy to the metalanguage entailment $\models$. However, this conception needs some corrections. We can understand statements $A \to B$ in two ways.

The first way of interpreting $A \to B$ suggests that it is an expression of the metalanguage entailment, i.e. $A \models B$, or $\models (A \to B)$. Such statements are statements about correlations between propositions within the language, in other words, their objects themselves are propositions (formulae) of a language. More precisely, the metastatements $A \models B$ are about forms A and B. Thus, taking into account well-known logico-semantical speculations, it would be better to write 'A' $\models$ 'B' instead of $A \models B$. Truth values of such statements do not depend on the structures of individual worlds, but on the sense of A and B and in general on the character of the set M of possible worlds.

Things are different when $A \to B$ is a formula of language. Firstly, it expresses the relation not between formulae A and B but between situations which are supposed to happen in the world and are described by these formulae. When related to some *s.d.*, $A \to B$ becomes a proposition in the language and the truth values are given to its atomic variables. The intensional implication $A \to B$ means "the situation A determines the situation B

in a world α". Now $A \to B$ has an additional essential content in comparison with the metalanguage $A \models B$. The determination itself can be a purely logical determination or can be a determination in conjunction with some scientific theory Th (i.e. in the case when $Th, A \models B$ where B depends on some part of Th, we have $Th \models A \to B$).

In the first case $A \to B$ is logically necessary. In the second case it is ontologically necessary (in the sense of this scientific theory with the relevant interpretation of entailment).[5]

Exactly in such process of linking $A \to B$ with some world, questions about its truth evaluation in this *s.d.* arise. Here a specifics of $\to$ (in comparison to metalanguage $\models$) as an intensional implication can be found. In this case $\to$ is an analogy to the natural language *if...then* when it is used for an expression of necessary statements, for example, scientific laws.

Thus, we can interpret meaning of $A \to B$ in the two ways: as a language expression of the metalanguage entaiment $A \models B$ or as an expression of the object-language, related to some worlds α. The usage of $A \to B$ under the second consideration seems to be essential when the language in terms of which $A \to B$ has been formulated, is a syntactical basis for some logic applied to some scientific theory. Exactly in this case truth conditions of $A \to B$ in some world are important.

For our task of the semantical justification of the system E as a deductive system, it is sufficient to accept the first of the described interpretations of $A \to B$, i.e. in the metalanguage aspect. (It is correct at least because of the following: correctness of proofs does not depend on the concrete contents of propositions). Thus, we do not need concepts $T(A \to B)/\alpha$ and $F(A \to B)/\alpha$, except those defined for the first degree case of $A \to B$.[6] $A \to B$, according to Df2, is true in α if and only if for any generalised *s.d.* β (and therefore for α itself) $TA/\beta \dot{\to} TB/\beta$ holds. Hence, we have $TA/\alpha \dot{\to} TB/\alpha$

[5] In classical logic $\models A \to B$ also determines $Th \models A \to B$. Here any logical necessity is an ontological necessity. It is however inacceptable for the relevant derivations. In relevant logic we can only have that if $\Gamma \models C$ then $\Gamma' \models C$, where Γ is not empty and Γ' is any extension of Γ.

[6] Note that exactly this first degree case of $A \to B$ is important when the system E (EQ) is applied as a tool to analyse some scientific theory. In the formulation $\forall x\,(A(x) \to B(x))$ of scientific laws $A(x)$ and $B(x)$ themselves do not contain expressions on entailment, i.e. do not have occurrences of $\to$.

and then $\dot{\neg} TA/\alpha \ \dot{\vee}\ TB/\alpha$. According to the contrapositivity[7] of entailment, i.e. $A \rightarrow B$ iff $\neg B \rightarrow \neg A$, we also have $T\neg B/\alpha \ \dot{\rightarrow}\ T\neg A/\alpha$ and then $\dot{\neg} T\neg B/\alpha \ \dot{\vee}\ T\ \neg A/\alpha$. Linking $A \rightarrow B$ with some particular α, in other words considering $\rightarrow$ as a specific logical connective meaning "the situation A determines the situation B", we are led to extend these truth conditions by the following: we require to have in α some information concerning either A or B. More precisely, it is sufficient to suggest that α satisfies A or $\neg A$ or B or $\neg B$. Thus, we have to accept:

$$T(A \rightarrow B)/\alpha \text{ iff } (\dot{\neg} TA/\alpha \ \dot{\vee}\ TB/\alpha) \ \dot{\wedge}\ (\dot{\neg} T\neg B/\alpha \ \dot{\vee}\ T\ \neg A/\alpha) \ \dot{\wedge}\ (TA/\alpha \ \dot{\vee}\ T\neg A/\alpha \ \dot{\vee}\ TB/\alpha \ \dot{\vee}\ T\neg B/\alpha)$$

As our metalanguage $\dot{\wedge}$, $\dot{\vee}$ and $\dot{\neg}$ are classical, this is equivalent to

$$T(A \rightarrow B)/\alpha \text{ iff } (\dot{\neg} TA/\alpha \ \dot{\vee}\ TB/\alpha) \ \dot{\wedge}\ (\dot{\neg} T\neg B/\alpha \ \dot{\vee}\ T\ \neg A/\alpha) \ \dot{\wedge}\ (T\neg A/\alpha \ \dot{\vee}\ TB/\alpha).$$

Now based on some simple natural reasoning, for $\neg(A \rightarrow B)$ we have:

$$F(A \rightarrow B) \text{ iff } (TA/\alpha \ \dot{\wedge}\ \dot{\neg} TB/\alpha) \ \dot{\vee}\ (T\ \neg B/\alpha \ \dot{\wedge}\ \dot{\neg} T\ \neg A/\alpha) \ \dot{\vee}\ (TA/\alpha \ \dot{\wedge}\ T\ \neg A/\alpha \ \dot{\wedge}\ TB/\alpha \ \dot{\wedge}\ T\neg B/\alpha)$$

which is equivalent to:

$$F(A \rightarrow B) \text{ iff } (TA/\alpha \ \dot{\wedge}\ \dot{\neg} TB/\alpha) \ \dot{\vee}\ (T\ \neg B/\alpha \ \dot{\wedge}\ \dot{\neg} T\ \neg A/\alpha) \ \dot{\vee}\ (TA/\alpha \ \dot{\wedge}\ T\ \neg B/\alpha)$$

This extends our Df1. As for the definition of the truth conditions for $A \rightarrow B$ for any A and B in extensional *s.d.*, it seems that this is in general impossible as such definition would require an introduction into *s.d.* some conditions of intensional character. It would be interesting investigation to try to reveal these conditions. But this seems to be very difficult problem.[8] We

[7]Contrapositivity is justified in [6], part 1, section 4, for the extensional A and B. It can be also established on the basis of the known law $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$, which will be considered below.

[8]This for example is indicated in the definitions of formal semantics of Routley-Meyer and Maximova [5]. Here the structures of *s.d.* α are only partly considered. Concepts of truthfullness and falseness of $A \rightarrow B$ (more precisely, truthfullness of $A \rightarrow B$ and $\neg(A \rightarrow B)$ as the concept of falseness is simply absent in these semantics) represent some conditions of interpreting subformulae in the formulations of some logical laws in some world independently of its structure. These conditions determining 3-placed accesability relation $R(\alpha, \beta, \gamma)$ have been artificially chosen to justify these laws.

have already mentioned that this general definition are not necessary for our analisys. Our definition Df2 is still valid for any formulae. However, this generalisation will require special methods of analysis, without referring to the concepts TA/α and FA/α. We only note here that the minimal condition of the truthfullness of $A \rightarrow B$ in some α is that fact of informative certainty of α in relation to any propositional variable p_i occuring in $A \rightarrow B$. For example, $p \rightarrow p$ is not true in the world α where $\dot{\neg} Tp/\alpha$ and $\dot{\neg} T \neg p/\alpha$. (Note also that if α satisfies both p and $\neg p$ then $p \rightarrow p$ is not only true in such world but also false in it.)

Our definition of $T(A \rightarrow B)/\alpha$ for the extensional A and B can be used to establish *E-11* (see appendix): $(A \rightarrow B) \rightarrow (\neg A \vee B)$. This is provided by simply considering the meaning of $A \rightarrow B$ as a statement about the correlation between the situations A and B in some world α: since the situation A in α determines the situation B, we exclude the case of the presence of A and the absence of B, i.e. α should satisfy $\neg(A \wedge \neg B)$. This last is "relevantly" equivalent to $\neg A \vee B$ (see the proof in the next section).

The scheme *E-11* indicates some correlation between intensional and material implication. Taking into account also contrapositivity of *E-11*, we are led to the important methodological conclusion about the correlation between the extensional and intensional laws. If $\models A$ with the intensional A then $\models A'$ where the extensional A' is obtained from A by substituting its subformulae $C \rightarrow D$ with $\neg C \vee D$. We can say that $\models A'$ is an extensional analogy of $\models A$.[9] To conclude this part we have to distinguish two general ways of using the formulae $A \rightarrow B$ in the language of E:

- for an expression of the law $\models (A \rightarrow B)$ — the case of established entailment $A \models B$ (as we have in axioms of the system E)
- for an expression of the assumption that a formula of this type is a law

In this sense it is important to remember that propositional variables are introduced into the propositional language as abbreviations for some statements the structure of which we abstracting from. We term these propositions (more precisely speaking, the situations described by them in possible worlds) *intensional values of propositional variables*. Their existence is implicitely suggested in the rules of substitution. Formula $A \rightarrow B$ is a law if and only if $A \models B$ under all intensional values of A and B, in other

[9]This correlation is expressed in the fact that any proof of an intensional theorem A in axiomatic formulation of E becomes a proof of its extensional analogy A in the classical logic. A' can be obtained from A using the same method of substituting all occurences of $\rightarrow$ with $\supset$ and interpretation of $A \supset B$ as $\neg A \vee B$. It is obvious that this is a superfluous formalisation of the *CPL*. In the system E itself any classical law can be obtained as a consequence of some intensional law considered in Df3.

words, under all substitions for these variables. An assumption that $A \rightarrow B$ is a law, which is for instance, introduced into the deductive reasoning, means that it is a law, but only under some values of its propositional variables, i.e. under some interpretations, which can be expressed as "let A and B be such that $A \models B$". In the first case variables A and B are considered in the interpretation of "totality", in the second case — in the conditional interpretation. By analogy we can speak about these two types of interpretation of metalanguage expressions A, B and C, in formulae schemes and so relatively differentiate schemes of factually established laws and assumptions that $A \rightarrow B$ is a law.

The formulae $p \rightarrow q$ and $q \rightarrow r$ in the formulation of the transitivity law $(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$ are assumptions, while $p \rightarrow r$ is their consequence. $p \rightarrow q$ is true for example when the intensional value of p is $p \wedge q$ (i.e. as a result of the substitution of p with $p \wedge q$); the scheme of assumption $A \rightarrow (A \rightarrow A)$ represents a scheme of a law $(A \rightarrow A) \rightarrow ((A \rightarrow A) \rightarrow (A \rightarrow A))$ when $A \rightarrow A$ is a proper intensional value of A (this last means $\models (A \rightarrow A) \rightarrow ((A \rightarrow A) \rightarrow (A \rightarrow A))$).

Obviously we have to allow the case when a variable p_i is an intensional value of itself (a case of trivial substitution of a variable p_i for itself). This is the proper intensional value of p_i. $A \rightarrow B$ is a law iff $A \models B$ under the proper values of A and B. Thus $A \rightarrow B$ is always used as a statement on entailment.[10] For example if $\models A$ and $\models B$ then $\neg(A \rightarrow \neg B)$.

We have analysed the usage of extensional *s.d.* of possible worlds for establishing relations of the logical entailment and laws of the type $A \rightarrow B$. To establish the relevant entailment, i.e. of the type $i(B) \preccurlyeq i(A)$, we need another methods. These methods, namely principles of *informational weakenings* of propositions, will be proposed in the second part of the paper. Some of them (WP1-3) are based on the established entailments, in the other cases (WP3-5) entailment is established via analysing the structures of propositions (possibly with established relations $i(A_1) \preccurlyeq i(A_2)$).

[10]Some logicians reject a possibility to charactarise $\rightarrow$ as an analogy to metalanguage $\models$. Their argumentation is often based exactly on the fact that not any occurrence of $\rightarrow$ expresses entailment, and thus, only main occurence of $\rightarrow$ can be interpreted in such way. However, here the possibility to consider $A \rightarrow B$ as an assumption on entailment between A and B is ignored. We also have to remember the following differenece between two types of systems. In the first case, as in the classical logic, only the main occurrence of implication (material) in a law can represent entailment. In the systems of another type, which as we think are "normal", any occurrence of implication is a language analogy of entailment. This happens in the systems *E* or *S4*. In *E* implication is an analogy of relevant entailment, while implication of *S4* is an analogy of classical entailment.

2. *Semantics of informational weakenings of formulae*

As the basis for weakening strategy let us introduce the following set of laws (entailment relations) of the previous part (with extensional A and B as we generalised this concept above).

IV. 1. $A \to A$ 2. $(A \wedge B) \to A$ 3. $(A \wedge B) \to B$ 4. $A \to (A \vee B)$
5. $B \to (A \vee B)$ 6. $A \to \neg\neg A$ 7. $\neg\neg A \to A$
8. $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$ 9. $A \to (A \wedge A)$
10. $(A \vee A) \to A$ 11. $(A \to B) \to (\neg A \vee B)$.

Here, together with *E-11*, we have the justification for schemes *E-1, E-2, E-3, E-4, E-7, E-8, E-9*. For the justification of the other schemes and rules of the system E we use principles of the informational weakening of propositions.

Our concept of informational weakenings proceeds from the rather obvious statements that p is informationally weaker than $p \wedge q$, while $p \vee q$ is informationally weaker than p. For their negations we have that $\neg(p \wedge q)$ is informationally weaker than $\neg p$, while $\neg p$ is informationally weaker than $\neg(p \vee q)$. To generalise this we use known definitions of positive and negative occurrences of subformulae within a formulae.[11] True, or considered as true, formula $C \to D$ points out that D is informationally weaker than C or informationally equivalent with C (we shall use an expression "weaker" supposing that it covers both cases). Formula A is informationally weakened if some of its positive subformulae are replaced by weaker formulae (according to some logical law of the type $C \to D$) or if some of its negative subformulae are substituted with more stronger formulae. For instance, $(p_1 \wedge p_2) \to p_1$ as well as $p_1 \to p_1 \vee p_2$ is weaker than $p_1 \to p_1$, etc. The set of formulae in *IV* gives some points of departure to develop a theory on the informational relations between formulae. The fact that these relations have been established for the extensional propositons, i.e. that their subformulae of the type $C \to D$ were considered as elemantary, can not be an obstacle in prospective application of the weakening principles. We will see that when applying a weakening of some formula on the basis of $A \to B$ we do not deal with the structures of A and B.

[11] Recall the concept of positive and negative instances of one formula within another one: 1. Each formula occurs positively in itself. 2. If a formula of the type $A \wedge B$ or $A \vee B$ positively (negatively) occurs in C then both A and B positively (negatively) occur in C. 3. If $\neg A$ positively (negatively) occurs in C then A negatively (positively) occurs in C. 4. If $A \to B$ positively (negatively) occurs in C then A negatively (positively) and B positively (negatively) occurs in C.

2.1 *Main principles of informational weakening of propositions*

Principle of weakening on the basis of assumptions
Let G be a formula of the type $(A_1 \rightarrow B_n) \wedge ... \wedge (A_k \rightarrow B_k)$, $k \geq 1$ (i.e. an admission of the law described in Df3)[12], C — be any formula with some positive instances of some A_i and negative instances of some B_j, $1 \leq i \leq k$, $1 \leq j \leq k$. Then C^w_G — (a weakening of C on the basis of G) is a result of substituting in C of at least one positive entry of A_i with B_i and (or) some negative entry of B_j with A_j. If C has any subformula of the type $E \wedge F$ or $E \vee F$, then some of the substitutions mentioned should be carried out in both E and F or should not be carried out at all (*condition R*). We now can define several rules of weakening as follows:

WP1 $\models (G \rightarrow (C \rightarrow C^w_G))$
where C^w_G is a result of weakening C on the basis of G.

If some $(A_i \rightarrow B_i)$ in G is simply $(A_i \rightarrow A_i)$ then we call the substitution trivial; however we consider substitutions done in all positive and negative occurrences of A_i in C. Note the important role the law of identity plays in establishing of the entailment relation. For instance, $(A \rightarrow B) \rightarrow ((A \wedge D) \rightarrow (B \wedge D))$ can not be deduced according to WP1 since in this case condition R is not satisfied. Note that the formula $(A \rightarrow B) \rightarrow ((A \wedge D) \rightarrow (B \wedge D))$ by weakening (see below WP4a) of $(A \wedge D) \rightarrow (B \wedge D)$ gives $(A \wedge D) \rightarrow D$ which itself leads to $(A \rightarrow B) \rightarrow ((A \wedge D) \rightarrow D)$. This last obviously represents irrelevance between antecedent and consequent. But simple addition of $D \rightarrow D$ to the antecedent yields a relevant formula $((A \rightarrow B) \wedge (D \rightarrow D)) \rightarrow ((A \wedge D) \rightarrow (B \wedge D))$.

Weakening principle WP1 is used to justify axioms *E-10* and *E-13*. For both schemes we have $A \rightarrow B$ with G as a basis for weakening; further, $\neg B$ for *E-10* plays the role of C, and thus we derive $\neg A$ by the weakening $\neg B$ on the basis of $A \rightarrow B$. In turn, in the case of *E-13* by weakening $B \rightarrow C$ we deduce $A \rightarrow C$ (i.e. A is derived by strengthening B on the basis of $A \rightarrow B$). Simple modification of WP1 leads to:

WP2 $\models ((G \wedge C) \rightarrow C^w_G)$. we call this *the 'principle of assumption' self-weakening*. Obviously here we obtain C^w_G similarly to WP1; C^w_G, being weaker than C, is weaker than $G \wedge C$. As me mentioned, WP2 is implied by WP1; but their correlation is stronger — WP1 and WP2 are relevantly equivalent because they have the same information that C^w_G is weaker than C on the basis of relations indicated in G. This equiva-

[12]For the semantical justification of E it is enough to consider $k \leq 2$.

lence becomes more transparent if we represent WP2 according to Df2 and Df3 in the following way:

WP3 $\models (G \rightarrow (C \rightarrow C^{w}_{G})) \rightarrow ((G \wedge C) \rightarrow C^{w}_{G})$

If in WP1 we have $\models G$, i.e. G is true under any substitution, then it also means that its consequent is universally valid. As a result we are led to the following principle, for the formulation of which we use L instead of G, to indicate that in this case we are dealing with a law;

WP4 $\models C \rightarrow C^{w}_{L}$

here *condition* R is not needed since we can always consider the proper law of identity in L. No related substitutions are necessary here. In the case of $\models C$ the following principle holds:

WP4a $\models C^{w}_{L}$

weakening of a *logical law* gives a law, we call this the *L-weakening*-"principle of logical weakening". Furthermore

WP5 $(G \rightarrow (C \rightarrow C^{w}_{G})) \rightarrow (C \rightarrow C^{w}_{G})$ if $\models G$

WP6 $(C \rightarrow C^{w}_{L}) \rightarrow C^{w}_{L}$ if $\models C$

Obviously WP5 and WP6 can be generalised as follows: $\models (A \rightarrow B) \rightarrow B$ where $\models A$.

Principles WP5 and WP6 justify *E-12*, depending on the conditions under which its antecedent (as an assumption) represents entailment $A \rightarrow A \models B$. This can happen due to the WP1, when B is equal to $(C \rightarrow C^{w}_{(A \rightarrow A)})$. In this case *E-12* is justified according to WP5 as we have $\models G$, i.e. $\models A \rightarrow A$. If this antecedent as an assumption represents entailment based on the *WP4*, i.e. $\models C \rightarrow C^{w}_{L}$, where both C and L are $A \rightarrow A$, we have $\models (A \rightarrow A) \rightarrow (A \rightarrow A)^{w}_{A \rightarrow A}$. In this case *E-12* is justified by WP6.[13] By WP1, together with WP4a schemes *E-3* and *E-6* are justified. According to WP1, we have $\models (((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow ((A \wedge A) \rightarrow (B \wedge C)))$, then by L-weakening on the basis of 9 of *IV*, following WP4a, *E-3* is derived. *E-6* has an analogous proof with 10 of *IV* as the basis for L-weakening.

The justification of *E-14* proceeds as follows: there is only one case when its antecedent can be true. This is the case of A with a structure of G of

[13] A question about the possibility of other approaches can arise. But if the semantical justification of all theorems of E can be successfully provided based only on our interpretation, then it is obvious that any other approach is not necessary. Note also that principles of weakening allow to establish laws of the type $A \rightarrow B$ indicating intensional values of syntactical variables in schemes of assumptions about entailment between some formulae.

WP1 and B equal to A^{w}_{A}. Then we have $(A \to (A \to B)) \to ((A \land A) \to B)$ as a type of WP3; now, according to WP4a, with *IV-9* as L, we derive *E-14*.

On the basis of WP1 and WP5 it is possible to generalise *E-11* established for the extensional A and B: with L as this scheme we have $C \to C^{w}_{L}$, where C^{w}_{G} is a result of substitution of any occurrence of $E \to F$ by $\neg E \lor F$.

Therefore all axiom schemes have been justified. To justify the rule *modus ponens* we, firstly, represent WP2 as the rule G, $C \models \; C^{w}_{G}$ which, according to Df2, is equivalent to WP2. If G is $C \to B$ then we have the formulation of *modus ponens*.[14] However, it was a special semantical case of *m.p.* Another E rule, introduction of conjunction, can be justified similarly using Df2a.

This establishes the semantical justification of E and hence its soundness has been shown.

For the proof of completeness we underline the main laws of the Df3 type and those derived from them, i.e. laws of the two types: $\models \; \neg(A \to B)$ and of the type of classical laws. Main laws, as we have already observed, are derived either according to Df2 (perhaps together with Df2a) or by direct establishment of the entailment relation or according to the weakening principles. All other laws are derivable from the main laws by weakening pronciples. For instance, the law of the type $\neg(A \to B)$ can be derived from some law C by the L-weakening $\neg\neg C$ (which is, on the other hand, strengthening of $\neg C$ in $\neg\neg C$). Here $\to$ in $\neg(A \to B)$ appears due to the strengthening of $\neg C$ (which gives $L \to \neg C$) on the basis of a law $(L \to B) \to B$. This last in turn represents a kind of generalization of a law introduced by *E-12*, where L is any semantical law.[15] Alternatively, the same result can be obtained by the strengthening of $\neg C$ (which gives $L \to \neg C$) according to the law $(A \to \neg A) \to A$, which as we will see is equivalent to *E-11*.

It is obvious that all laws of *CPL* can be derived by a weakening of some law indicated in Df3 based on the scheme *E-11* (where some members A_i

[14]Generally speaking, we have already applied this rule when we generated weakening principle WP4 from WP1 with $\models \; G$.

[15]In [6] one can find a description of the system E_t which is equivalent to E. Each theorem in E_t is derived by logical weakenings from some conjunction, perhaps degenerate, of the laws of identity. Taking into account that any such conjunction is derived from $A \to A$ by WP5, where $A \to A$ plays a role of $C \to C^{w}_{L}$, we can in any case generate some law L (theorem in E) as a result of weakening of $A \to A$ in *E-12* according to the principles of E. This statement can be intepreted in terms of the justification of recently mentioned law in the described semantics where weakening principles of E_t are considered semantically (in E_t originally they are understood as syntactical rules).

$\rightarrow B_i$ are simply laws of identities $A_i \rightarrow A_i$), and possibly together with *E-1, E-2, E-3, E-7*.

Taking into consideration an interpretation of a classical law A in conjunctive normal form (CNF)[16] one can show that each conjunctive member of CNF contains $\neg p_i \vee p_i$ for some i, and hence can be derived from $p_i \rightarrow p_i$ by using axioms *E-4* and *E-5*. Moreover, the whole conjunction is deduced from the conjunction of all laws $p_i \rightarrow p_i$, which correspond to each member of CNF of A, according to *E-3*. A law of the type $A \vee B$, where A or B are laws, can be derived by WP4a on the basis of *E-4* and *E-5*.

Consequently, we have to show that all main laws introduced by Df3 are theorems. Now the strategy is as follows: firstly, to show that any law of the type $A \rightarrow B$ for the extensional A and B is a theorem in E and, secondly, that so are all weakening principles considered as syntactical statements. Then, since we have the rule of the introduction of $\wedge$ in the system E, any Df3-type law is a theorem.

For the laws with extensional A and B it is known that each of them is a tautological implication in the sense of section 15 of [2], (see also [6], section 4), i.e. $A \rightarrow B$ (where A and B have no occurrences of $\rightarrow$, or, as we can now say, possibly contain formulae of the type $C \rightarrow D$ considered as elementary) is tautological if for any member A_i of the DNF A and any member B_j of the CNF B there is at least one common member in any pair $<A_i, B_j>$.

Any tautological implication is a theorem in E — this is obvious since any tautological implication is a theorem in the E_{fde}— subsystem of E, as it is shown in [2], and hence is a theorem in E. All of the weakening principles can be justified in E. Principle WP1 can be proved by induction based on the number of logical connectives in C. Other principles are proved according to the concept of derivability in E.

We now illustrate the effectiveness of the introduced semantical concepts. Consider the well-known principle of relevance ([2]), according to which no formula of the type $A \rightarrow (B \rightarrow C)$ can be a theorem in E if A does not contain an $\rightarrow$. This principle becomes obvious in the light of WP1, since from the point of view of weakening, it can be interpreted as "C is a result of weakening B on the basis of A", but nothing here indicates that A has the structure which is required for G by WP1. The same reason can be found to establish that $A \rightarrow (B \rightarrow B)$ is not a theorem of E, recall that the addition of this formula to E as an axiom transforms E into the modal system *S4*. Obviously the characteristic axiom of *EM* (E-Mingle) $(A \rightarrow B) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow B))$ also does not satisfy ideology of weakenings.

16 Semantical justification of the transformation of extensional formulae of E to their conjunctive and disjunctive normal forms can be easily approved using Df1.

Though its consequent is itself a theorem in E, it is not relevant to its antecedent, since it can not be deduced as a result of weakening $(A \rightarrow B)$, which should be the case according to WP1. First intuition does not indicate anything against $(A \rightarrow B) \rightarrow (A \rightarrow (A \wedge B))$. However, according to WP1, its consequent $(A \wedge B)$ is not a result of weakening A (which plays the role of C in the formulation of this principle). Note that excepting this formula as a law we also have to except its consequence $(A \rightarrow B) \rightarrow (A \rightarrow A)$ which is obviously irrelevant.

Application of weakening principles significantly simplifies proofs of many theorems in E. For example, consider the following proof of $\neg(A \wedge \neg B) \models (\neg A \vee B)$:

$$\neg(A \wedge \neg B) \models \neg(\neg\neg A \wedge \neg B) \models \neg(\neg(\neg A \vee B) \wedge \neg(\neg A \vee B)) \models \neg\neg(\neg A \vee B) \models (\neg A \vee B).$$

Usually in the formulation of E $(A \rightarrow \neg A) \rightarrow \neg A$ is used instead of *E-11*. It is easy to show their equivalence in the sense of weakenings. Indeed, having $(A \rightarrow \neg A) \rightarrow \neg A$ and using WP4 and WP4a we construct the following proof: $A \rightarrow B \models A \rightarrow \neg\neg B \models A \rightarrow \neg(A \wedge \neg B) \models (A \wedge \neg B) \rightarrow \neg(A \wedge \neg B) \models \neg(A \wedge \neg B)$. The last step is derived by the scheme $(A \rightarrow \neg A) \rightarrow \neg A$. Thus, using the scheme justified above we have $(A \rightarrow B) \models (\neg A \vee B)$.

In the opposite direction:

$(A \rightarrow \neg A) \models \neg(A \wedge \neg\neg A) \models \neg(A \wedge A) \models \neg A$, where the first step is obtained by the scheme $A \rightarrow B \models \neg(A \wedge \neg B)$.

It is obvious that after establishing some law of the type of Df3 we can use weakening WP4a with this law as L. As an example let us consider the proof of the distributivity law $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$ by weakenings. We begin with the law $(A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow (B \rightarrow C))$ and logically weaken its consequent obtaining $((A \rightarrow B) \rightarrow (A \rightarrow C))$:

1. $(A \rightarrow (B \rightarrow C)) \rightarrow (((B \rightarrow C) \rightarrow (A \rightarrow C)) \rightarrow (A \rightarrow (A \rightarrow C)))$ — weakening by the transitivity law.
2. $(A \rightarrow B) \rightarrow (A \rightarrow (A \rightarrow C))$ — weakening of the previous formula by the strengthening its antecedent again by the transitivity law.
3. $(A \rightarrow B) \rightarrow (A \rightarrow C))$ — weakening of the consequent of 2 on the basis of the contraction (scheme *E-14*).

A possible extension of E mentioned at the beginning of this article can be developed by weakening WP2. One can easily see that both restrictions accompanying WP1 are not necessary in the formulation of WP2. Indeed, if C^w_G in WP2 is equal to C (i.e. no weakenings on the basis of G have been provided), then we have a law $((A \wedge B) \rightarrow B)$. Moreover, even if *condition R* is not satisfied while C is weakened on the basis of G, it does not lead to any irrelevance, unlike the outcome for WP1. Among formulae that become valid under these new conditions, we discover formulae, which are valid in the Urquhart system U, but not in E:

$((A \rightarrow B) \wedge (C \rightarrow (A \vee D))) \rightarrow (C \rightarrow (B \vee D))$ as well as its dual:
$((A \rightarrow B) \wedge ((B \wedge D) \rightarrow C)) \rightarrow ((A \wedge D) \rightarrow C))$
and many other formulae.

3. *Possible weakenings of the concept of relevance*

It is natural now to consider the problem in what sense some other systems usually considered as relevant are indeed relevant. It is not enough just to claim that the absence of paradoxes is a sufficient reason to term a system as a relevant system. It is essential to investigate the character of entailment and implication in that system. In this section we want to focus our attention to the famous systems R (of relevant implication) and NR (of Relevant Implication with necessity). There is rather wide-spread opinion that R is more useful than E as a relevant logic. This statement is often based only on the claim that R is simpler than E (we allow ourselves the following counter-argument: a physicist will not, in general, prefer classical mechanics to relativistic). As for the second system, NR, some logicians including the developers of the system R. Sylvan and R. Meyer, consider NR rather than E the most applicable theory of relevant entailment. According to them, entailment is represented in NR by its theorems of the type $N(A \rightarrow B)$, where $\rightarrow$ is an implication of R and N is a necessity of $S4$ type; NR itself is obtained from R by the addition to R of postulates that define N.

As it is known, R differs from E by containing the law $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$ — which allows unrestricted permutation of antecedents. Thus if $(A \rightarrow (B \rightarrow C))$ is a law then $(B \rightarrow (A \rightarrow C))$ is also a law. But if B does not contain $\rightarrow$, then we have no reason to believe that the information of the consequent is a part of the information of the antecedent. Here B plays the role of a garantor of A. Relevance here can only be seen in the fact that this relation between the truth values of propositions is determined by their logical contents. This is relevance in some very weak sense.

However if R is used as a logical framework for the analysis of some concrete theory, we can expect cases of the failure of relevance even in a very weak sense. For example, from a particular case of the identity law, $(A \rightarrow B) \rightarrow (A \rightarrow B)$, we can derive the statement $A \rightarrow ((A \rightarrow B) \rightarrow B)$.

Suppose that A and B are statements of Newtonian mechanics. Let A be "A body a at a moment t is affected by forces whose resultant is not equal to zero" and B express "A body a at a moment t is moving with some acceleration (or possibly with deceleration)". Let A be true. Then by *m.p.* we have $(A \rightarrow B) \rightarrow B$, where "$A \rightarrow B$" is a statement of necessary character, a particular case of one of the laws of Newtonian mechanics, and B is a factual proposition about a unique, even occasional fact. It is not clear from the point of view of simple intuition what sense can be attached to the statement that some individual occasional event is determined by a law. We can accept "$(A \rightarrow B) \rightarrow B$, since A is true", but this points out only that it is impossible to deduce A by *modus ponens* if A (in $A \rightarrow ((A \rightarrow B) \rightarrow B)$) appeared as a result of unacceptable in E permutation of antecedents. Moreover, the statement $(A \rightarrow B) \rightarrow B$ can be derived even if $A \rightarrow B$ is false, as happens if A means "A body a is effected at the moment t by some forces". It is very uncertain in what sense one can assert the truth conditions of such a statement.

All these implications fail in NR. $N(A \rightarrow B)$ is true in NR only in cases where $\rightarrow$ is the intensional implication of E, which means that $A \rightarrow B$ is a theorem in E, or on the other hand, can be obtained from some theorem of E by some permutation of antecedents, again, unacceptable in E. In this last case $A \rightarrow B$ means only "if A is true, then it is necessary that B is true", and this necessity itself means nothing more, than that the connection between the truth values of A and B is determined by the logical contents of A and B. The next question which arises now is: how useful can this necessity be? The point is that when the logical system has only one goal, to provide true consequences of true premises, the paradoxical cases in classical logic are rather inoffensive. Exclusion of these superfluous cases restricts the deductive facilities of a system.

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Appendix

Axiomatic System for E

1. $(A \wedge B) \rightarrow A$
2. $(A \wedge B) \rightarrow B$
3. $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$
4. $A \rightarrow (A \vee B)$
5. $B \rightarrow (A \vee B)$
6. $((A \rightarrow C) \wedge (B \rightarrow C)) \rightarrow ((A \vee B) \wedge C)$
7. $(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$
8. $A \rightarrow \neg\neg A$
9. $\neg\neg A \rightarrow A$
10. $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$
11. $(A \rightarrow B) \rightarrow (\neg A \vee B)$
12. $((A \rightarrow A) \rightarrow B) \rightarrow B$
13. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
14. $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$

Rules of inference:
1.) $(A \rightarrow B), A \vdash B$ — modus ponens
2.) $A, B \vdash (A \wedge B)$ — introduction of conjunction
Note that *E-11* is equivalent to $(A \rightarrow \neg A) \rightarrow \neg A$, while *E-14* is equal to $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$.