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EQUALITY IN LINEAR LOGIC

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1. *Quantales*

In this section we introduce the basic definitions and results of the theory of quantales (a good reference is [Ros]). Quantales were introduced by Mulvey ([Mul]) as an algebraic tool for studying representations of non-commutative C^*-algebras. Informally, a quantale is a complete lattice Q equipped with a product distributive over arbitrary sup's. The importance of quantales for Linear Logic is revealed in Yetter's work ([Yet]), who proved that semantics of classical linear logic is given by a class of quantales, named Girard quantales, which coincides with Girard's phase semantics. An analogous result is obtained for a sort of non-commutative linear logic, as well as intuitionistic linear logic without negation, which suggest that the utilisation of the theory of quantales (or even weaker structures, such that *-autonomous posets) might be fruitful in studying the semantic of several variants of linear logic.

As usual, we denote the order in a lattice by $\leq$, while $\bigvee$ and $\bigwedge$ denote the operations of sup and inf, respectively. We write $\top$ for the largest element in a lattice and $\mathbf{0}$ for its smallest element.

Definition 1.1 A quantale *is a complete lattice Q with an associative binary operation $\otimes$: $Q \times Q \longrightarrow Q$, which distributes on the right and on the left of arbitrary sup's, i.e.:*

[$Q1$] $a \otimes (b \otimes c) = (a \otimes b) \otimes c$, *for every* $a,b,c \in Q$

[$Q2$] $a \otimes (\bigvee_{i\in I} a_i) = \bigvee_{i\in I}(a \otimes a_i)$, $(\bigvee_{i\in I} a_i) \otimes a = \bigvee_{i\in I}(a_i \otimes a)$

A quantale Q is unital *if it has an element $\mathbf{1} \in Q$ such that $a \otimes \mathbf{1} = \mathbf{1} \otimes a = a$, for every $a \in Q$. A quantale Q is* commutative *if $a \otimes b = b \otimes a$, for every $a, b \in Q$.*

A morphism of quantales *is an operator between quantales which preserves $\otimes$ and arbitrary sup's.*

It's easily seen that the above axioms imply that $\otimes$ is increasing in both coordinates, that is

If $a \leq b$ then, $\forall c \in Q$, $a \otimes c \leq b \otimes c$ and $c \otimes a \leq c \otimes b$

We register a classic result:

Proposition 1.2 The endomorfisms $a \otimes \cdot$, $\cdot \otimes a : Q \longrightarrow Q$ have right adjoints, denoted by $a \rightarrow_r \cdot$ and $a \rightarrow_l \cdot$, respectively. Thus,

$$a \otimes c \leq b \text{ iff } c \leq a \rightarrow_r b \qquad c \otimes a \leq b \text{ iff } c \leq a \rightarrow_l b$$

and consequently

$$a \rightarrow_r b = \bigvee \{c \in Q : a \otimes c \leq b\}$$
$$a \rightarrow_l b = \bigvee \{c \in Q : c \otimes a \leq b\}.$$

Definition 1.3 Let Q be a quantale. A map $j : Q \longrightarrow Q$ is said to be a
a) quantic nucleus *if it satisfies:*

[$NQ1$] $a \leq b$ *implies* $j(a) \leq j(b)$ [$NQ2$] $a \leq j(a)$
[$NQ3$] $j(j(a)) = j(a)$ [$NQ4$] $j(a) \otimes j(b) \leq j(a \otimes b)$

b) quantic conucleus *if it satisfies:*

[$CNQ1$] $a \leq b$ *implies* $g(a) \leq g(b)$ [$CNQ2$] $g(a) \leq a$
[$CNQ3$] $g(g(a)) = g(a)$ [$CNQ4$] $g(a) \otimes g(b) \leq g(a \otimes b)$

Quantic nuclei and conuclei are important, because they determine the quotients and subobjects in the category of quantales.

Definition 1.4 Let Q be a quantale. A subset $S \subseteq Q$ is a subquantale *of Q if it is closed under $\otimes$ and arbitrary sup's.*

Proposition 1.5 [*Ros*]: *(a)If $Q \xrightarrow{j} Q$ is a quantic nucleus, then $Q_j = \{ x \in Q : j(x) = x \}$ is a quantale where the operations $\otimes^j$, $\bigvee^j$ and $\bigwedge^j$ in Q_j are given by:*

$$a \otimes^j b = j(a \otimes b) \quad \bigvee^j_{i \in I} a_i = j(\bigvee_{i \in I} a_i) \quad \bigwedge^j_{i \in I} a_i = \bigwedge_{i \in I} a_i .$$

Moreover, the map $j : Q \longrightarrow Q_j$ given by $a \mapsto j(a)$, is a surjective morphism of quantales. Further, every surjective morphism of quantales can be represented in this form.

(b) If $Q \xrightarrow{g} Q$ *is a quantic conucleus, then* $Q_g = \{ x \in Q : g(x) = x \}$ *is a subquantale where, with notation as in (a),* $\bigwedge^g_{i\in I} a_i = g(\bigwedge_{i\in I} a_i)$. *Moreover, every subquantale is of this form, i.e.,*

if $S \subseteq Q$ *is a subquantale, then there exists a quantic conucleus g in Q such that* $S = Q_g$.

Definition 1.6 An element $\bot \in Q$ *is* dualizing *if:*

$$(a \to_r \bot) \to_l \bot = a = (a \to_l \bot) \to_r \bot, \textit{for every } a \in Q.$$

An element $s \in Q$ *is* cyclic *if:*

$$a \to_r s = a \to_l s, \textit{for every } a \in Q.$$

Proposition 1.7 [*Yet*]: *Let Q be a quantale and s,* $\bot$ *be elements of Q.*

a) $s \in Q$ *is cyclic iff for all* $a_1, ..., a_n$ *in Q,*

$$a_1 \otimes \cdots \otimes a_n \leq s \textit{ implies } a_{\pi(1)} \otimes a_{\pi(2)} \otimes \cdots \otimes a_{\pi(n)} \leq s,$$
for all cyclic permutations π *of* $\{1, ..., n\}$.

b) If $\bot \in Q$ *is dualizing, then Q is unital and we have:*

$$\mathbf{1} = \bot \to_r \bot = \bot \to_l \bot.$$

Definition 1.8 A Girard quantale *is a quantale which has a cyclic dualizing element* $\bot$. *The operator* $\cdot \to_l \bot = \cdot \to_r \bot \equiv_{def} \cdot \to \bot$ *is called* linear negation, *and we write* $a^\bot = a \to \bot$ *(note that* $\mathbf{1} = \bot^\bot$, $\bot = \mathbf{1}^\bot$, *and* $a = a^{\bot\bot}$*).*

Next proposition is of frequent use when computing in a Girard quantale.

Proposition 1.9 [*Ros*]: *Let Q be a Girard quantale with a cyclic dualizing element* $\bot$ *and let a,b* $\in Q$. *Then:*

(1)	$a \to_l b = (a \otimes b^\bot)^\bot$	(2)	$a \to_r b = (b^\bot \otimes a)^\bot$
(3)	$a \otimes b = (a \to_l b^\bot)^\bot$	(4)	$b \otimes a = (a \to_r b^\bot)^\bot$
(5)	$a \to_r b = b^\bot \to_l a^\bot$	(6)	$a \to_l b = b^\bot \to_r a^\bot$

Proposition 1.10 [*Yet*]: *Let Q be an unital quantale and* $s \in Q$ *cyclic. Then,* $j : Q \longrightarrow Q$, *given by* $j(a) = (a \to s) \to s$ *is a quantic nucleus, and* $Q_j = \{ a \in Q : j(a) = a \}$ *is a Girard quantale, where* $\bot = s \in Q_j$.

Example 1.11: The Phase Quantales (Girard)

Let $(M\ ,\ \cdot\ ,\ 1)$ *be a monoid. We define* $A\cdot B = \{\ a\cdot b : a \in A\ ,\ b \in B\ \}$ $\forall A,B \subseteq M$. *Let* $\perp \subseteq M$ *be such that* $a\cdot b \in \perp$ *implies* $b\cdot a \in \perp$ *(for example,* $\perp$ *can be a semiprime ideal or the complement of a prime ideal of* M*).*

Thus, $\wp(M)$ *is a quantale with the product defined above, and with sup's e inf's calculated as unions* $(\bigcup)$ *and intersections* $(\bigcap)$*. In fact, it's an unital quantale, where* $\mathbf{1} = \{\ 1\ \}$ *and* $\perp$ *is cyclic. Then, by proposition 1.10,* $\wp(M)_j$ *is a Girard's quantale containing* $\perp$*. Their elements are called* facts, *and we have:*

$$A \otimes B = (A\cdot B)^{\perp\perp}$$
$$\bigvee_{i\in I} A_i = (\bigcup_{i\in I} A_i)^{\perp\perp};\ \textit{in particular, } A \vee B = (A \cup B)^{\perp\perp} \equiv_{def} A \oplus B,$$
$$\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i;\ \textit{in particular, } A \wedge B = A \cap B \equiv_{def} A \;\&\; B.$$

The next result tells us that every Girard quantale is of this form.

Proposition 1.12 [*Ros*]: *If* Q *is a Girard's quantale, then* Q *is isomorphic to a phase quantale.*

We are now going to relate the exponentials treated by Yetter ([Yet]) and Avron ([Avr]) with certain concepts in the theory of quantales.

Definition 1.13 [*Yet*]: *An* open modality *in a quantale* Q *is a map* $\mu : Q \longrightarrow Q$ *satisfying:*

$[M1]$ $\mu(a) \leq a$	$[M2]$ $a \leq b$ *implies* $\mu(a) \leq \mu(b)$
$[M3]$ $\mu(\mu(a)) = \mu(a)$	$[M4]$ $\mu(\mu(a) \otimes \mu(b)) = \mu(a) \otimes \mu(b)$.

An open modality μ *is said to be*

– central *if* $b \otimes \mu(a) = \mu(a) \otimes b$ *for every* $a,b \in Q$.
– idempotent *if* $\mu(a) \otimes \mu(a) = \mu(a)$ *for every* $a \in Q$.
– weak *if* Q *is unital,* $\mu(\mathbf{1}) = \mathbf{1}$, *and* $\mu(a) \leq \mathbf{1}$ *for every* $a \in Q$.

Let $M(Q) = \{\mu\colon \mu$ *is an open modality in* $Q\}$, *partially ordered by pointwise order.*

Proposition 1.14 [*Yet*]: *Let* Q *be a quantale (resp. unital quantale). Then, there exists a unique maximal open modality in* Q *central (resp. central, idempotent and weak) denoted by* c_m *(resp.* $!_m$*) given by:*

$$c_m(x) = \bigvee \{\ a \in Q : a \leq x\ ,\ a \in Z(Q)\ \}$$
$$!_m(x) = \bigvee \{\ a \in Q : a \leq x \wedge \mathbf{1}\ ,\ a = a \otimes a\ ,\ a \in Z(Q)\ \},$$

where $Z(Q) = \{ a \in Q : a \otimes b = b \otimes a,$ *for every* $b \in Q \}$.

Definition 1.15 [*Avr*]: *Let Q be an unital quantale. A map* $B: Q \longrightarrow Q$ *is a* modal operation *if it satisfies, for all* $x, y \in Q$:

[*B*1] $B(\mathbf{1}) = \mathbf{1}$ [*B*2] $B(x) \leq x$
[*B*3] $B(B(x)) = B(x)$ [*B*4] $B(x) \otimes B(y) = B(x \wedge y)$.

Proposition 1.16 Let Q be a quantale, and $\mu : Q \longrightarrow Q$ *a map.*
(i) μ *is an open modality iff* μ *is a quantic conucleus.*
(ii) If Q is unital, then μ *is an open, idempotent and weak modality iff* μ *is a modal operation.*

Proof: (i) We must prove that [*M*4] is equivalent to [*CNQ*4]. Let μ be an open modality. Because $\mu(a) \otimes \mu(b) \leq a \otimes b$, we get $\mu(a) \otimes \mu(b) = \mu(\mu(a) \otimes \mu(b)) \leq \mu(a \otimes b)$.

Conversely, if μ is a quantic conucleus, then

$$\begin{aligned}\mu(a) \otimes \mu(b) &= \mu(\mu(a)) \otimes \mu(\mu(b)) \\ &\leq \mu(\mu(a) \otimes \mu(b)) \leq \mu(a) \otimes \mu(b).\end{aligned}$$

(ii) Assume that μ is an open, weak and idempotent modality. Then, it clearly satisfies conditions [*B*1],[*B*2],[*B*3]. Because $\mu(a) \leq a$ and $\mu(b) \leq \mathbf{1}$, we get $\mu(a) \otimes \mu(b) \leq a \otimes \mathbf{1} = a$. Similarly, $\mu(a) \otimes \mu(b) \leq b$ and therefore, $\mu(a) \otimes \mu(b) \leq a \wedge b$. Thus, $\mu(a) \otimes \mu(b) = \mu(\mu(a) \otimes \mu(b)) \leq \mu(a \wedge b)$.

Now, since μ is increasing and $a \wedge b \leq a, b$, then $\mu(a \wedge b) \leq \mu(a), \mu(b)$. Thus,

$$\mu(a \wedge b) = \mu(a \wedge b) \otimes \mu(a \wedge b) \leq \mu(a) \otimes \mu(b),$$

and so μ satisfies [*B*4].

Conversely, if μ is a modal operation, then [*M*1] is just [*B*2]; [*M*3] is [*B*3], while [*M*4] is equivalent to [*CNQ*4], in the presence of [*M*1], [*M*2], [*M*3], by item (i) above.

Condition [*M*2] is item (5) of Lemma 4.2 in [Avr] and μ satisfies [*CNQ*4] by item 8 of Lemma 4.2 in [Avr]. It follows from items 4 and 1 in that same Lemma, that $\mu(x) \leq \mathbf{1}$ for every $x \in Q$. Thus, [*B*1] yields that μ is weak. □.

Definition 1.17 A frame *(or* Complete Heyting algebra*) is a quantale where* $\otimes = \wedge$.

Definition 1.18 Let Q be a quantale and let $\top$ *be the maximum of Q. We say that* $x \in Q$ *is*

a)	right-sided (in Q)	*if* $x \otimes \top \leq x$;
b)	left-sided (in Q)	*if* $\top \otimes x \leq x$;
c)	two-sided (in Q)	*if it is right and left sided in Q.*
d)	idempotent	*if* $x \otimes x = x$.

Proposition 1.19 [*Ros*]: *Let Q be a quantale, and g a quantic conucleus in Q. Are equivalent:*

(*a*) Q_g *is a frame.* (*b*) $g(a) \otimes g(b) = g(a \wedge b)$.
(*c*) *every* $x \in Q_g$ *is idempotent and two-sided (in* Q_g*).*

Definition 1.20 A map g satisfying the conditions above is called a localic conucleus *and* Q_g *is called a* localic subquantale.

Since open, weak idempotent modalities are localic conuclei μ such that $\mu(\mathbf{1}) = \mathbf{1}$, we have

Corollary 1.21 Let Q be an unital quantale, and let μ *be an open, weak and idempotent modality. Then,* Q_μ *is a frame.*

Thus, the fixed points of the interpretation of the modality ! (of course) lie in a localic subquantale (complete Heyting algebra) of any quantale in which linear logic is interpreted.

2. *Linear Calculus with Equality*

In this section we discuss the laws for a binary predicate representing equality. The goal is to define a reflexive, symmetric and transitive predicate satisfying the substitution (Leibnitz's) rule for the class of all formulas. We may assume, just as in Classical Logic (*CL*), that we have substitution for atomic formulas. With this model in mind, we shall define a prototype of a linear calculus with equality, called (LLE_1). Our formulation will use sequents in Linear Logic (*LL*). Analogously, we will set down a calculus with equality for the (*MALL*) fragment, i.e., the fragment without exponentials, indicated by (LLE_0). Starting from the property of substitution for elementary formulas, we prove that $(\cdot = \cdot)$ must be $\leq \mathbf{1}$ and idempotent in (*MALL*), and open in the general case; in other words, $(\cdot = \cdot)$ must be intuitionistic.

Definition 2.1 *A* first order linear language with equality $\mathbb{L}$ *consists in a countable set of predicate symbols* $\wp = \{ P_n : n \in \omega \} \cup \{ = \}$ *(where "=" is binary), a countable set of variables* $V = \{ v_n : n \in \omega \}$, *together with the symbols:*

$$\mathbf{1}, \bot, \top, \mathbf{0}, {}^{\bot}, \bigvee, \bigwedge, \oplus, \&, \otimes, \sqcup\ , !\ , ?.$$

Definition 2.2 The formulas *of* $\mathbb{L}$, $FOR(\mathbb{L})$, *are defined recursively:*

[$F1$] $\mathbf{1}, \bot, \top, \mathbf{0} \in FOR(\mathbb{L})$.
[$F2$] *If* $P \in \wp$ *is a predicate of arity n and* $x_1,...,x_n$ *are variables, then* $P(x_1,...,x_n)$ and $P(x_1,...,x_n)^{\bot} \in FOR(\mathbb{L})$.
[$F3$] *if* $F, G \in FOR(\mathbb{L})$ *then* $F \otimes G$, $F \sqcup G$, $F \,\&\, G$, $F \oplus G \in FOR(\mathbb{L})$.
[$F4$] *if* $F \in FOR(\mathbb{L})$, *then* $!F$, $?F \in FOR(\mathbb{L})$.
[$F5$] *if* $F \in FOR(\mathbb{L})$ *and* $x \in V$, *then* $\bigwedge x.F$ *and* $\bigvee x.F \in FOR(\mathbb{L})$.

Every occurrence of a variable x in a formula F is free *except in a subformula of the type* $\bigwedge x.G$ *or* $\bigvee x.G$ *(which are* bounded *occurrences). We shall write*

$ELF(\mathbb{L}) = \{ P(x_1,...,x_n) : P \in \wp - \{ = \} , x_i \in V \}$, *the set of* elementary formulas

and

$ELF^{+}(\mathbb{L}) = ELF(\mathbb{L}) \cup \{ (x = y) : x , y \in V \}$, *the* extended *set of elementary formulas.*

Definition 2.3 The syntactic linear negation *is a map* $\bot : FOR(\mathbb{L}) \longrightarrow FOR(\mathbb{L})$ *given by the usual rules*:

[$NL1$] $\bot(\mathbf{1}) = \bot$, $\bot(\bot) = \mathbf{1}$, $\bot(\mathbf{0}) = \top$, $\bot(\top) = \mathbf{0}$
[$NL2$] $\bot(P(x_1,...,x_n)) = P(x_1,...,x_n)^{\bot}$, $\bot((x = y)) = (x = y)^{\bot}$,
$\bot(P(x_1,...,x_n)^{\bot}) = P(x_1,...,x_n)$, $\bot((x = y)^{\bot}) = (x = y)$
(here, $P(x_1,...,x_n) \in ELF(\mathbb{L})$*)*
[$NL3$] $\bot(F \otimes G) = \bot(F) \sqcup \bot(G)$, $\bot(F \sqcup G) = \bot(F) \otimes \bot(G)$,
$\bot(F \,\&\, G) = \bot(F) \oplus \bot(G)$, $\bot(F \oplus G) = \bot(F) \,\&\, \bot(G)$
[$NL4$] $\bot(!F) = ?\bot(F)$, $\bot(?F) = !\bot(F)$
[$NL5$] $\bot(\bigwedge x.F) = \bigvee x.\bot(F)$, $\bot(\bigvee x.F) = \bigwedge x.\bot(F)$

We write $\bot(F) = F^{\bot}$; *clearly* $F = F^{\bot\bot}$ *for every* $F \in FOR(\mathbb{L})$.

Definition 2.4 Let A a formula; we define A[y/x] to be the formula obtained from A by replacing all bound occurrences of y by z, where z is the first variable (in the natural order of V) not occurring in A, and then replacing all free occurences of variable x by y.

Definition 2.5 (Girard) The calculus (LL) for first order commutative linear logic is defined by the axioms and rules below (here, A, B denote formulas, and Γ, Δ denote multisets of formulas):

$[AX1]\ \vdash A^{\perp}, A \qquad [AX2]\ \vdash \top, \Gamma \qquad [AX3]\ \vdash \mathbf{1}$

$[CUT]\ \dfrac{\vdash \Gamma, A \qquad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}$

$[EXCH]$ *If* Δ *is a permutation of* Γ, *then* $\dfrac{\vdash \Gamma}{\vdash \Delta}$

$[\&]\ \dfrac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \,\&\, B,\ \Gamma}$

$[\oplus 1]\ \dfrac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma} \qquad [\oplus 2]\ \dfrac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}$

$[\perp]$ *If* Γ *is not empty, then* $\dfrac{\vdash \Gamma}{\vdash \perp, \Gamma}$

$[\otimes]\ \dfrac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B,\ \Gamma,\ \Delta} \qquad [\sqcup\,]\ \dfrac{\vdash A, B, \Gamma}{\vdash A \sqcup\ B\,, \Gamma}$

$[dereliction]\ \dfrac{\vdash A, \Gamma}{\vdash ?A, \Gamma} \qquad [weakening]\ \dfrac{\vdash \Gamma}{\vdash ?A, \Gamma}$

$[contraction]\ \dfrac{\vdash ?A, ?A, \Gamma}{\vdash ?A,\ \Gamma} \qquad [!]\ \dfrac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}$

$[\bigvee]\ \dfrac{\vdash A[y/x], \Gamma}{\vdash \bigvee x.A, \Gamma} \qquad [\bigwedge]$ *If x is not free in* Γ, *then* $\dfrac{\vdash A,\ \Gamma}{\vdash \bigwedge x.A, \Gamma}$

We can consider as defined connectives the linear implication *and the* linear equivalence, *given by:*

$(A \multimap B) \equiv_{def} (A^{\perp} \sqcup\ B) = (A \otimes B^{\perp})^{\perp}$
$(A \multimapboth B) \equiv_{def} (A \multimap B) \,\&\, (B \multimap A).$

Definition 2.6 The linear calculus with equality (LLE_0) *for the* (*MALL*) *fragment of* (*LL*) *is defined by adding the axioms below to those in the preceding definition:*

$[R=] : \vdash (x = x);$ $\qquad [S=] : \vdash (x = y)^{\perp}, (y = x);$
$[T=] : \vdash (x = y)^{\perp}, (y = z)^{\perp}, (x = z).$

$[SUBST] : \vdash (x_1 = y_1)^{\perp}, \ldots, (x_n = y_n)^{\perp},$
$F(x_1,\ldots,x_n)^{\perp}, F(x_1 \wr y_1,\ldots, x_n \wr y_n),$
where $F(x_1,\ldots,x_n)$ *denotes an elementary formula with (eventually)* $x_1,\ldots,x_n$ *free and* $F(x_1 \wr y_1,\ldots, x_n \wr y_n)$ *is obtained from* $F(x_1,\ldots,x_n)$ *by replacing some occurrences of* x_i *(which are not in the scope of a* y_i*-quantifier) by* y_i.

$[I=] : \vdash (x = y)^{\perp}, (x = y) \otimes (x = y)$ $\qquad [\leq \mathbf{1}] : \vdash (x = y)^{\perp}, \mathbf{1}$

Definition 2.7 The linear calculus with equality (LLE_1) *for* (*LL*) *is defined by adding, besides the axioms* [*R*=], [*S*=], [*T*=], [*SUBST*] *the following rule*

$[!=] : \vdash (x = y)^{\perp}, !(x = y)$

Obviously, all the rules (with exception of [*R*=]) could be formulated as linear implications, for example [*T*=] could be stated as

$\vdash (x = y) \otimes (y = z) \multimap (x = z)$

We have as well that (LLE_1) is equivalent to the calculus obtained from (LLE_1) by replacing [*S*=] and [!=] by the rule:

$[S!=] \vdash (x = y)^{\perp}, !(y = x)$

3. *Semantics*

In this section we develop interpretations for the calculi described above. It will be necessary to extend the definitions in [Yet] so that the axioms involving equality are verified.

Definition 3.1 An interpretation *of a language* $\mathbb{L}$ *is a triple* $<\mathcal{A}, \mathcal{V}, |\cdot|_{\mathcal{A}}>$ *where*

- $\mathcal{A}$ *is an algebra for the theory with constants* $\mathbf{1}, \bot, \top, \mathbf{0}$;
- *a set of unary operations* $\{\ ^{\bot}, !, ?\ \}$ *(the interpretation of the modalities)*;
- *binary operations* $\otimes, \sqcup, \&, \oplus$, *and infinitary operations* $\bigvee$ and $\bigwedge$;
- $\mathcal{V} \subseteq \mathcal{A}$, *the set of* valids *elements*; *and*
- $|\cdot|_{\mathcal{A}} : ELF^{+}(\mathbb{L}) \longrightarrow \mathcal{A}$ *is a map.*

It is straightforward to see that there exists a (unique) extension of $|\cdot|_{\mathcal{A}}$ *to* $FOR(\mathbb{L})$, *also denoted by* $|\cdot|_{\mathcal{A}}$. *When there is no risk of confusion, we write* $|\cdot|$ *for* $|\cdot|_{\mathcal{A}}$.

Definition 3.2 A semantics *for* $\mathbb{L}$ *is a class of interpretations. A semantics is* sound *(with respect to a linear calculus P) if:*

$\vdash A_1,\ldots,A_n$ *is provable in P implies* $|A_1 \sqcup \ldots \sqcup A_n|_{\mathcal{A}} \in \mathcal{V}$
for every $< \mathcal{A}, \mathcal{V}, |\cdot|_{\mathcal{A}} >$ *in the semantics.*

A semantics for $\mathbb{L}$ *is* complete *(with respect to P) if:*

$|F|_{\mathcal{A}} \in \mathcal{V}$, *for every* $<\mathcal{A}, \mathcal{V}, |\cdot|_{\mathcal{A}}>$ *in the semantics,*
implies $\vdash F$ *is provable in P.*

Definition 3.3 An interpretation of quantales *for* $\mathbb{L}$ *is a Girard quantale Q, together with an assignment* $! \mapsto \mu \in M(Q)$ *and a map* $|\cdot|_Q : ELF^{+}(\mathbb{L}) \longrightarrow Q$, *where* $\mathcal{V} = \{ a \in Q : a \geq \mathbf{1} \}$ *such that*

- $\forall a, b \in Q, a \sqcup b \equiv_{def} (a^{\bot} \otimes b^{\bot})^{\bot}$ *and* $?a \equiv_{def} (\mu(a^{\bot}))^{\bot}$;
- *We interpret* $\bigwedge x.A$ *and* $\bigvee x.A$ *as* $\bigwedge_{y \in V} |A[y/x]|_Q$ *and* $\bigvee_{y \in V} |A[y/x]|_Q$, *respectively.*

Recall that $M(Q)$ *is the lattice of* open *modalities in Q (Definition 1.13).*

Definition 3.4 The semantics of quantales f*or (commutative) Linear Logic with equality is the class of interpretation of quantales for* $\mathbb{L}$ *such that:*

[*S*1] *Q is commutative.* [*S*2] $\mu \equiv_{def}\ !$ *is idempotent and weak.*

[*S*3] $|\cdot|_Q : ELF^{+}(\mathbb{L}) \longrightarrow Q$ *satisfies:*

(a) For the Calculus (LLE_0):

$[=1]: |x = x| \geq \mathbf{1}$; $[=2]: |x = y| \otimes |y = z| \leq |x = z|$;
$[=3]: |x = y| = |y = x|$;
$[=4]: |x_1 = y_1| \otimes \cdots \otimes |x_n = y_n| \otimes |F(x_1,\ldots,x_n)| \leq |F(x_1 \wr y_1,\ldots,x_n \wr y_n)|$
(same notation as [SUBST]);
$[=5]: |x = y| \leq |x = y| \otimes |x = y|$; $[=6]: |x = y| \leq \mathbf{1}$

(b) For the Calculus (LLE_1), *conditions* [=5] *and* [=6] *are replaced by the stronger:*

$[=7]: |x = y| \leq\, !|x = y|$.

The next result shows that it is always possible to define a map $|\cdot|_Q$ satisfying the above conditions.

Proposition 3.5 If $|\cdot|_Q : ELF(\mathbb{L}) \longrightarrow Q$ *is a map, then it can be extended to* $ELF^+(\mathbb{L})$ *satisfying, in the (MALL) case, conditions* [=1],...,[=6] *and, in the general case, conditions* [=1],...,[=4],[=7].

Proof: Let's begin by the general case, in which $! \in M(Q)$ is idempotent and weak.

Let $|\cdot|_Q : ELF(\mathbb{L}) \longrightarrow Q$ be a map, and let $x,y \in V$, $x \neq y$. Define $A_{x,y}$ and $B_{x,y}$ as:

$A_{x,y} = \{ F \in ELF(\mathbb{L})$: neither x nor y occur in $F\}$
$B_{x,y} = \{ F \in ELF(\mathbb{L})$: x or y occur in $F\}$

In $B_{x,y}$ consider the relation:

$F \sim G$ iff G is obtained *from* F by replacing some occurrences of x by y and/or some occurrences of y by x

For example, $P(x,z,y) \sim P(x,z,x) \sim P(y,z,y) \sim P(y,z,x)$. Clearly, $\sim$ is an equivalence relation. Now define,

$$T_{x,y} = \bigwedge \{ |F|_Q \to |G|_Q : F,G \in B_{x,y}, F \sim G \}.$$

Fact 1: If $|\cdot|$ satisfies [=7] and [=3], then it satisfies [=4] iff $|x = y| \leq T_{x,y}$ for $x \neq$ y.

To see this, assume that $F,G \in B_{x,y}$, $F \sim G$. Note that there is $H \in ELF(\mathbb{L})$ such that $H = F(x \wr y)$, $G = H(y \wr x)$.

Since ! is idempotent, [=7] implies [=5] and so, by [=3],[=4] and [=5] we have that $|x = y| \otimes |F| \leq |H|$; thus,

$$|x=y| \otimes |F| \leq |x=y| \otimes (|x=y| \otimes |F|) \leq |x=y| \otimes |H| \leq |G|.$$

Therefore, $|x=y| \leq |F| \rightarrow |G|$, proving that $|x=y| \leq T_{x,y}$.
Conversely, suppose that $|x = y| \leq T_{x,y}$ for $x \neq y$. Then, if $F \in \cup_{i=1}^{n} B_{x_i,y_i}$,

$$|x_n = y_n| \leq |F(x_1,...,x_n)| \rightarrow |F(x_1,...,x_{n-1},x_n \wr y_n)|$$

and therefore, $|x_n = y_n| \otimes |F(x_1,...,x_n)| \leq |F(x_1,...,x_{n-1},x_n \wr y_n)|$.
Analogously,

$$\begin{aligned} |x_{n-1} = y_{n-1}| \otimes (|x_n = y_n| \otimes |F(x_1,...,x_n)|) & \\ \leq |x_{n-1} = y_{n-1}| \otimes |F(x_1,...,x_{n-1},x_n \wr y_n)| & \\ \leq |F(x_1,...,x_{n-2},x_{n-1} \wr y_{n-1},x_n \wr y_n)|. & \end{aligned}$$

We may proceed by induction to get:

$$|x_1 = y_1| \otimes \cdots \otimes |x_n = y_n| \otimes |F(x_1,...,x_n)| \leq |F(x_1 \wr y_1,...,x_n \wr y_n)|,$$

as desired.
If $F \in \cap_{i=1}^{n} A_{x_i,y_i}$ then, since [=7] implies [=6], we have:

$$|x_1 = y_1| \otimes \cdots \otimes |x_n = y_n| \otimes |F| \leq (\mathbf{1} \otimes \cdots \otimes \mathbf{1}) \otimes |F| = |F|,$$

showing that[=4] is valid and proving Fact 1.
By induction, define a map $|\cdot|$ as follows: for $x, y \in V$, $x \neq y$, set

$$\begin{aligned} |x=y|_0 &= T_{x,y}; \\ |x=y|_{n+1} &= \bigwedge_{z \neq x,y} (|x=z|_n \leftrightarrow |y=z|_n) \\ & \qquad (a \leftrightarrow b \text{ means } (a \rightarrow b) \wedge (b \rightarrow a)); \\ |x=y|_\infty &= !(\bigwedge_{n \in N} |x=y|_n); \text{ finally, set } |x=x|_\infty = \mathbf{1}. \end{aligned}$$

Fact 2: $|\cdot|_\infty$ satisfies the properties required for the full logic.
In fact: condition [=1] is clear, while [=3] is verified because it's true for $|x=y|_n$, for all $n \in \omega$.

Since $|x=y|_\infty = !|x=y|_\infty$, [=7] is satisfied (the case $x = y$ is valid too, because $!\mathbf{1} = \mathbf{1}$). Observe that, for every $n \geq 0$ and $z \neq x,y$:

$$\begin{aligned} & |x=y|_{n+1} \otimes |x=z|_n \leq |y=z|_n \text{ and therefore,} \\ & |x=y|_\infty \otimes |x=z|_\infty \leq |x=y|_{n+1} \otimes |x=z|_n \leq |y=z|_n \text{ for every } n \geq 0. \end{aligned}$$

Thus,

$$|x = y|_\infty \otimes |x = z|_\infty \leq \bigwedge_{n \in N} |y = z|_n \text{; now [=7] and [}M4\text{] yield [=2].}$$

Since $|x = y|_\infty \leq |x = y|_0 = T_{x,y}$, Fact 1 guarantees that $|\cdot|_\infty$ satisfies [=4].

For the ($MALL$) fragment, it's enough to take $! = !_m$ (see Proposition 1.14) in the above computations. □

Discussion 3.6 Are requirements [=5],[=6] *(resp.* [=7]*) too strong ?*

The motivation for them is, starting with substitution for elementary formulas, to have substitution for every formula. Obviously, they are sufficient, but indeed, *they are also necessary*. To see this, note that, if $a = |x = y|$, $b = |P(x)|$ and $c = |P(x \wr y)|$, then we must have that $a \otimes b \leq c$ and $a \otimes c \leq b$, (i.e. $a \leq (b \leftrightarrow c)$) must imply $a \otimes !b \leq !c$ (and, of course, that $a \otimes !c \leq !b$). Since $\mathbf{1}$ is a formula, we must also have $a = a \otimes \mathbf{1} \leq \mathbf{1}$. The critical cases are $\otimes$ and !. We have

Lemma 3.7 Let Q be a Girard Quantale and let a be an element of Q such that $a \leq \mathbf{1}$. (a) Are equivalent:

(*i*) $\forall\, b, c, d \in Q, \quad a \leq (b \leftrightarrow c), a \leq (d \leftrightarrow e)$
implies $\quad a \leq ((b \otimes d) \leftrightarrow (c \otimes e))$.
(*ii*) $a \leq a \otimes a$

(*b*) *Assume that* $! \in M(Q)$ *is idempotent and weak. Are equivalent:*

(*i*) $\forall\, b, c \in Q, \quad a \leq (b \leftrightarrow c)$ *implies* $\quad a \leq (!b \leftrightarrow !c)$.
(*ii*) $a = !a$

(*c*) $\forall\, b, c \in Q, a \leq (b \leftrightarrow c)$ *implies* $\quad a \leq (b^\perp \leftrightarrow c^\perp)$.

Proof: (a) (*i*) $\Rightarrow$ (*ii*): since $a \leq \mathbf{1}$, then $a \leq (\mathbf{1} \leftrightarrow a)$ and so $a \leq ((\mathbf{1} \otimes \mathbf{1}) \leftrightarrow (a \otimes a))$; thus, $a = a \otimes (\mathbf{1} \otimes \mathbf{1}) \leq a \otimes a$.
(*i*) $\Leftarrow$ (*ii*): If $a \leq (b \leftrightarrow c), a \leq (d \leftrightarrow e)$, then we have $a \otimes b \leq c$ and $a \otimes d \leq e$, and so $a \otimes (b \otimes d) \leq (a \otimes a) \otimes (b \otimes d) = (a \otimes b) \otimes (a \otimes d) \leq c \otimes e$. Analogously, $a \otimes (c \otimes e) \leq b \otimes d$.

(b) (*i*) $\Rightarrow$ (*ii*): since $a \leq \mathbf{1}$ and $!\mathbf{1} = \mathbf{1}$, then $a \leq (\mathbf{1} \leftrightarrow a)$ and therefore $a \leq (\mathbf{1} \leftrightarrow !a)$; thus, $a = a \otimes \mathbf{1} \leq !a \leq a$.

(*i*) $\Leftarrow$ (*ii*): suppose that $a \leq (b \leftrightarrow c)$; thus, $a \otimes b \leq c$ and then $a \otimes !b \leq a \otimes b \leq c$. Thus, by [$M4$], we have $a \otimes !b = !a \otimes !b = !(!a \otimes !b) = !(a \otimes !b) \leq !c$. Analogously, we can prove that $a \otimes !c \leq !b$.

(c) Since Q is commutative, then $(x \rightarrow y) = (y^\perp \rightarrow x^\perp)$, by Proposition 1.9. □

We can now show that we have substitution for every formula:

Proposition 3.8 Let $|\cdot|$: *ELF*($\mathbb{L}$) $\longrightarrow Q$ *satisfying* [=1] – [=6] *(*[=1] – [=4], [=7] *resp.). Then, the extension to FOR*($\mathbb{L}$) *verifies:*

$$|x_1 = y_1| \otimes \cdots \otimes |x_n = y_n| \otimes |\phi(x_1,\ldots,x_n)| \leq |\phi(x_1 \wr y_1,\ldots,x_n \wr y_n)|$$

where $\phi(x_1,\ldots,x_n) \in FOR(\mathbb{L})$ *has (eventually) free occurrences of variables* $x_1,\ldots,x_n$, *and* $\phi(x_1 \wr y_1,\ldots,x_n \wr y_n)$ *is obtained from* $\phi(x_1,\ldots,x_n)$ *by replacing some occurrences of* x_i *(not in the scope of a* y_i*-quantifier) by* y_i.

Proof: By induction in the complexity of ϕ. As a first step, we have two cases to consider.

i) $\phi \in ELF^+(\mathbb{L})$.
It's true by [=2],[=3],[=4] and [=6] (or [=7] and ! weak).

ii) $\phi \in \{ \mathbf{1}, \bot, \top, \mathbf{0} \}$. It's true by [=6] (or [=7] and ! weak).

To proceed with the induction, assume that substitution holds true for every φ with complexity $\leq k$ and let ϕ be a formula with complexity $k + 1$. We have the following cases:

a) $\phi = \alpha \otimes \beta$.
It's immediate from [=5] (common to both systems) and lemma 3.7.

b) $\phi = \alpha^\bot$.
Since $|\alpha^\bot| = |\alpha|^\bot$, the conclusion follows from Lemma 3.7.

c) $\phi = \bigwedge x.\alpha\,(x_1,\ldots,x_n,x)$. Given $z \in V$, we have:

$$\begin{aligned}\otimes_{i=1}^n |x_i = y_i| \otimes \textstyle\bigwedge_{y\in V} |\alpha\,(x_1,\ldots,x_n,x)[y/x]| \\ \leq \otimes_{i=1}^n |x_i = y_i| \otimes |\alpha\,(x_1,\ldots,x_n,x)[z/x]| \\ \leq |\alpha\,(x_1 \wr y_1,\ldots,x_n \wr y_n,x)[z/x]|, \text{ and then}\end{aligned}$$

$$\begin{aligned}\otimes_{i=1}^n |x_i = y_i| \otimes \textstyle\bigwedge_{y\in V} |\alpha\,(x_1,\ldots,x_n,x)[y/x]| \\ \leq \textstyle\bigwedge_{y\in V} |\alpha\,(x_1 \wr y_1,\ldots,x_n \wr y_n,x)[y/x]|.\end{aligned}$$

d) $\phi = \alpha \,\&\, \beta$. Similar to c).

e) (full logic) $\phi = !\alpha$. It follows from [=7] and Lemma 3.7.

Since the other connectives are defined by duality, the proof is complete. □

Now, we shall extend the results in [Yet] to the calculus with equality.

Theorem 3.9 (Soundness) The semantics of quantales for commutative Linear Logic with equality is sound with respect to the given calculus.

Proof: We prove the validity of the new axioms (for the other rules, consult [Yet]). Define $|\vdash A_1,...,A_n| = |A_1 \sqcup \cdots \sqcup A_n|$. By observing that, for every $a,b \in Q$, $a \leq b$ iff $a^{\perp} \sqcup b \geq \mathbf{1}$, the validity of each axiom is guaranteed by [=1]−[=6] (resp. [=1]−[=4],[=7]). □

Theorem 3.10 (Completeness) The semantic of quantales for commutative Linear Logic with equality is complete with respect to the given calculus.

Proof: The proof is an extension of the proofs by [Yet] and [Gir]. Let M_1 be the set of finite sequence of formulas in $\mathbb{L}$; M_1 is a monoid with the operation of concatenation (and identity the null sequence). Let M be the (commutative) monoid obtained by identifying sequences which are distinct only by a permutation of their elements. Just as in example 1.11, $\wp$(M) is a commutative quantale and we set $\perp = \{ \Gamma : \vdash \Gamma \text{ is provable } \} \in \wp(M)$.

Since M is commutative, $\perp$ is cyclic and so we can consider the phase quantale $Q = \wp(M)_j$, where $j : \wp(M) \longrightarrow \wp(M)$ is given by $j(A) = (A \rightarrow \perp) \rightarrow \perp$.

Let $Pr : FOR(\mathbb{L}) \longrightarrow \wp(M)$ defined by:

$$Pr(A) = \{ \Gamma : \vdash A, \Gamma \text{ is provable } \}.$$

By theorem 3.4 in [Yet], Pr factores through Q, i.e., $Pr(FOR(\mathbb{L})) = Q$.

Let $|\cdot| = Pr\,|_{ELF+(\mathbb{L})}$; clearly, the unique extension of $|\cdot|$ to $FOR(\mathbb{L})$ is Pr, once we have defined in Q the open, weak and idempotent modality ! as:

$$!(x) = \bigvee \{ Pr(!A) : Pr(!A) \leq x \}$$

Fact: $!\mathbf{1} = \mathbf{1}$ (where, by definition, $\mathbf{1} = \bigwedge \{ Pr(A) : \vdash A \text{ is provable}\}$).

To see this, Let $\Gamma \in PR(!\mathbf{1})$; thus, $\vdash !\mathbf{1}, \Gamma$ (i.e., $\vdash (?\perp)^{\perp}, \Gamma$) is provable and let $A \in FOR(\mathbb{L})$ be such that $\vdash A$ is provable. Then:

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash A}}{\dfrac{\vdash A, \perp}{\vdash A, ?\perp}} \qquad \dfrac{\vdots}{\vdash (?\perp)^{\perp}, \Gamma}}{\vdash A, \Gamma}$$

i.e., $\Gamma \in Pr(A)$, and so $Pr(!\mathbf{1}) \leq \bigwedge \{ Pr(A) : \vdash A \text{ is provable} \} = \mathbf{1}$.

Thus, $Pr(!\mathbf{1}) \leq \bigvee \{ Pr(!A): Pr(!A) \leq \mathbf{1} \} = !\mathbf{1}$. In fact, equality holds because $\vdash !\mathbf{1}$ is provable, and so $\mathbf{1} \leq Pr(!\mathbf{1}) \leq !\mathbf{1} \leq \mathbf{1}$, i.e. $\mathbf{1} = !\mathbf{1}$.

We shall now show that $|\cdot|$ satisfies the required properties. Since $\vdash (x = x)$ is provable, $\mathbf{1} \leq Pr((x = x)) = |x = x|$. This verifies [= 1].

To prove [=2], let $\Gamma\ \Delta \in |x = y| \cdot |y = z|$. Then we have:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \vdash (x = y), \Gamma\end{matrix} \qquad \vdash (x = y)^{\perp}, (y = z)^{\perp}, (x = z)}{\vdash (y = z)^{\perp}, (x = z), \Gamma} \qquad \begin{matrix}\vdots \\ \vdash (y = z), \Delta\end{matrix}}{\vdash (x = z), \Gamma, \Delta}$$

i.e., $\Gamma\ \Delta \in |x = z|$. Thus, $|x = y| \cdot |y = z| \leq |x = z|$ and so

$$|x = y| \otimes |y = z| = (|x = y| \cdot |y = z|)^{\perp\perp} \leq |x = z|.$$

To prove [=3], let $\Gamma \in |x = y|$. We have:

$$\dfrac{\begin{matrix}\vdots \\ \vdash (x = y), \Gamma\end{matrix} \qquad \vdash (x = y)^{\perp}, (y = x)}{\vdash (y = x), \Gamma}$$

i.e., $\Gamma \in |y = x|$ and thus $|x = y| \leq |y = x|$

Properties [=4], [=5], [=6], [=7] are verified in a similar way. This shows that we have an interpretation of quantales such that $|A| \geq \mathbf{1}$ iff $\vdash A$ is provable, completing the proof. □

4. *Generalisations of the Calculus*

There are alternatives to the treatment of equality given above, using the exponentials of Linear Logic. For example, instead of requiring $(\cdot = \cdot)$ to be open, we could establish that its characteristic properties be valid in the interior of $(\cdot = \cdot)$. Thus, we define the calculus $(P_!)$ by the axioms:

[$R!$] $\vdash !(x = x)$ [$S!$] $\vdash !(x = y) \multimap !(y = x)$

[$T!$] $\vdash !(x = y) \otimes !(y = z) \multimap !(x = z)$

For every predicate symbol P,

$$[SUBS!] \quad \vdash\ !(x_1 = y_1) \otimes \cdots \otimes !(x_n = y_n) \otimes P(x_1,\ldots,x_n) \multimap P(x_1 \wr y_1,\ldots,x_n \wr y_n)$$

Clearly, we have substitution for every formula, and we can say that $!(x = y)$ has the behaviour of a linear equality. An even weaker form of substitution can be defined as follows:

$$\vdash\ !(x_1 = y_1) \otimes \cdots \otimes !(x_n = y_n) \otimes !P(x_1,\ldots,x_n) \multimap\ !P(x_1 \wr y_1,\ldots,x_n \wr y_n)$$

or, even

$$\vdash\ !(x_1 = y_1) \otimes \cdots \otimes !(x_n = y_n) \otimes !P(x_1,\ldots,x_n) \multimap\ ?P(x_1 \wr y_1,\ldots,x_n \wr y_n).$$

We can modify each axiom combining the modal operators in all possible ways. It is straightforward to verify that (incorporating *id*, the identity operation on formulas), there are only seven modal operations obtained by successive applications of $\{!, ?\}$, namely, the set

$\mathcal{M} = \{\ !, !?!, ?!, !?, ?!?, ?, id\ \}$

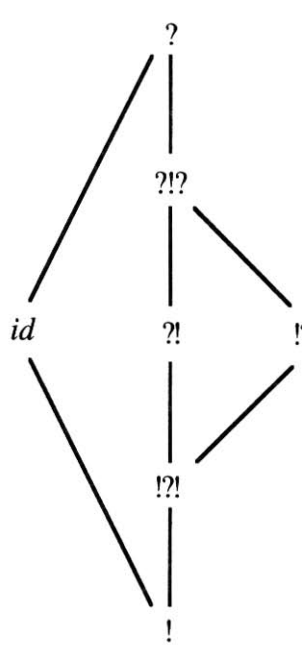

fig. 1

with its natural lattice structure (see figure 1). This is in complete analogy with Kuratowski's problem, in which we can prove that, given a topological space X and $A \subseteq X$, there are only 14 distinct sets that can be obtained from A by combinations of taking complements and closure (in our case, we do not consider modalities of the form $m'(x) = m(x)^{\perp}$ with $m \in \mathcal{M}$). Starting with this, we can define a general scheme to create a linear calculus with equality.

Definition 4.1 Let $(m_i)_{i=1}^{10}$ *be a sequence in* $\mathcal{M}$. *A* linear calculus with equality $P((m_i))$ *is defined by the following axioms:*

$$[R] \quad \vdash m_1(x = x) \qquad\qquad [S] \vdash m_2(x = y) \multimap m_3(y = x)$$
$$[T] \quad \vdash m_4(m_5(x = y) \otimes m_5(y = z)) \multimap m_6(x = z)$$

For each predicate symbol P, the axiom

$$[SUBS] \quad \vdash m_7((\otimes_{i=1}^{n} m_8(x_i = y_i)) \otimes m_9(P(x_1,\ldots,x_n))) \multimap m_{10}(P(x_1 \wr y_1,\ldots,x_n \wr y_n)).$$

Remarks 4.2 (a) If $m_3 \leq m_2$, *then* [S] *implies that, for* $m_3 \leq r,s \leq m_2$, $r(x = y)$, $s(x = y)$, $r(y = x)$ *and* $s(y = x)$ *are all equivalent.*

(b) If $m_1 = id$ *or* $m_1 = !$, *then* $\vdash m(x = x)$ *is provable for every* $m \in \mathcal{M}$.

(c) If $m_7 = id$, $m_8 \in \{ !, !?, !?! \}$ *and* $m_9 \geq m_{10}$, *then substitution holds for every formula* ϕ *built up from formulas of the form* $m_9(P)$, *where* $P \in ELF(\mathbb{L})$.

(d) For each $m_i \in \{!, !?, !?! \}$, *we have* $m(\otimes_{i=1}^{n} m_i(A_i)) = m!(\otimes_{i=1}^{n} m_i(A_i))$.

Examples 4.3 (a) The system (LLE_1) *of section 2 is obtained by the assignment*

$m_3 = !$ *and* $m_i = id$, *for every* $i \neq 3$.

(b) If we set

$m_1 = m_2 = m_3 = m_5 = m_6 = m_8 = !$ *and* $m_4 = m_7 = m_9 = m_{10} = id$,

we get the system $(P_!)$, *in which "equality" is the interior of* $(\cdot = \cdot)$. *This system satisfies substitution for every formula.*

(c) If we set

$$m_1 = m_2 = m_5 = m_6 = m_8 = ?, \quad m_4 = m_7 = m_9 = m_{10} = id,$$

and

$$m_3 = !?$$

then we have a system in which "equality" corresponds to the closure of $(\cdot = \cdot)$*; from [S] will follow that that* $?(\cdot = \cdot)$ *is "clopen". This system satisfies the substitution rule for every formula.*

(d) By considering

$$m_1 = m_3 = m_6 = m_{10} = ?!, \; m_2 = m_5 = m_8 = m_9 = !?!$$

and

$$m_4 = m_7 = !? \text{ or } !?!$$

we get a system containing a translation of the classical theory of equality inside (LL), to be studied in next section.

(e) Another translation of classical equality in (LL) can be obtained by considering the assignment

$$m_1 = id, m_2 = m_4 = m_5 = m_7 = m_8 = m_9 = ?$$

and

$$m_3 = m_6 = m_{10} = !.$$

Now, we would like to identify the classes of equivalent calculi. Instead of doing a complete classification of the possible calculi, we present quantale theoretic techniques that are useful in deciding this problem. We start with reflexivity ([R]):

Proposition 4.4 Let Q be a Girard's quantale,and let $x \in Q$. Then:

(a) Are equivalent:

$$(i) \; ?x \geq \mathbf{1} \qquad (ii) \; !?x \geq \mathbf{1} \qquad (iii) \; ?!?x \geq \mathbf{1}$$

(*b*) *Are equivalent:*

(*i*) $x \geq \mathbf{1}$ (*ii*) $!x \geq \mathbf{1}$

(*c*) *Are equivalent:*

(*i*) $?!x \geq \mathbf{1}$ (*ii*) $!?!x \geq \mathbf{1}$

Proof: (*a*): If $?x \geq \mathbf{1}$, then $!?x \geq !\mathbf{1} = \mathbf{1}$. If $!?x \geq \mathbf{1}$, then $?!?x \geq ?\mathbf{1} \geq \mathbf{1}$. Finally, $?!?x \geq \mathbf{1}$ implies $?x \geq \mathbf{1}$, because $? \geq ?!?$.
(*b*) comes directly from $!\mathbf{1} = \mathbf{1}$, while (*c*) is consequence of (*b*). □

Thus, introducing the notation $r \equiv s$ to denote that calculi obtained with $m_1 = r$ and $m_1 = s$ are equivalent, we have:

(*a*) $? \equiv !? \equiv ?!?$ (*b*) $id \equiv !$ (*c*) $?! \equiv !?!$

and so there are only three non equivalent possibilities for [*R*].

With respect to axiom [*S*], consider $\mathcal{M}_1 = \{?, ?!, ?!?\} = \{?m\colon m \in \mathcal{M}\}$, together with $\mathcal{M}_2 = \{!, !?, !?!\} = \{!m\colon m \in \mathcal{M}\}$. Write $(r,s) \equiv (r',s')$ to denote that calculi obtained with $m_2 = r$, $m_3 = s$ and $m_2 = r'$, $m_3 = s'$ are equivalent. Then we have:

Proposition 4.5 (*a*) *If* $m_3 \in \mathcal{M}_1$, *then:*

(*i*) $(!,m_3) \equiv (?!,m_3) \equiv (!?!,m_3)$ (*ii*) $(!?,m_3) \equiv (?!?,m_3)$
(*iii*) $(id,m_3) \equiv (?,m_3)$

(*b*) *If* $m_2 \in \mathcal{M}_2$, *then:*

(*i*) $(m_2,?) \equiv (m_2,!?) \equiv (m_2,?!?)$ (*ii*) $(m_2,?!) \equiv (m_2,!?!)$
(*iii*) $(m_2,id) \equiv (m_2,!)$

Therefore, there are only 26 non equivalent cases for [*S*]:

(1): $(!,!?) \equiv (!,?!?) \equiv (!,?) \equiv (!?!,?!?) \equiv (?!,?!?) \equiv (!?!,!?) \equiv (!?!,?) \equiv (?!,?)$
(2): $(!?,!?) \equiv (!?,?!?) \equiv (!?,?) \equiv (?!?,?!?) \equiv (?!?,?)$
(3): $(!,?!) \equiv (!?!,?!) \equiv (?!,?!) \equiv (!,!?!) \equiv (!?!,!?!)$
(4): $(id,?) \equiv (?,?)$ (5): $(!?,?!) \equiv (?!?,?!) \equiv (!?,!?!)$
(6): $(id,?!) \equiv (?,?!)$ (7): $(id,?!?) \equiv (?,?!?)$ (8): $(!,id) \equiv (!,!)$
(9): $(!?,!) \equiv (!?,id)$ (10): $(!?!,!) \equiv (!?!,id)$ (11): (id,id)
(12): $(id,!)$ (13): $(id,!?)$ (14): $(id,!?!)$ (15): $(?,id)$

(16): (?,!?)	(17): (?,!?!)	(18): (?,!)	(19): (?!?,*id*)
(20): (?!?,!?)	(21): (?!?,!?!)	(22): (?!?,!)	(23): (?!,*id*)
(24): (?!,!?)	(25): (?!,!?!)	(26): (?!,!)	

For transitivity ([*T*]), we shall write $(r,s,t) \equiv (r',s',t')$ when calculi obtained with $m_4 = r$, $m_5 = s$, $m_6 = t$ and $m_4 = r'$, $m_5 = s'$, $m_6 = t'$ are equivalent. Then we have:

Proposition 4.6 If $m_5 \in \mathcal{M}_2$, *then:*
(*i*) *If* $m_6 \in \mathcal{M}_1$, *then* $(r,m_5,m_6) \equiv (s,m_5,m_6)$ *for every* $r,s \in \mathcal{M}$
(*ii*) $(r,m_5,?) \equiv (s,m_5,?) \equiv (id,m_5,!?) \equiv (!,m_5,!?) \equiv (!?,m_5,!?) \equiv (!?!,m_5,!?) \equiv (t,m_5,?!?) \equiv (u,m_5,?!?)$ *for every* $r,s,t,u \in \mathcal{M}$.
(*iii*) $(r,m_5,?!) \equiv (s,m_5,?!) \equiv (id,m_5,!?!) \equiv (!,m_5,!?!) \equiv (!?,m_5,!?!) \equiv (!?!,m_5,!?!)$ *for every* $r,s \in \mathcal{M}$.

The results above imply that, for $m_5 \in \mathcal{M}_2$ fixed, we have only 8 possibilities for pairs (m_4,m_6) (in contrast with 26 possibles ones), where the numbers refer to the pairs described above.

[1]:	(1) ≡ (2) ≡ (4) ≡ (7) ≡ (13)	[2]:	(3) ≡ (5) ≡ (6) ≡ (14)
[3]:	(8) ≡ (11) ≡ (12)	[4]:	(9) ≡ (10)
[5]:	(15) ≡ (19) ≡ (23)	[6]:	(16) ≡ (20) ≡ (24)
[7]:	(17) ≡ (21) ≡ (25)	[8]:	(18) ≡ (22) ≡ (26)

With respect to substitution, if we fix m_8, $m_9 \in \mathcal{M}_2$, then there are again, for pairs (m_7,m_{10}), only the 8 cases above. Thus, these are the non equivalent calculi that can be constructed with the exponentials, satisfying the usual rules of equality. We register that this corresponds to only 8% of the original universe of possible calculi.

5. *Interpreting Classical Equality in Linear Logic with Equality*

It is well know that the exponentials of linear logic (*LL*) are important in interpreting intuitionistic logic (*IL*) and classical logic (*CL*) inside *LL*. For each of these logics, we have two translations, one of them based in the fact that every (commutative) Girard's quantale contains a frame (complete Heyting algebra) (corollary 1.21) and a complete Boolean algebra:

$$\mathcal{H} = \{ \, !x \colon x \in Q \, \} \quad \text{and} \quad \mathcal{B} = \{ \, ?!x \colon x \in Q \, \},$$

respectively. Operations and constants in each of these algebras are given by:

(H). For the frame $\mathcal{H}$:

$$\mathbf{0}_{\mathcal{H}} = \mathbf{0},\ \top_{\mathcal{H}} = \mathbf{1}, \quad (x \wedge_{\mathcal{H}} y) = (x \otimes y) = !(x \wedge y), \quad (x \vee_{\mathcal{H}} y) = (x \vee y);$$
$$(x \Rightarrow y) = !\,(x \rightarrow y), \quad \neg\, x = (x \Rightarrow \mathbf{0}_{\mathcal{H}}) = !\,(x \rightarrow \mathbf{0}),$$
$$\textstyle\bigwedge_{\mathcal{H}} S = !\bigwedge S, \quad \bigvee_{\mathcal{H}} S = \bigvee S.$$

(B). For the complete Boolean algebra $\mathcal{B}$:

$$\mathbf{0}_{\mathcal{B}} = \perp,\ \top_{\mathcal{B}} = ?\mathbf{1}, \quad (x \wedge_{\mathcal{B}} y) = ?(!x \otimes !\,y) = ?!(x \wedge y),$$
$$(x \vee_{\mathcal{B}} y) = (x \sqcup y) = (x^{\perp} \otimes y^{\perp})^{\perp};$$
$$(x \Rightarrow y) = (!x \rightarrow y), \neg\, x = (x \Rightarrow \mathbf{0}_{\mathcal{B}}) = (!x \rightarrow \perp) = ?(x^{\perp});$$
$$\textstyle\bigwedge_{\mathcal{B}} S = ?!\bigwedge S, \quad \bigvee_{\mathcal{B}} S = ?\bigvee \{\ !x : x \in S\ \}.$$

Thus, we can interpret $A \in FOR(IL)$ as $A^{i} \in FOR(\mathbb{L})$ by the rules:
$A^{i} = !A$ if A is atomic; proceed by induction on complexity using the operations in (H) to define A^{i} for all intuitionistic formulas (here, $FOR(IL)$ denote the set of intuitionistic formulas).

For classical logic, we have our first translation:

(*) $A^{c} = ?!A$ if A is atomic; then proceed by induction on complexity, using the rules in (B).

Thus, if we assume two-hand sequents for (LL), we have ([Gir]):

$A_1,\ldots,A_n \vdash_{IL} A$ is provable in intuitionistic logic
iff $(A_1)^{i},\ldots,(A_n)^{i} \vdash A^{i}$ is provable in LL,

and

$(1^{st}\ T)$: $A_1,\ldots,A_n \vdash_{CL} A$ is provable in classical logic iff $!(A_1)^{c},\ldots,!(A_n)^{c} \vdash A^{c}$ is provable in LL.

Another interpretation for classical logic is constructed from polarities for formulas. Thus, given a sequent for classical logic $\Gamma \vdash_{CL} \Delta$, we'll say that occurrences of formulas $A \in \Gamma$ are *positive* (denoted by pA) and occurrences of formulas $B \in \Delta$ are *negative* (denoted by nA). We have the following rules of a second translation (**):

$pA = A = nA$ if A is atomic.
$p(\neg A) = (nA)^{\perp}$, $n(\neg A) = (pA)^{\perp}$,
$p(A \vee B) = p(A) \oplus p(B)$, $n(A \vee B) = !n(A) \sqcup !n(B)$,
$p(A \wedge B) = ?p(A) \otimes ?p(B)$, $n(A \wedge B) = n(A) \,\&\, n(B)$,
$p(A \Rightarrow B) = n(A)^{\perp} \oplus p(B)$, $n(A \Rightarrow B) = ?p(A) \multimap !n(B)$,

$p((\forall x)A) = \bigwedge x.?p(A)$, $n((\forall x)A) = \bigwedge x.n(A)$,
$p((\exists x)A) = \bigvee x.p(A)$, $n((\exists x)A) = \bigvee x.!n(A)$.

With this definition, we have ([Gir]):

(**) $A_1,...,A_k \vdash_{CL} A$ is provable in (CL) iff $!n(A_1),...,!n(A_k) \vdash ?p(A)$ is provable in (LL).

We now turn to the question of determining a linear theory of equality such that, given a classical theory of, its translation into linear logic is defines a linear theory contained in the original linear theory of equality.

For this, we need to relate the deduction of a sequent in (LL) from a set of sequent-axioms and the deduction of a formula from a multiset of hypothesis (in the left-hand side of the sequent), called *external* and *internal* relations of consequence (respectively) by [Avr].

Definition 5.1 Let $A \in FOR(\mathbb{L})$ *with exactly* $x_1,...,x_n$ *as free variables. The* universal closure *of* A, *denoted by* $\bigwedge A$ *is the formula obtained from* A *by quantifying universally all the free variables of* A, *i.e.*, $\bigwedge A = \bigwedge x_1. \cdots \bigwedge x_n.A$. *Analogously, we define* $\bigvee A$, *the* existential closure *of* A.

It is straightforward to prove next result:

Proposition 5.2 Let $A_1,...,A_n$ *be formulas,* $B = !\bigwedge A_1 \,\&\, \cdots \,\&\, !\bigwedge A_n$, *and* Δ *a multiset of formulas. Are equivalent:*

(*i*) $\vdash \Delta$ *is provable from the sequent-axioms* $\vdash A_1,...,\vdash A_n$;
(*ii*) *there exists* $k \geq 0$ *such that* $\underbrace{B,\ldots,B}_{k\ times} \vdash \Delta$ *is provable;*
(*iii*) $!\bigwedge A_1,...,!\bigwedge A_n \vdash \Delta$ *is provable.*

Now, assume that $\mathcal{A}$ is the set of axioms (without free variables) of a theory of (CL) in a language without functional symbols. It follows from Proposition 5.2 that if $A_1,...,A_n \in \mathcal{A}$ and A is a formula, then the following are equivalent, for the first translation (*) of CL into LL:

(i) $A_1,...,A_n \vdash_{CL} A$;
(ii) $!(A_1)^c,...,!(A_n)^c \vdash A^c$ is provable in (LL);
(iii) There is a proof of $\vdash A^c$ from the axioms $\vdash (A_1)^c,...,\vdash (A_n)^c$ (recall that the A_i's have no free variables).

Thus, to each axiom $A \in \mathcal{A}$ of a classical theory, corresponds a sequent-axiom $\vdash A^c$ in (LL). Similarly, for the second translation (**), it can be seen that to each axiom $A \in \mathcal{A}$, there corresponds a sequent-axiom $\vdash n(A)$.

Now assume that $\mathcal{A}$ defines the classical theory of equality. Thus, $\mathcal{A}$ consists in the following axioms:

Now assume that $\mathcal{A}$ defines the classical theory of equality. Thus, $\mathcal{A}$ consists in the following axioms:

$(\forall x)(x = x);$
$(\forall x)(\forall y)((x = y) \Rightarrow (y = x));$
$(\forall x)(\forall y)(\forall z)((x = y) \wedge (y = z) \Rightarrow (x = z));$
$(\forall x_1)(\forall y_1)...(\forall x_n)(\forall y_n)((x_1 = y_1) \wedge \cdots \wedge (x_n = y_n) \wedge P(x_1,...,x_n)$
$\Rightarrow P(x_1 \wr y_1,...,x_n \wr y_n))$
(where P varies over every predicate symbol of arity n of the language).

For the first translation (*), we get a calculus (E_1) with the following axioms:

$[R1]$ $\vdash ?!\bigwedge x\, ?!(x = x)$ $[S1]$ $\vdash ?!\bigwedge x\, ?!\bigwedge y\, (!?!(x = y) \multimap ?!(y = x))$
$[T1]$ $\vdash ?!\bigwedge x\, ?!\bigwedge y\, ?!\bigwedge z\, (!?(!?!(x = y) \otimes !?!(y = z)) \multimap ?!(x = z))$
$[SUBST1]$ If P is a n-ary predicate symbol of the language
$\vdash ?!\bigwedge x_1\, ?!\bigwedge y_1 \cdots ?!\bigwedge x_n\, ?!\bigwedge y_n (!?(!?!(x_1 = y_1) \otimes \cdots \otimes$
$!?!(x_n = y_n) \otimes !?!P(x_1,...,x_n)) \multimap ?!P(x_1 \wr y_1,...,x_n \wr y_n))$

For the second translation (**), we get a calculus (E_2) with the following axioms:

$[R2]$ $\vdash \bigwedge x\, (x = x)$ $[S2]$ $\vdash \bigwedge x \bigwedge y\, (?(x = y) \multimap !(y = x))$
$[T2]$ $\vdash \bigwedge x \bigwedge y \bigwedge z\, (?(?(x = y) \otimes ?(y = z)) \multimap !(x = z))$
$[SUBST2]$ If P is a n-ary predicate symbol in the language
$\vdash \bigwedge x_1 \bigwedge y_1 \cdots \bigwedge x_n \bigwedge y_n\, (?(?(x_1 = y_1) \otimes \cdots \otimes$
$?(x_n = y_n) \otimes ?P(x_1,...,x_n)) \multimap !P(x_1 \wr y_1,...,x_n \wr y_n)).$

Consider the linear theory (P_1) determined by the axioms:

$[LR1]$ $\vdash ?!(x = x)$ $[LS1]$ $\vdash ?!(x = y) \multimap ?!(y = x)$
$[LT1]$ $\vdash !?!(x = y) \otimes !?!(y = z) \multimap ?!(x = z)$
$[LSUBST1]$ $\vdash !?!(x_1 = y_1) \otimes \cdots \otimes !?!(x_n = y_n) \otimes !?!P(x_1,...,x_n))$
$\multimap ?!P(x_1 \wr y_1,...,x_n \wr y_n).$

Clearly, (P_1) contains (E_1) (because every axiom in (E_1) is deducible in (P_1), using the equivalences of last section) and we have that (P_1) defines a linear theory of equality in the sense of section 4.

Similarly, define the theory (P_2) by the axioms:

$$[LR2] \ \vdash (x = x) \qquad [LS2] \ \vdash ?(x = y) \multimap !(y = x)$$
$$[LT2] \ \vdash ?(?(x = y) \otimes ?(y = z)) \multimap !(x = z)$$
$$[LSUBST2] \quad \vdash ?(?(x_1 = y_1) \otimes \cdots \otimes ?(x_n = y_n) \otimes ?P(x_1,\ldots,x_n)) \multimap !P(x_1 \wr y_1,\ldots,x_n \wr y_n).$$

We can reformulate (P_2) as:

$$[LR2'] \ \vdash (x = x) \qquad [LS2'] \ \vdash (x = y) \multimap (y = x)$$
$$[LT2'] \ \vdash (x = y) \otimes (y = z) \multimap (x = z)$$
$$[LSUBST2'] \quad \vdash ?((x_1 = y_1) \otimes \cdots \otimes (x_n = y_n) \otimes ?P(x_1,\ldots,x_n)) \multimap !P(x_1 \wr y_1,\ldots,x_n \wr y_n)$$
$$[? \leq !] \ \vdash ?\,(x = y) \multimap !(x = y).$$

It is straightforward to check that (P_2) and (E_2) are equivalents. In fact, (P_2) is the strongest (and therefore more restricted as far as semantics is concerned) of the calculi already defined.

6. *Equality and intuitionistic Linear Logic*

In this section we shall study how to recover classical logic and intuitionistic logic from a linear calculus without exponentials, as well as analyse another extensions of the relationship between linear logic and quantales. Our basic system is (commutative) first order intuitionistic linear logic, from now on denoted by *(LLI)*. The appropriate semantics will turn out to be furnished by commutative unital quantales. Similarly, the semantics for non-commutative first order intuitionistic linear shall be proven to be given by unital quantales. From this, it will be seen that, adding appropiate axioms, we can recover intuitionistic logic, classical logic and classical linear logic. Thus, an intuitionistic linear theory of equality provides an intuitionistic theory of equality, a classical theory of equality, and a classical linear theory of equality, simultaneously. The most natural candidate is system (LLE_0), the simplest already defined.

We define a sequent calculus for (commutative) first-order intuitionistic linear logic without negation, simply by extending the system in [GiLa] and then proving soundness and completeness for this system.

Definition 6.1 The language $\mathbb{L}_i$ for commutative first-order intuitionistic linear logic consist of a countable set of predicate symbols, $\wp = \{ P_n : n \in \omega \}$, a countable set of variables $V = \{ v_n : n \in \omega \}$, the symbols $\otimes$, $\multimap$, &, $\oplus$, $\bigvee$, $\bigwedge$ and the same rules as in Definition 2.2 for the formation of the set of formulas $FOR(\mathbb{L}_i)$. For $A \in FOR(\mathbb{L}_i)$ and $x,y \in V$, $A[y/x]$ is as in Definition 2.4.

Definition 6.2 The calculus (LLI) for commutative first order intuitionistic linear logic consist in the following rules and axioms (A,B,C denote formulas, and Γ, Δ denote multisets of formulas):

$$[AX1]\quad A \vdash A \qquad\qquad [AX2]\quad \Gamma \vdash \top$$

$$[AX3]\quad \vdash \mathbf{1} \qquad\qquad [AX4]\quad \Gamma, \mathbf{0} \vdash A$$

$$[CUT]\quad \frac{\Gamma \vdash A \qquad A, \Delta \vdash B}{\Gamma, \Delta \vdash B} \qquad [EXCH]\quad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$$

$$[\mathbf{1}L]\quad \frac{\Gamma \vdash A}{\mathbf{1}, \Gamma \vdash A} \quad [\otimes R]\quad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \quad [\otimes L]\quad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

$$[\& R]\quad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B} \quad [\& 1]\quad \frac{\Gamma, A \vdash C}{\Gamma, A \& B \vdash C} \quad [\& 2]\quad \frac{\Gamma, B \vdash C}{\Gamma, A \& B \vdash C}$$

$$[\oplus L]\quad \frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C} \quad [\oplus 1]\quad \frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} \quad [\oplus 2]\quad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}$$

$$[\multimap R]\quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad [\multimap L]\quad \frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}$$

$$[\bigvee R]\quad \frac{\Gamma \vdash A[y/x]}{\Gamma \vdash \bigvee x.A} \quad [\bigvee L]\quad \frac{\Gamma, A \vdash B}{\Gamma, \bigvee x.A \vdash B} \text{ if } x \text{ not occurs free in } \Gamma, B$$

$$[\bigwedge L]\quad \frac{\Gamma, A[y/x] \vdash B}{\Gamma, \bigwedge x.A \vdash B} \qquad [\bigwedge R]\quad \frac{\Gamma \vdash A}{\Gamma \vdash \bigwedge x.A} \text{ if } x \text{ not occurs free in } \Gamma$$

Definition 6.3 An interpretation of quantales *for (LLI) is a commutative unital quantale* $(Q, \bigvee, *, \mathbf{1})$ *and a map* $|\cdot|_Q : FOR(\mathbb{L}_i) \longrightarrow Q$ *satisfying*:

1. $|\mathbf{0}| = \mathbf{0}$, $|\mathbf{1}| = \mathbf{1}$, $|\top| = \top$.
2. $|A \otimes B| = |A| * |B|$, $|A \multimap B| = |A| \to |B|$.
3. $|A \& B| = |A| \wedge |B|$, $|A \oplus B| = |A| \vee |B|$.
4. $|\bigwedge x.A| = \bigwedge y \in VAR(\mathbf{L}_i)\ |A[y/x]|$,
   $|\bigvee x.A| = \bigvee y \in VAR(\mathbf{L}_i)\ |A[y/x]|$.

We say that $A \in FOR(\mathbb{L}_i)$ *is* valid in Q *if* $|A|_Q \geq \mathbf{1}$.
A sequent $\Gamma \vdash A$ *is* valid in Q *if* $|\Gamma|_Q \leq |A|_Q$, *where*

$$|\Gamma|Q = \begin{cases} \mathbf{1} & \text{if } \Gamma = \varnothing \\ \otimes_{i=1}^{n} |A_i|_Q & \text{if } \Gamma = A_1, \ldots, A_n \end{cases}$$

Before stating and proving soundness we establish the following simple

Lemma 6.4 Let Q be a quantale. Then, $a \otimes \mathbf{0} = \mathbf{0} \otimes a = \mathbf{0}$ for every $a \in Q$.

Proof: $\mathbf{0} \leq a \rightarrow_r \mathbf{0}$ implies $a \otimes \mathbf{0} \leq \mathbf{0}$; $\mathbf{0} \leq a \rightarrow_l \mathbf{0}$ implies $\mathbf{0} \otimes a \leq \mathbf{0}$.□

Theorem 6.5 (Soundness) If $\Gamma \vdash A$ is provable in (LLI), then it is valid in every interpretation of quantales, i.e., $|\Gamma|_Q \leq |A|_Q$ for every unital quantale Q.

Proof: By induction on least lenght n of a proof of $\Gamma \vdash A$.

Let Q be a quantale. If $n = 1$, [$AX1$],[$AX2$] and [$AX3$] are immediate, while [$AX4$] is a consequence of Lemma 6.4.

Assume the thesis holds for all sequents with a proof of length $\leq n$ ($n \geq 1$ fixed) and let $\Lambda \vdash D$ be a sequent admitting a proof of minimum length $n + 1$. We discuss the last rule applied in the proof of $\Lambda \vdash D$:

[CUT]: We have $|\Gamma| \leq |A|$ and $|A| * |\Delta| \leq |B|$ by the induction hypothesis; thus, $|\Gamma| * |\Delta| \leq |A| * |\Delta| \leq |B|$.

The passage through the rules [$EXCH$] and [$\mathbf{1}L$] follow from the fact that Q is commutative and that $\mathbf{1}$ is the unit of Q.

[$\otimes R$]: $|\Gamma| \leq |A|$ and $|\Delta| \leq |B|$ (induction hypothesis) yield, then $|\Gamma| * |\Delta| \leq |A| * |B|$.

[$\otimes L$]: This works by definition of interpretation.

[$\& R$]: By induction, $|\Gamma| \leq |A|$ and $|\Gamma| \leq |B|$, and so $|\Gamma| \leq |A| \wedge |B|$.

[& 1] and [& 2]: We have $|\Gamma| * |A| \leq |C|$ (induction hypothesis); thus

$$|\Gamma| * (|A| \wedge |B|) \leq |\Gamma| * |A| \leq |C|.$$

The other rule is similar.

[$\oplus L$],[$\oplus$ 1],[$\oplus$ 2]: Same argument as in &, using $\vee$ in place of $\wedge$.

[$\multimap R$]: Induction yields $|\Gamma| * |A| \leq |B|$; thus, by adjointness $|\Gamma| \leq |A| \rightarrow |B|$.

[$\multimap L$]: The induction hypothesis yields $|\Gamma| \leq |A|$ and $|\Delta| * |B| \leq |C|$. Thus,

$$(|A| \rightarrow |B|) * |\Gamma| * |\Delta| \leq (|A| \rightarrow |B|) * |A| * |\Delta| \leq |B| * |\Delta| \leq |C|.$$

[$\bigvee R$]: By induction, there is a variable y such that $|\Gamma| \leq |A[y/x]|$; but then

$$|\Gamma| \leq |A[y/x]| \leq \bigvee_y |A[y/x]|.$$

We now state a Fact whose proof is routine:

Fact: If Γ , $A \vdash B$ is provable in n steps and x is not free in either Γ or B, then Γ , $A[y/x] \vdash B$ is provable in n steps for every $y \in V$.

[$\bigvee L$]: Suppose that last rule applied was [$\bigvee L$]. Since x is not free in Γ, B, by induction and the Fact above, we may assume that $|\Gamma| * |A[y/x]| \leq |B|$, for every $y \in V$.

Thus, $|A[y/x]| \leq |\Gamma| \rightarrow |B|$ for every $y \in V$ and so $|\bigvee x A| = \bigvee_y |A[y/x]| \leq |\Gamma| \rightarrow |B|$, i.e. $|\Gamma| * |\bigvee x.A| \leq |B|$.

[$\bigwedge L$]: This is treated just as [$\bigvee R$], above.

[$\bigwedge R$]: Suppose that last rule applied was [$\bigwedge R$]. Since x is not free in Γ, induction and the Fact above allow us to assume that $|\Gamma| \leq |A[y/x]|$, for every $y \in V$. It is then clear that $|\Gamma| \leq \bigwedge_y |A[y/x]|$, ending the proof. □

Before we prove completeness, we need a general result about quantales (generalizing an analogous result about Heyting algebras in [Mir]).

Definition 6.6 An autonomous poset *is a partially ordered set P with a binary associative operation* $\otimes$ *such that the endomorfisms* $a \otimes \cdot$ *and* $\cdot \otimes a$ *have right adjoints, denoted by* $a \rightarrow_r \cdot$ *and* $a \rightarrow_l \cdot$, *respectively.*

A [$\otimes$, $\bigvee$]-autonomous lattice *L is an autonomous poset where L is a lattice and such that* $\otimes$ *is distributive for all* $\bigvee$'s *which exists in L, i.e., if* $S \subseteq L$ *such that exists* $\bigvee S$ *in L, then, for every* $a \in L$, $\bigvee s \in S (a \otimes s)$ and $\bigvee s \in S\ (s \otimes a)$ *exists in L, and we have that* $\bigvee s \in S (a \otimes s) = a \otimes (\bigvee S)$, $\bigvee s \in S\ (s \otimes a) = (\bigvee S) \otimes a$.

If S, T are subsets of L, define

$$S \cdot T = \{a \otimes b : a \in S \text{ and } b \in T\}.$$

Definition 6.7 Let L be a lattice and $I \subseteq L$ *an ideal (i.e., if* $x \in I$ *and* $y \leq x$ *then* $y \in I$; *if* $x,y \in I$ *then* $x \vee y \in I$). *We say that I is* complete *if it satisfies*:

$$\text{if } S \subseteq I \text{ such that } \bigvee_L S \text{ exists, then } \bigvee_L S \in I.$$

Since an arbitrary meet of complete ideals is again a complete ideal,

$CI(L) = \{ I \subseteq L : I$ *is a complete ideal in L* $\}$, *is a complete lattice ordered by inclusion, and containing, for each* $a \in L$,

$$a^{\leftarrow} = \{x \in L : x \leq a \}.$$

If $a \in L$ *and* $S \subseteq L$ *is a subset of L, define*

$$a \rightarrow_r S \equiv_{def} \bigcup_{c \in S} (a \rightarrow_r c)^{\leftarrow} \quad \text{and} \quad a \rightarrow_l S \equiv_{def} \bigcup_{c \in S} (a \rightarrow_l c)^{\leftarrow}.$$

Lemma 6.8 Let L be a $[\otimes, \bigvee]$*-autonomous lattice. Then*
(a) If $a \in L$ *and* $I \in CI(L)$*, then* $a \rightarrow_r I$ *and* $a \rightarrow_l I$ *are in* $CI(L)$.
(b) For every $S, T \subseteq L$ *and* $K \in CI(L)$*, we have*

$$S \cdot T \subseteq K \text{ iff } S \subseteq \bigcap_{a \in T} (a \rightarrow_l K) \text{ iff } T \subseteq \bigcap_{a \in S} (a \rightarrow_r K).$$

Proof: (a) If $x \in a \rightarrow_r I$, then there is $c \in I$ such that $x \leq a \rightarrow_r c$; thus, if $y \leq x$ then $y \leq a \rightarrow_r c$ and therefore $y \in a \rightarrow_r I$.

If $x, y \in a \rightarrow_r I$, then there are $c, d \in I$ such that $x \leq a \rightarrow_r c$, and $y \leq a \rightarrow_r d$; therefore

$$x \vee y \leq (a \rightarrow_r c) \vee (a \rightarrow_r d) \leq a \rightarrow_r (c \vee d),$$

yields $x \vee y \in a \rightarrow_r I$, because $c \vee d \in I$.

Let $S \subseteq a \rightarrow_r I$ be such that $c = \bigvee_L S$ exists. For every $s \in S$ there is $c_s \in I$ such that $s \leq a \rightarrow_r c_s$, i.e., $a \otimes s \leq c_s$. Since I is an ideal, $a \otimes s \in I$, for all $s \in S$. Then,

$$a \otimes c = a \otimes \bigvee_{s \in S} s = \bigvee_{s \in S} a \otimes s \in I,$$

because I is complete. But this means that $c \in a \rightarrow_r I$. A similar computation will prove that $a \rightarrow_l I \in CI(L)$.

(b) Let $x \in S$ and $a \in T$ and assume that $S \cdot T \subseteq K$; since $x \otimes a = c \in K$, then $x \leq a \rightarrow_l c$ and thus $x \in a \rightarrow_l K$, for every $a \in T$.

Now, suppose that $S \subseteq \bigcap_{a \in T} (a \rightarrow_l K)$, $x \in S$ and $a \in T$; since $x \in a \rightarrow_l K$, there is $c \in K$ such that $x \leq a \rightarrow_l c$, i.e. $x \otimes a \leq c$. Since K is an ideal, we get $x \otimes a \in K$. The other equivalence can be handled similarly. □

Theorem 6.9 (The completion of a $[\otimes, \bigvee]$*-autonomous lattice)*

Let L be a $[\otimes, \bigvee]$*-autonomous lattice with* $\mathbf{0}$, $\top$*. Then, there exists a quantale Q and an injective map* $\phi : L \longrightarrow Q$ *such that*

1. $\phi(\mathbf{0}) = \mathbf{0}$, $\phi(\top) = \top$. *Moreover,* ϕ *preserves all* $\bigvee$*'s and* $\bigwedge$*'s existing in L, i.e., if* $\bigvee_{i \in I} a_i$ *and* $\bigwedge_{j \in J} b_j$ *exists in L, then*

$$\phi(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} \phi(a_i) \text{ and } \phi(\bigwedge_{j \in J} b_j) = \bigwedge_{j \in J} \phi(b_j).$$

2. *For all* $a, b \in L$,

   (i) $\phi(a \otimes b) = \phi(a) \otimes \phi(b)$
   (ii) $\phi(a \rightarrow_r b) = \phi(a) \rightarrow_r \phi(b)$ *and* $\phi(a \rightarrow_l b) = \phi(a) \rightarrow_l \phi(b)$.

3. *If* $a \otimes b = b \otimes a$ *for every* $a, b \in L$*, then Q is commutative.*

4. If L is unital, i.e., there is $\mathbf{1} \in L$ *such that* $a \otimes \mathbf{1} = a = \mathbf{1} \otimes a$, $\forall a \in L$, *then Q is unital and* $\phi(\mathbf{1}) = \mathbf{1}_Q$.

5. If in addition, L has a cyclic dualizing element $\bot$, *then Q is a Girard quantale and* $\phi(\bot) = \bot$.

Proof: Set $Q = Cl(L)$ and define in Q the operation

(I) $\quad I * J \equiv_{def} \cap \{ K \in Q : K \supseteq I \cdot J \}$,
where $I \cdot J = \{ a \otimes b : a \in I, b \in J \}$.

Clearly $*$ is increasing in both variables. By Lemma 6.8, $\cdot * I$ and $I * \cdot$ have right adjoints $I \rightarrow_r J$ and $I \rightarrow_l J$, given by

$$\cap_{a \in I} (a \rightarrow_r J) \text{ and } \cap_{a \in I} (a \rightarrow_l J),$$

respectively. Thus, $\forall I, J \in Q$,

(II) $\quad I * J \subseteq K$ iff $I \subseteq J \rightarrow_l K$ iff $J \subseteq I \rightarrow_r K$.

Formulas (I) and (II) will be of constant use. To show that $*$ is associative, we need

Fact 1: Let $I, J, K \in Q$. Then

$$\mathcal{A} = \{ H \in Q : H \supseteq (I * J) \cdot K \} = \{ H \in Q : H \supseteq I \cdot (J * K) \} = \mathcal{B}.$$

Proof of Fact 1: We have, using Lemma 6.8, the following sequence of implications

$$H \supseteq (I * J) \cdot K \Rightarrow H \supseteq (I \cdot J) \cdot K = I \cdot (J \cdot K)$$
$$\Rightarrow J \cdot K \subseteq I \rightarrow_r H \Rightarrow J * K \subseteq I \rightarrow_r H.$$

The last term implies $H \supseteq I \cdot (J * K)$ and $\mathcal{A} \subseteq \mathcal{B}$. Similar reasoning yields $\mathcal{B} \subseteq \mathcal{A}$.

It follows directly from the Fact that $*$ is associative. We now must show that $*$ distributes over suprema.

Let $I \in Q$ and $\{I_\alpha\}_{\alpha \in A} \subseteq Q$; we shall prove that $I * \bigvee I_\alpha = \bigvee (I * I_\alpha)$. Observe that $\bigvee I_\alpha = \cap \{ H \in Q : H \supseteq \cup_\alpha I_\alpha \}$, as well as that $\bigvee (I * I_\alpha) \subseteq I * \bigvee I_\alpha$ is always true. Now, we have the following sequence of implications

$$K \supseteq \cup (I * I_\alpha) \Rightarrow K \supseteq I * I_\alpha, \forall \alpha \in A \Rightarrow I_\alpha \subseteq I \rightarrow_r K, \forall \alpha \in A$$
$$\Rightarrow \cup I_\alpha \subseteq I \rightarrow_r K \Rightarrow \bigvee I_\alpha \subseteq I \rightarrow_r K \Rightarrow I * \bigvee I_\alpha \subseteq K,$$

and so $I * \bigvee I_\alpha \subseteq \bigvee (I * I_\alpha)$. Right distributivity is similar and so Q is indeed a quantale.

Before defining the map ϕ, we state

Fact 2: For $a, b \in Q$

(i) $a^{\leftarrow} \rightarrow_r b^{\leftarrow} = (a \rightarrow_r b)^{\leftarrow}$ and $a^{\leftarrow} \rightarrow_l b^{\leftarrow} = (a \rightarrow_l b)^{\leftarrow}$.

(ii) If $S \subseteq L$ is such that $a = \bigvee S$ exists in L, then $\bigvee_{s \in S} s^{\leftarrow} = a^{\leftarrow}$.

(iii) If $S \subseteq L$ such that $a = \bigwedge S$ exists in L, then $a^{\leftarrow} = \bigwedge_{s \in S} s^{\leftarrow} = \cap_{s \in S} s^{\leftarrow}$.

(iv) $a^{\leftarrow} * b^{\leftarrow} = (a \otimes b)^{\leftarrow}$.

Proof of Fact 2: (i) Let $K \in Q$ be such that $a^{\leftarrow} * K \subseteq b^{\leftarrow}$; for $x \in K$, we have $a \otimes x \leq b$ and so, $x \leq a \rightarrow_r b$. Thus, $K \subseteq (a \rightarrow_r b)^{\leftarrow}$, proving that $a^{\leftarrow} \rightarrow_r b^{\leftarrow} \subseteq (a \rightarrow_r b)^{\leftarrow}$.

For the reverse inclusion, consider $x \leq a$ and $y \leq a \rightarrow_r b$; then $x \otimes y \leq a \otimes (a \rightarrow_r b) \leq b$ and so $a^{\leftarrow} \cdot (a \rightarrow_r b)^{\leftarrow} \subseteq b^{\leftarrow}$, i.e., $a^{\leftarrow} * (a \rightarrow_r b)^{\leftarrow} \subseteq b^{\leftarrow}$. This last relation implies $(a \rightarrow_r b)^{\leftarrow} \subseteq a^{\leftarrow} \rightarrow_r b^{\leftarrow}$, as needed. For the operation $\rightarrow_l$, the argument is similar.

(ii) If $\bigvee S = a$, then $s^{\leftarrow} \subseteq a^{\leftarrow}$ for every $s \in S$, and so it is clear that $\bigvee_{s \in S} s^{\leftarrow} \subseteq a^{\leftarrow}$.

If $H \supseteq \cup_{s \in S} s^{\leftarrow}$, since H is complete, it follows that $a \in H$.

(iii) Is similar to (ii).

(iv) If $x \leq a, y \leq b$ then $x \otimes y \leq a \otimes b$ implies $a^{\leftarrow} \cdot b^{\leftarrow} \subseteq (a \otimes b)^{\leftarrow}$. Thus, $a^{\leftarrow} * b^{\leftarrow} \subseteq (a \otimes b)^{\leftarrow}$.

Now, if $H \supseteq a^{\leftarrow} \cdot b^{\leftarrow}$ then $a \otimes b \in H$ and we get $(a \otimes b)^{\leftarrow} \subseteq H$. By the definition of $*$ (formula (I)), this yields $(a \otimes b)^{\leftarrow} \subseteq a^{\leftarrow} * b^{\leftarrow}$, as desired. This ends the proof of Fact 2.

We now define

$$\phi: L \longrightarrow Q \text{ by } \phi(a) = a^{\leftarrow}.$$

Clearly ϕ is injective and, by Fact 2.(ii) and (iii), preserves all existing $\bigvee$'s and $\bigwedge$'s in L. Furthermore, $\phi(\mathbf{0}) = \{ \mathbf{0} \}$ (the smallest complete ideal in L) and $\phi(\top) = L = \top_Q$. The preservation of the other operations is guaranteed by Fact 2.(i) and (iv). This shows that ϕ has the properties in items 1 and 2 of the statement.

It is quite clear from the definition of $*$ (see (I), above) that Q will be commutative if $*$ is commutative in L. Moreover, if L is unital, straightforward computation will show that $\phi(\mathbf{1}) = \mathbf{1}^{\leftarrow}$ is the unit of Q.

(d) Let $\perp \in L$ be a cyclic dualizing element in L. Clearly $\perp^{\leftarrow} \in Q$ is cyclic and therefore $Q_j = \{ I \in Q : j(I) = I \}$ is a Girard quantale, where

$j : Q \longrightarrow Q$ given by $j(I) = (I \to \bot^{\leftarrow}) \to \bot^{\leftarrow}$ (Proposition 1.10). Moreover, $a^{\leftarrow} \in Q_j$, for every $a \in L$. Thus, the map $j \circ \phi : L \longrightarrow Q_j$ is an embedding of L in a Girard quantale, with all the required properties. □

We can now prove

Theorem 6.10 (Completeness) If $|\Gamma|_Q \leq |A|_Q$ *for all interpretations in quantales Q, then* $\Gamma \vdash A$ *is provable in (LLI).*

Proof: Our method will be to show that the Lindenbaum algebra L of (*LLI*) can be embedded in a commutative unital quantale Q, such that $|\Gamma|_Q \leq |A|_Q$ iff $\Gamma \vdash A$ is provable.

Define in $FOR(\mathbb{L}_i)$ the relation:

$$A \sim B \qquad \text{iff} \qquad A \vdash B \text{ and } B \vdash A \text{ are provable.}$$

It is easily seen that $\sim$ is an equivalence relation. Its equivalence classes shall be denoted by $A/_\sim$.

Let L be the set of equivalence classes, i.e., $L = \{ A/_\sim : A \in FOR(\mathbb{L}_i) \}$. In L, define the relation:

$$A/_\sim \leq B/_\sim \quad \text{iff} \quad A \vdash B \text{ is provable in } (LLI).$$

By [*CUT*], $\leq$ is a partial order in L. Moreover, for formulas A, B in *LLI*, we have

Fact 1: If A, B are formulas in (*LLI*), then

1. $(A \,\&\, B)/_\sim = (A/_\sim) \wedge (B/_\sim)$ and $(A \oplus B)/_\sim = (A/_\sim) \vee (B/_\sim)$.
2. $\bigwedge_{y \in V} (A[y/x]/_\sim) = (\bigwedge x.A)/_\sim$ and $\bigvee_{y \in V} (A[y/x]/_\sim) = (\bigvee x.A)/_\sim$.
3. $\mathbf{0} = \mathbf{0}/_\sim$ and $\top = \top/_\sim$.

Proof of Fact 1: All these equalities can be read off the corresponding rules of the calculus. We do the first one in each of items 1. and 2. in some detail, just naming the rules that should be used for the other cases.

1. From [*AX*1], [&1] and [&] we get $A \,\&\, B \vdash A, B$ and so $(A \,\&\, B)/_\sim \leq A/_\sim$, $(A \,\&\, B)/_\sim \leq B/_\sim$. On the other hand, if $C \vdash A$ and $C \vdash B$, then [&*R*] yields $C \vdash A \,\&\, B$. This means that

$$C/_\sim \leq A/_\sim,\ C/_\sim \leq B/_\sim \Rightarrow C/_\sim \leq (A \,\&\, B)/_\sim,$$

proving that $(A \,\&\, B)/_\sim = (A/_\sim) \wedge (B/_\sim)$ in L.

Similarly, [*AX*1], [$\oplus$ 1], [$\oplus$ 2] and [$\oplus L$] will yield $(A \oplus B)/_\sim = (A/_\sim) \vee (B/_\sim)$.

2. By [$AX1$] and [$\bigwedge L$], $(\bigwedge x.\, A)/_{\sim} \leq A[y/x]/_{\sim}$, for every $y \in V$.

If $B/_{\sim} \leq A[y/x]/_{\sim}$, for every y, let z be a variable not occurring in B. Then, $B/_{\sim} \leq A\,[z/x]/_{\sim}$ and so [$\bigwedge R$] yields $B/_{\sim} \leq (\bigwedge x\ A)/_{\sim}$. This proves that $\bigwedge_{y \in V} (A[y/x]/_{\sim}) = (\bigwedge x.A)/_{\sim}$.

For the existential quantifier, the reasoning is the same, using [$AX1$], [$\bigvee R$] and [$\bigvee L$].

3. This follows directly from [$AX2$] and [$AX4$], ending the proof of Fact 1.

Define in L a binary operation * by:

$$(A/_{\sim}) * (B/_{\sim}) \equiv_{def} (A \otimes B)/_{\sim}.$$

Clearly, * is well defined, is associative, commutative, increasing in both variables and $\mathbf{1}/_{\sim}$ is the unit. We have

Fact 2: With the operation * defined above, L is a [$\otimes$, $\bigvee$] autonomous lattice.

Proof of Fact 2: It remains to be shown that * has right adjoints and distributes over the suprema existing in L.

If A, B, C are formulas in (LLI), then

(+) $A \otimes B \vdash C$ is provable iff $A, B \vdash C$ is provable
iff $A \vdash B \multimap C$ is provable.

To see this, note that [$\otimes R$] yields $\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \otimes B}$

Thus if $A \otimes B \vdash C$ is provable, [CUT] implies that A, $B \vdash C$ is provable.

The converse comes directly from [$\otimes L$] as $\dfrac{A, B \vdash C}{A \otimes B \vdash C}$.

From [$\multimap R$] we get $\dfrac{A, B \vdash C}{A \vdash B \multimap C}$.

To show that the last clause in (+) implies the second, first note that $B, B \multimap C \vdash C$ is provable, because we can use [$\multimap R$] a $\dfrac{B \vdash B \qquad C \vdash C}{B, B \multimap C \vdash C}$.

Thus, the provability of $A \vdash B \multimap C$ and $B, B \multimap C \vdash C$ and [CUT] yield that of $A, B \vdash C$.

It is clear from (+) that $(A \multimap \cdot)/_{\sim}$ is the rigth adjoint to $A/_{\sim} * \cdot = \cdot * A/_{\sim}$ in L.

To show that $*$ distributes over the sup's in L, let $S \subseteq L$ be such that there is $A/_{\sim} \in L$ satisfying $A/_{\sim} = \bigvee_L S$. Fix $B/_{\sim} \in L$. Since $*$ is increasing, $(B/_{\sim}) * (X/_{\sim}) \leq (B/_{\sim}) * (A/_{\sim})$, for every $X/_{\sim} \in S$.

Let $F/_{\sim} \in L$ such that $(B/_{\sim}) * (X/_{\sim}) \leq F/_{\sim}$, for every $X/_{\sim} \in S$. By (+) we have that, for $X/_{\sim} \in S$,

$$B \otimes X \vdash F \text{ is provable} \quad \text{iff} \quad X \vdash (B \multimap F) \text{ is provable,}$$

$\forall X/_{\sim} \in S$; taking sup's, we get $A \vdash (B \multimap F)$ is provable. Thus, $B \otimes A \vdash F$ is provable, and so $(B/_{\sim}) * (A/_{\sim}) \leq F/_{\sim}$. But this means that $\bigvee_{X/\sim \in S}((B/_{\sim})*(X/_{\sim}))$ exists in L, and it's equal to $(B/_{\sim}) * \bigvee_L S$, ending the proof of Fact 2.

Therefore, L is a commutative $[\otimes, \bigvee]$-autonomous lattice and by theorem 6.9 it has a completion $\phi : L \longrightarrow Q$, where Q is an unital commutative quantale. Moreover, ϕ preserves $\otimes$, $\multimap$, $\mathbf{0}$, $\top$, $\mathbf{1}$ and all existing $\bigvee$'s and $\bigwedge$'s in L. Consequently, if X, Y are formulas in (*LLI*) we have

$$\phi(X/_{\sim}) \leq \phi(Y/_{\sim}) \qquad \text{iff} \qquad X/_{\sim} \leq Y/_{\sim}.$$

It follows from these preservation properties that the map $|\cdot|_Q : FOR(\mathbb{L}_i) \longrightarrow Q$ given by $|A|_Q = \phi(A/_{\sim})$ is an interpretation of quantales such that $|\Gamma| \leq |A|$ iff $\Gamma \vdash A$ is provable, as required to complete the proof of the Theorem. □

Before we give logical applications of the preceding results, we need to investigate when is it that a quantale becomes a frame. From Lemma 6.4 comes:

Corollary 6.11 Let Q be a quantale. If Q has a largest localic subquantale L, then

$$L = I(Q) = \{\, a \in Q : a^2 = a \,\}.$$

Proof: We must have $L \subseteq I(Q)$ because every $x \in L$ is idempotent (proposition 1.19).

For $a \in I(Q)$, set $L_a = \{\, \mathbf{0}, a \,\}$. By Lemma 6.4, L_a is closed under $\otimes$; it is clearly closed under arbitrary $\bigvee$'s, and so L_a is a subquantale of Q. Since a is idempotent, it follows that L_a is localic. Thus, $L_a \subseteq L$, i.e., $a \in L$, and therefore $I(Q) \subseteq L$. □

The next result is a correction of Theorem 3.4.1 in [Ros], which is false as it stands. In fact, the usual quantale of phases of Linear Logic is a counterexample to that result.

Proposition 6.12 Let Q be a quantale. Are equivalent:
(1) Q has a largest localic subquantale.
(2) $I(Q)$ is a localic subquantale of Q.
(3) For all every $a,b \in I(Q)$, $a \otimes b = b \otimes a \leq a \wedge b$ ($\wedge$ in Q or in $I(Q)$).

Proof: (1) $\Rightarrow$ (2) comes from Corollary 6.11, while (2) $\Rightarrow$ (3) is immediate.

(3) $\Rightarrow$ (1): Because $I(Q)$ is commutative, it is closed under $\otimes$. To show that $I(Q)$ is closed under sup's, let $\{ a_i \}_{i \in I} \subseteq I(Q)$. Then:

$$(*) \quad \bigvee_{i \in I} a_i = \bigvee_{i \in I} (a_i \otimes a_i) \leq \bigvee_{i,j \in I} (a_i \otimes a_j) = (\bigvee_{i \in I} a_i) \otimes (\bigvee_{j \in I} a_j);$$
$$(**) \quad (\bigvee_{i \in I} a_i) \otimes (\bigvee_{j \in I} a_j) = \bigvee_{i \in I} (\bigvee_{j \in I} (a_i \otimes a_j)) \leq \bigvee_{i \in I} a_i,$$

because $a_i \otimes a_j \leq a_i$ for every $j \in I$. From (*) and (**) it follows that $\bigvee_{i \in I} a_i$ is idempotent and so $I(Q)$ is a subquantale of Q.

To show that $I(Q)$ is localic, observe that it follows from 3. that $a \otimes b \leq a$, for every $b \in I(Q)$. Thus, $\forall\, a \in I(Q)$,

$$a \otimes \bigvee I(Q) = \bigvee_{b \in I(Q)} (a \otimes b) \leq a \text{ and } (\bigvee I(Q)) \otimes a \leq a,$$

which shows that all elements of $I(Q)$ are idempotent and two-sided *in* $I(Q)$. Now, Proposition 1.19 guarantees that $I(Q)$ is localic. If $L \subseteq Q$ is a localic subquantale then every $x \in L$ is idempotent (Proposition 1.19) and so $L \subseteq I(Q)$. □

Thus, in a commutative Girard quantale Q, $I(Q) \cap \mathbf{1}^{\leftarrow}$ is the *largest frame* contained in $\mathbf{1}^{\leftarrow}$. From the point of view of Logic, we have a privileged localic subquantale, that corresponds to intuitionistic Logic. However, the situation is quite different in the non commutative case: in general, a Girard quantale will not have a largest localic subquantale. If one considers Proposition 1.14, one realizes the importance of commutativity: there is a largest interpretation for ! in a Girard's quantale Q, corresponding to the largest among frames L satisfying

(i) $L \subseteq Z(Q)$ (ii) $L \subseteq \mathbf{1}^{\leftarrow} = \{ x \in Q : x \leq \mathbf{1} \}$

In fact, $Q_! = I(Q) \cap Z(Q) \cap \mathbf{1}^{\leftarrow}$, where $Z(Q)$ is the center of Q.

The quantale theoretic setting will be helpful in finding axioms that, when added to various systems of Linear Logic, produce classical intuitionism and classical logic. We have the simple (compare with proposition 1.19),

Lemma 6.13 Let L be a complete lattice and $* : L \times L \longrightarrow L$ *a binary operation on L. Are equivalent:*

(*i*) $* = \wedge$.
(*ii*) (*a*) $x * x = x$ *for every* $x \in L$.
(*b*) $x * \top \leq x$, $\top * x \leq x$ *for every* $x \in L$.
(*c*) $*$ *is increasing in both variables.*

Proof: (i) $\Rightarrow$ (ii) is clear.
(ii) $\Rightarrow$ (i): We have $a * b \leq a * \top \leq a$; $a * b \leq \top * b \leq b$ and therefore, $a * b \leq a \wedge b$.
On the other hand, since $*$ is increasing $a \wedge b = (a \wedge b) * (a \wedge b) \leq a * b$.
□

An analogous (and dual) result holds for the operation $\vee$. From the preceding results we get

Corollary 6.14 Let Q be a quantale. Are equivalent:
(*i*) *Q is a frame, i.e.,* $\otimes = \wedge$.
(*ii*) *every* $x \in Q$ *is idempotent and two-sided.* □

This Corollary and the completeness Theorem 6.10 yield

Proposition 6.15 (a) The system (LI) obtained from (LLI) by adding the axioms

[*ID*1] $\vdash A \otimes A \multimap A$ [*ID*2] $\vdash A \multimap A \otimes A$
[2*S*] $\vdash A \otimes \top \multimap A$

determines intuitionistic logic, and therefore if we add to (LI) the axiom

$[\neg\neg]\ (A \multimap \mathbf{0}) \multimap \mathbf{0} \vdash A$

we get classical logic, where $\mathbf{1}$ *is equivalent to* $\top$ *and* $\mathbf{1} \multimap \mathbf{0}$ *is equivalent to* $\mathbf{0}$.

Thus, adding the axioms [*ID*1],[*ID*2],[2*S*] *and* [*R*=], [*S*=], [*T*=], [*SUBST*], [$\leq \mathbf{1}$] *(see section 2) to (LLI) we obtain a intuitionistic theory of equality,* *which becomes first order classical logic with equality upon the addition of* $[\neg\ \neg]$.

(b) The system (MALL1) obtained from (LLI) by adding the constant formula $\bot$ *to the language and the axiom*

$$[\neg\ \neg'] \ (A \multimap \bot) \multimap \bot \vdash A,$$

is equivalent with (MALL) in the following sense:

$\vdash A$ *is provable in (MALL) iff* $\vdash A$ *is provable in (MALL1),*

where $A^{\bot}$ *and* $A \sqcup B$ *are interpreted as* $A \multimap \bot$ *and* $((A \circ \bot) \otimes (B \multimap \bot)) \multimap \bot$, *respectively. Thus, adding* $\bot$ *to the language we have that*

(LLI) + the axioms $[\neg\ \neg'],[R=],[S=],[T=],[SUBST],[I=],[\leq \mathbf{1}]$
is equivalent to (LLE_0),

where (LLE_0) *is described in Definition 2.6.*

(c) Adding [ID1], [ID2] and [2S] to (MALL1) produces classical logic, where $\mathbf{1}$ *is equivalent to* $\top$, *and* $\mathbf{0}$ *is equivalent to* $\bot$. □

Remark 6.16 The possibility of obtaining (MALL) from (LLI) in part (b) appears in [Dos] proved by a different method: one interprets $A \multimap B$, $A \,\&\, B$ *and* $\bigwedge x.A$ *as* $(A \otimes (B \multimap \bot)) \multimap \bot$, $((A \multimap \bot) \oplus (B \multimap \bot)) \multimap \bot$ *and* $(\bigvee x.(A \multimap \bot)) \multimap \bot$, *respectively, together with the interpretation for* $\sqcup$ *made in (b).*

Since $(a \otimes b) \rightarrow c = a \rightarrow (b \rightarrow c)$ *and* $b = \mathbf{1} \rightarrow b$, *in every unital commutative quantale, it can be shown that his interpretation of linear implication is equivalent to adding* $[\neg\ \neg']$ *to (LLI) to obtain (MALL) (translations for & and* $\bigwedge$ *are derived and a consequence of the ones given).*

As a matter of fact, we can extend this result to non-commutative intuitionistic linear logic, simply by observing that each rule of (*LLI*) determines an algebraic property of the Lindenbaum algebra of the calculus. For a general unital quantale, we need two implications, $\multimap_l$ and $\multimap_r$, to discard the exchange rule [*EXCH*], as well as to modify the rules in order to describe the properties of each operation. We can define a linear calculus which describes unital quantales, essentially the same as in [Abr].

Definition 6.17 The calculus (NCLLI) for noncommutative first-order linear intuitionistic logic consist in the following rules and axioms:

[$AX1$] $A \vdash A$ [$AX2$] $\Gamma \vdash \top$

[$AX3$] $\vdash \mathbf{1}$ [$AX4$]. $\Gamma, \mathbf{0}, \Delta \vdash A$

[CUT] $$\frac{\Gamma \vdash A \qquad \Sigma, A, \Delta \vdash B}{\Sigma, \Gamma, \Delta \vdash B}$$ [1L] $$\frac{\Gamma, \Delta \vdash A}{\Gamma, \mathbf{1}, \Delta \vdash A}$$

[$\otimes R$] $$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$$ [$\otimes L$] $$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \otimes B, \Delta \vdash C}$$

[$\& R$] $$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \& B}$$ [$\oplus L$] $$\frac{\Gamma, A, \Delta \vdash C \qquad \Gamma, B, \Delta \vdash C}{\Gamma, A \oplus B, \Delta \vdash C}$$

[$\& L1$] $$\frac{\Gamma, A, \Delta \vdash C}{\Gamma, A \& B, \Delta \vdash C}$$ [$\& L2$] $$\frac{\Gamma, B, \Delta \vdash C}{\Gamma, A \& B, \Delta \vdash C}$$

[$\oplus R1$] $$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}$$ [$\oplus R2$] $$\frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}$$

[$\multimap_l L$] $$\frac{\Gamma \vdash A \qquad \Sigma, B, \Delta \vdash C}{\Sigma, A \multimap_r B, \Gamma, \Delta \vdash C}$$ [$\multimap_r R$] $$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap_r B}$$

[$\multimap_r L$] $$\frac{\Gamma \vdash A \qquad \Sigma, B, \Delta \vdash C}{\Sigma, \Gamma, A \multimap_r B, \Delta \vdash C}$$ [$\multimap_r R$] $$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \multimap_r B}$$

[$\bigvee R$] $$\frac{\Gamma \vdash A[y/x]}{\Gamma \vdash \bigvee x.A}$$

[$\bigvee L$] *If x is not free in Γ, Δ, B then* $$\frac{\Gamma, A, \Delta \vdash B}{\Gamma, \bigvee x.A, \Delta \vdash B}$$

[$\bigwedge L$] $$\frac{\Gamma, A[y/x], \Delta \vdash B}{\Gamma, \bigwedge x.A, \Delta \vdash B}$$

[$\bigwedge R$] *If x is not free in Γ then* $$\frac{\Gamma \vdash A}{\Gamma \vdash \bigwedge x.A}$$

With the same method employed in Theorems 6.5 and 6.10, Theorem 6.9 yields that unital quantales are a complete and sound class of models for (*NCLLI*):

Theorem 6.18 (Completeness and Soundness for (NCLLI))
A sequent $\Gamma \vdash A$ *is provable in [NCLLI] iff* $\Gamma \vdash A$ *is valid in every interpretation of quantales.*

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