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GENTZEN'S SECOND CONSISTENCY PROOF AND STRONG CUT-ELIMINATION

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1. *Introduction*

It has been observed that an important difference between Natural Deduction and Sequent Calculus is that while in Natural Deduction we have a natural concept of local reduction, in Sequent Calculus one can only talk about a "cut-elimination procedure". The fact that one can define reduction steps for Natural Deduction Systems has made it natural to think of different orders for the application of reduction steps, thus setting up the stage for strong normalizability. By contrast, the traditional cut-elimination procedure has not been thought of as a set of independent reduction steps, which could be applied in a certain order, but rather as a global method for bringing a given sequent calculus proof into its cut free form. This in turn has made it conceptually difficult (not to say impossible!) to think of a strong cut-elimination result. A partial solution to this problem was given in 1974 by Zucker, who showed how to analyse Gentzen's cut-elimination procedure into atomic steps that could be used in the definition of independent reductions steps (see [Zucker74]. Zucker proved strong cut-elimination for a fragment of **LJ** using strong normalizability for the corresponding fragment in **NJ** and a translation between the systems (for a similar result cf. also [Pottinger77]). A proof of strong cut-elimination built up directly for Sequent Calculus (i.e., without the use of strong normalizability for a correspondent Natural Deduction system) was obtained by Dragalin, who again used a set of reductions that resulted from a decomposition of Gentzen's *Hauptsatz* into atomic independent reduction steps (see [Dragalin88]). An immediate consequence of an analysis of Gentzen's cut-elimination procedure into atomic steps[1] is that among reductions resulting from this analysis we find permutative reductions corresponding to procedures used in order to attain a decrease in the rank of a given proof. It is trivial to generate infinite reduction sequences from applications of permutative reductions. In order to avoid these infinite reduction sequences, Dragalin

[1] For a recent and detailed account of strong cut-elimination techniques see [Bittar96].

imposed a restriction on the definition of reduction steps of this kind. This restriction corresponds to the introduction of an order in the application of local permutative reductions (a similar problem concerning permutative reductions occurs in Zucker's proof).

The aim of the present paper[2] is to provide a new proof, directly for sequent calculus, of strong cut-elimination for classical and intuitionistic propositional logic. This result differs from those mentioned above in the following points:

1. Instead of using a set of reductions that resulted from Gentzen's *Hauptsatz*, we shall work with reductions defined by Gentzen in his second published consistency proof. Gentzen's reductions were designed to be applied to a certain part (the so-called end-part) of any derivation of the empty sequent (in a sequent calculus for first order arithmetic). Furthermore, any application of what Gentzen called an operational reduction was shown to entail a decrease in the ordinal associated with a given derivation of the empty sequent. The consistency proof was then completed with the proof of the accessibility of ordinals in the ε_0-fragment In this paper we shall show that, in the case of classical and intuitionistic propositional logic, these reductions do not depend either on the form of the end-sequent of derivations or on where they should be applied.
2. We don't have any kind of permutative reduction in this new set of reductions. This fact allows us to avoid difficulties that proofs based on Gentzen's *Hauptsatz* have to face.
3. The proof of strong cut-elimination uses induction over a natural number assignment extracted from Gentzen's original ordinal assignment. We show that if (1) π is a derivation in classical or intuitionistic propositional logic, (2) n is assigned to π and (3) π reduces to π', then the natural number n' associated with π' is such that $n' < n$. This natural number assignment provides a natural global measure which is shown to decrease through applications of reductions steps.
4. We shall be working with a modified concept of cut-free form. A derivation is said to be in cut-free form if all its cuts are atomic (for a similar concept of normal form see [Schwichtemberg77]).
5. In contrast with the proofs mentioned above, our proof works with the cut-rule instead of the mix-rule.

[2]A preliminary version of some of the results proved in this paper appeared in [Pereira83].

The paper is organized in the following way: in section 2 we introduce some basic definitions; in section 3 we define an assignment of natural numbers to derivations in the propositional fragment **LKP** of **LK**; in section 4 we define the reductions; in section 5 we prove strong cut-elimination for **LKP**; in section 6 we show how this result can be extended to intuitionistic propositional logic; and finally, in the concluding section 7, we make some brief comments on the possibility of extending these results to the first order case.

2. *Basic Definitions*

The system **LKP** (see Figure 1) is the propositional fragment of Gentzen's sequent calculus **LK** ([Gentzen35]). The notions of upper (lower) sequent, premiss, main formula, side formula, operational and structural inference are the usual ones. A proof Π in **LKP** is a formula tree that satisfies the following conditions:

- (i) The top nodes of Π are initial sequents.
- (ii) Every other node η, except the last one, is the upper sequent of an application of an inference rule, whose lower sequent occurs immediately below η in Π.
- (iii) The upper sequent(s) of an application of an inference α in Π is (are) separated from the lower sequent by means of an horizontal line called *line of inference*.

In any application of an inference rule, the side formulas occur both in the upper sequent and in the lower sequent. We say that these occurrences are connected. We shall also say that the premises of an application of contraction are connected with the main formula, and that the premiss A (B) in an application of permutation is connected with the main formula A (B). A *connected sequence* for a formula occurrence A in a proof Π is a sequence of formula occurrences $B_1,...,B_n$ in Π such that:

- (i) $A = B_n$;
- (ii) Each B_i occurs immediately above and is connected with B_{i+1}, $(1 \leq i < n)$; and
- (iii) B_1 occurs either in an initial sequent, or as the main formula of an operational inference, or as the main formula of an application of thinning.

The *connection* associated with a formula occurrence A in a proof Π is the set of all connected sequences for A in Π.

An *essential cut* is a cut that satisfies one of the following conditions:

- (1) There is at least one connected sequence both in the connection associated with the right cut formula and in the connection associated with the left cut formula whose first element is the main formula of an operational inference.
- (2) Either every connected sequence associated with the right cut formula A begins with the main formula of an application of thinning, or every connected sequence associated with the left cut formula A begins with the main formula of an application of thinning.

Let $d(A)$ denote the degree of a formula A. The degree of a proof Π, is defined as:

$$d(\Pi) = \max \{ d(A) \colon A \text{ is a cut formula in } \Pi \}.$$

The degree of an aplication of the cut rule is equal to the degree of its cut formula. A proof Π is called *normal*, if $d(\Pi) = 0$.

The *level* of a sequent S occurring in a proof Π is defined to be the greatest degree of an application of the cut rule whose lower sequent occurs below S. A line of inference in a proof Π is called a *level line* iff the level of the upper sequents is greater than the level of the lower sequent.

3. *The mapping G_3 from* **LKP** *derivations into the natural numbers*

The aim of this section is to define a mapping G_3 from derivations in **LKP** into natural numbers in such a way that we associate to each sequent and line of inference occurring in a derivation Π a certain natural number. The number $G_3(\Pi)$ associated with Π is the number associated with the last sequent of Π. From now on, the notation "$3_k(n)$" will be used to denote k-times iterated exponentiation with base 3, which can be defined as follows:

- (a) $3_0(n) = n$
- (b) $3_{k+1}(n) = 3^{3_k(n)}$

The correlation G_3 is defined as follows:

(1) $G_3(\Gamma \Rightarrow \Delta) = 1$ if $(\Gamma \Rightarrow \Delta)$ is an initial sequent.
(2) Let n (n_1, n_2) be the number(s) associated with the upper sequent(s) of an application α of an inference rule. The number $G_3(l)$ associated with the line l of the inference α is defined as:

- (a) $G_3(l) = n$, if α is a structural rule different from the cut rule.
- (b) $G_3(l) = n + 1$, if α is an operational rule with one upper sequent.
- (c) $G_3(l) = n_1 + n_2 + 1$, if α is an operational rule with two upper sequents.
- (d) $G_3(l) = n_1 + n_2$, if α is an application of the cut rule.

(3) Let n be the number associated with the line l of an application α of an inference rule in a derivation Π. The number associated with the lower sequent of l is defined as:

- (a) n, if l is not a level line.
- (b) $3_m(n)$, if l is a level line and the difference between the level of the upper sequents and the level of the lower sequent is equal to m.

4. *Reductions for* **LKP**

In this section, we introduce certain operations in order to transform a given derivation Π into a cut-free derivation Π' of the same end sequent. These operations are called *reductions* and are of two types: *structural reductions* and *operational reductions*. The reductions I shall be using here are the same as those used by Gentzen in the second published proof of the consistency of elementary arithmetic ([Gentzen38]).

4.1. *Structural Reductions*

Let Π be the following derivation:

$$\dfrac{\begin{matrix}\Pi_1 & & \Pi_2\\ \Gamma_1 \Rightarrow \Delta, A & & \Theta \Rightarrow \Lambda\end{matrix}}{\Gamma, \Theta^* \Rightarrow \Delta, \Lambda}$$
$$\Pi_3$$

where all connected sequences for the left cut-formula of Π have the form described in the definition of an essential cut of type (2).

We then say that Π reduces to Π' that assumes the following form:

$$\begin{array}{c} \Pi_1' \\ \Gamma \Rightarrow \Delta \\ \vdots \\ \Gamma, \Theta^* \Rightarrow \Delta, \Lambda \end{array}$$

where:

(1) Π' is obtained from Π through the elimination of the applications of thinning that introduce formulas in the connection associated with the left cut formula;

(2) Θ^* and Λ are introduced by applications of thinning.

The case where the right cut-formula fulfils the same condition is treated in a similar way.

4.2. *Operational Reductions*

The aim of these reductions is to eliminate essential cuts of type (1). There are several cases to be examined depending on the operational rule we use. I shall below consider just a representative case, the other cases being treated in a similar way.

(a) The cut-formula is $(A \wedge B)$ and Π has the following form:

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\Pi_1\\ \Gamma_1 \Rightarrow \Delta_1, A\end{array} \quad \begin{array}{c}\Pi_2\\ \Gamma_1 \Rightarrow \Delta_1, B\end{array}}{\Gamma_1 \Rightarrow \Delta_1, A \wedge B} \\ \Pi_3 \\ \Gamma \Rightarrow \Delta, A \wedge B\end{array} \qquad \begin{array}{c}\dfrac{\begin{array}{c}\Sigma_1\\ A, \Theta_1 \Rightarrow \Psi_1\end{array}}{A \wedge B, \Theta_1 \Rightarrow \Psi_1} \\ \Sigma_2 \\ A \wedge B, \Theta \Rightarrow \Psi\end{array}}{\begin{array}{c}\Gamma, \Theta \Rightarrow \Delta, \Psi \\ \Sigma_3\end{array}}}{\begin{array}{c}\Gamma_3 \Rightarrow \Theta_3 \\ \Sigma_4\end{array}}$$

Where:

(1) the uppermost left (right) occurrence of $(A \wedge B)$ is the first term of a connected sequence associated with the left (right) cut-formula of the indicated cut;

(2) the level of the indicated cut is ρ;

(3) the line immediately above $\Gamma_3 \Rightarrow \Theta_3$ is a level line;

(4) the level τ of the sequents that stand above this line is such that $d(A) < \tau \leq \rho$;
(5) the level of $\Gamma_3 \Rightarrow \Theta_3$ is $\sigma \leq d(A)$.

Π reduces to Π' that assumes the following form:

$$
\cfrac{\cfrac{\cfrac{\begin{array}{c}\cfrac{\begin{array}{cc}\Pi_1 & \Pi_2\\ \Gamma_1\Rightarrow\Delta_1,A & \Gamma_1\Rightarrow\Delta_1,B\end{array}}{\Gamma_1\Rightarrow\Delta_1,A\wedge B}\\ \Pi_3\\ \Gamma\Rightarrow\Delta,A\wedge B\end{array} \qquad \begin{array}{c}\Sigma_1\\ \cfrac{A,\Theta_1\Rightarrow\Psi_1}{A\wedge B,A,\Theta_1\Rightarrow\Psi_1}\\ \Sigma_2'\\ A\wedge B,A,\Theta\Rightarrow\Psi\end{array}}{\begin{array}{c}\Gamma,A,\Theta\Rightarrow\Delta,\Psi\\ \Sigma_3'\end{array}}}{A,\Gamma_3\Rightarrow\Theta_3}}{} \quad \cdots
$$

$$
\cdots \quad \cfrac{\cfrac{\cfrac{\begin{array}{c}\Pi_1\\ \cfrac{\Gamma_1\Rightarrow\Delta_1,A}{\Gamma_1\Rightarrow\Delta_1,A,A\wedge B}\\ \Pi_3'\\ \Gamma\Rightarrow\Delta,A,A\wedge B\end{array} \qquad \begin{array}{c}\Sigma_1\\ \cfrac{A,\Theta_1\Rightarrow\Psi_1}{A\wedge B,\Theta_1\Rightarrow\Psi_1}\\ \Sigma_2\\ A\wedge B,\Theta\Rightarrow\Psi\end{array}}{\begin{array}{c}\Gamma,\Theta\Rightarrow\Delta,A,\Psi\\ \Sigma_3''\end{array}}}{\Gamma_3\Rightarrow\Theta_3,A}}{}
$$

$$
\begin{array}{c}\Gamma_3,\Gamma_3\Rightarrow\Theta_3,\Theta_3\\ \vdots\\ \Gamma_3\Rightarrow\Theta_3\\ \Sigma_4\end{array}
$$

5. *Strong Cut-Elimination theorem*

A *reduction sequence* for a derivation Π is a sequence of proofs Π_1, ..., Π_n, Π_{n+1}, ... such that:

(i) Π_1 is Π;
(ii) Π_n reduces to Π_{n+1}.

A derivation Π satisfies strong cut-elimination iff it satisfies the following conditions:
(i) (Finitess) There is no infinite reduction sequence beginning with Π.
(ii) (Uniqueness) Every derivation Π has a unique cut-free form.

We shall prove in what follows the "Finitess" part of the Strong Cut-Elimination Theorem.

5.1 *Strong Cut-Elimination*

Theorem 5.1 (Strong Cut-Elimination - Finitess): Every reduction sequence for a derivation Π *in* **LKP** *is finite.*

Proof: induction over the value G_3 (Σ) associated with a proof Π. We shall show that if Π reduces to Π' then, $G_3(\Pi) < G_3(\Pi')$.

(a) Π reduces to Π' by means of an operational reduction. We shall only examine the case of conjunction presented above, the other cases being treated in a similar way. We can easily see that the numbers n_1 and n_2 associated with the lines of inference immediately above $\Gamma_3 \rightarrow \Theta_3, A$ and $A, \Gamma_3 \rightarrow \Theta_3$ in Π' are smaller than the number n_3 associated with the line of inference immediately above $\Gamma_3 \rightarrow \Theta_3$ in Π. The level χ of the upper sequents of the new cut is such that

$$\sigma \leq \chi < \tau$$

1. If $\sigma = \chi$ and $\tau - \chi = k$ then, the number $3_k(n_1) + 3_k(n_2)$ associated with $\Gamma_3 \rightarrow \Theta_3$ in Π' is smaller than the number $3_k(n_3)$ associated with $\Gamma_3 \rightarrow \Theta_3$ in Π. This difference is preserved through Π' till its end sequent and hence,

$$G_3(\Pi') < G_3(\Pi).$$

2. If $\chi > \sigma$, then $\chi = \sigma + k_1$ for some $k_1 \geq 1$. We also have that $\tau = \chi + k_2$ for some $k_2 \geq 1$. The numbers associated with the upper sequents of the new cut are $3_{k_2}(n_1)$ and $3_{k_2}(n_2)$. The number associated with the lower sequent is thus, $3_{k_1}(3_{k_2}(n_1) + 3_{k_2}(n_2))$. But this number is smaller than the number $3_{k_1+k_2}(n_3)$ associated with the sequent $\Gamma_3 \rightarrow \Theta_3$ in Π. This difference is preserved throughout Π' and hence, $G_3(\Pi') < G_3(\Pi)$.

(b) Π reduces to Π' by means of a structural reduction. Let Π be,

$$\dfrac{\begin{matrix}\Pi_1 \\ \Gamma_1 \Rightarrow \Delta, A\end{matrix} \quad \begin{matrix}\Pi_2 \\ \Theta \Rightarrow \Lambda\end{matrix}}{\Gamma, \Theta^* \Rightarrow \Delta, \Lambda}$$
$$\Pi_3$$

and let Π' be

$$\begin{matrix}\Pi_1' \\ \Gamma \Rightarrow \Delta \\ \vdots \\ \Gamma, \Theta^* \Rightarrow \Delta, \Lambda\end{matrix}$$

In order to show that $G_3(\Pi') < G_3(\Pi)$, we shall prove the general statement that if S' is the sequent in Π' corresponding to the sequent S in Π, and if the level of S' in Π' is lower than the level of S in Π by m, then $G_3(S') \leq 3_m(G_3(S))$. Let $l_1, ..., l_j$ be the level lines occurring in Π_1 such that between each l_i and $\Gamma \Rightarrow \Delta, A$ there is no other level line ($1 \leq i \leq j$). Let m be the change of level determined by l_i, and let $G_3(l_i) = n_i$.

Our general statement is obviously true for the sequents occurring above $l_1, ..., l_j$ in Π'_1, because we just have to remember that there is no change in the assignment for these sequents.

For the sequent S'_i occurring immediately below l_i, we have that:

$$G_3(S'_i) = 3_{m+m_i}(n_i) = 3_m(3_{m_i}(n_i)) = 3_m(G_3(S_i)).$$

If we now pass from S'_1 or S'_1, S'_2 to S' without any change of level, then $G_3(S') = G_3(S'_1) + 1$ or $G_3(S') = G_3(S'_1) + G_3(S'_2) + 1$ or $G_3(S') = G_3(S'_1) + G_3(S'_2)$.

By the induction hypothesis, we know that $G_3(S'_i) \leq 3_m(G_3(S_i))$ (for $i = 1, 2$). The result then follows directly from arithmetical properties of iterated exponentiation with base 3.

Now, it remains the case where we have a change of level when passing from S'_1, S'_2 to S'. This is the case of a cut,

$$\frac{S_1' \quad S_2'}{S'}$$

where the level of S' is lower than the level of the corresponding S in Π' by m'. Noticing that the difference of level determined by this cut is equal to $m' - m$, we verify that:

$$G_3(S') = 3_{m'-m}(G_3(S'_1) + G_3(S'_2)) \leq$$

$$\leq 3_{m'-m}(3_m(G_3(S_1)) + 3_m(G_3(S_2))) \leq$$

$$\leq 3_{m'-m+m}(G_3(S_1) + G_3(S_2)) \leq$$

$$\leq 3_{m'}(G_3(S_1) + G_3(S_2)) = 3_{m'}(G_3(S)).$$

This completes the proof of our general result.

Applying this result to our sequent $\Gamma \Rightarrow \Delta$ in Π', we check that $G_3(\Gamma \Rightarrow \Delta) \leq 3_m(G_3(\Gamma \Rightarrow \Delta, A))$, where m is the change of level determined by the cut we had chosen. Obviously, $G_3(\Gamma \Rightarrow \Delta) = G_3(\Gamma, \Theta^* \Rightarrow \Delta^*, \Lambda) < 3_m(G_3(\Gamma \Rightarrow \Delta, A) + G_3(A, \Theta \Rightarrow \Lambda))$.

6. *Strong Cut Elimination for the System FIL*

We can immediately see that this strong cut-elimination result cannot be extended to the propositional fragment **LJP** (see Figure 2) of the intuitionistic system **LJ**. The strong restriction imposed on the size of the consequents of LJ-sequents and the use of thinning in the definition of the operational reductions make it the case that **LJP** is not closed under reductions, i.e., the reduction Π' of a derivation Π in **LJP** may not be a derivation in **LJP**. This cardinality restriction can be liberalized in more than one way. In this section we shall show that our strong cut-elimination result can be extended to the intuitionistic propositional system **FIL** (**F**ull **I**ntuitionistic **L**ogic), where cardinality restrictions are replaced by explicit dependency restrictions.

The system **FIL** is a multiple consequent intuitionistic system[3], where an *indexing* device allows us to keep track of dependency relations between formulas. The point here is that these dependency relations determine the restriction on the formulation of the rule for introduction of implication on the right, which guarantees that *only* intuitionistic valid formulae are derivable.

[3] For a comprehensive account of **FIL** see [dePaivaPer96].

Let us first introduce some conventions and terminology we shall be using throughout this section.

A sequent in **FIL** is an expression of the form

$$A_1\ (n_1), \ldots, A_k\ (n_k) \Rightarrow B_1/S_1, \ldots, B_m/S_m$$

where

- A_i for $(1 \le i \le k)$ and B_j for $(1 \le j \le m)$ are propositional formulas;
- n_i for $(1 \le i \le k)$ are natural numbers;
- S_j for $(1 \le j \le m)$ are *finite sets* of natural numbers.

The main intuition behind the indexing device is that a succedent formula B/S depends on antecedent formulas with indexes in S. Capital Greek letters like Γ, Δ, Ψ, Θ, denote now sequences of indexed formulas. In order to describe the inference rules of **FIL** we need some notational conventions. Assume $\Delta = B_1/S_1, \ldots, B_m/S_m$,

- If S is any set of natural numbers, $\Delta\ (k|S)$ denotes the result of replacing each S_j in Δ such that $k \in S_j$ by $(S_j - \{k\}) \cup S$;
- $\Delta - \{n\}$ is the result of replacing each S_j in Δ such that $n \in S_j$ by $S_j - \{n\}$;
- $\Delta\ (k \Leftarrow S)$ denotes the result of replacing each S_j in Δ such that k is in S_j by $S_j \cup S$.

The system **FIL** is given by the axioms and rules of inference shown in Figure 3. We assume that in the case of the rules for conjunction on the right, disjunction on the left, implication on the left and Cut, the derivations of the upper sequents have no index in common. This is in fact no strong restriction since we can always *rename* the indexes.

Now, we can clearly see that the problem we faced with respect to **LJP** does not occur in the case of **FIL**. The fact that the occurrences of $A \wedge B$ introduced by thinning are associated with the empty set ($\varnothing$) and with a new index n' respectively, warrants the closure of **FIL** under reductions.

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\Pi_1 \\ \Gamma_1 \Rightarrow \Delta_1, A/S_1\end{matrix} \quad \begin{matrix}\Pi_2 \\ \Gamma_2 \Rightarrow \Delta_2, B/S_2\end{matrix}}{\begin{matrix}\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, A \wedge B/S_1 \cup S_2 \\ \Pi_3 \\ \Gamma \Rightarrow \Delta, A \wedge B/S\end{matrix}} \quad \dfrac{\begin{matrix}\Sigma_1 \\ A(n), \Theta_1 \Rightarrow \Psi_1\end{matrix}}{\begin{matrix}A \wedge B(n'), A(n), \Theta_1 \Rightarrow \Psi'_1 \\ \Sigma'_2 \\ A \wedge B(n^*), A(n), \Theta \Rightarrow \Psi\end{matrix}}}{\begin{matrix}\Gamma, A(n), \Theta \Rightarrow \Delta, \Psi(n^* | S) \\ \Sigma'_3\end{matrix}}}{A, \Gamma_3 \Rightarrow \Theta_3} \quad \ldots$$

$$\ldots \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\Pi_1 \\ \Gamma_1 \Rightarrow \Delta_1, A/S_1\end{matrix}}{\begin{matrix}\Gamma_1 \Rightarrow \Delta_1, A / S_1, A \wedge B / \varnothing \\ \Pi'_3 \\ \Gamma \Rightarrow \Delta, A / S_1 \cup S_2, A \wedge B / \varnothing\end{matrix}} \quad \dfrac{\begin{matrix}\Sigma_1 \\ A(n), \Theta_1 \Rightarrow \Psi_1\end{matrix}}{\begin{matrix}A \wedge B(n), \Theta_1 \Rightarrow \Psi_1 \\ \Sigma_2 \\ A \wedge B(k), \Theta \Rightarrow \Psi\end{matrix}}}{\begin{matrix}\Gamma, \Theta \Rightarrow \Delta, \Psi, A / S_1 \cup S_2 \\ \Sigma''_3\end{matrix}}}{\Gamma_3 \Rightarrow \Theta_3, A / S_1 \cup S_2}$$

$$\begin{matrix}\Gamma_3, \Gamma_3 \Rightarrow \Theta_3, \Theta_3 \\ \vdots \\ \Gamma_3 \Rightarrow \Theta_3 \\ \Sigma_4\end{matrix}$$

where,

1. $\Psi'_1 = \Psi_1(n \Leftarrow \{n'\})$.
2. $\Psi' = \Psi_1(n \Leftarrow \{n'\})$.
3. Σ'_2 is exactly like Σ_2 except for containing an extra occurence of A in the antecedent of some of its sequents. A similar remark holds for Σ'_3.
4. Π'_3 is exactly like Π_3 except for containing an extra occurence of A in the antecedent of some of its sequents. A similar remark holds for Σ''_3.
5. In order to simplify the notation we used the same indexes in the derivations of the upper sequents of the last cut. This is clearly no essential restriction, since the indexes could be easily renamed.

7. *Concluding Remarks*

Again, we can easily see that, for reasons similar to those mentioned above in the intuitionistic case, this strong cut-elimination result cannot be extended to the first-order case. The use of thinning that replaces the operational rule in the right-hand side of the original derivation violates the closure of the system under reductions. In order to circumvent this difficulty, it seems to be possible to use, in addition to the dependency indexes, bookkeeping devices that would indicate quantificational levels in the same spirit of indexes used in modal logic in order to indicate modal levels (see, for example, [Massini92], [Simpson93]). In a certain sense, the main idea is to explore the fact that sequent-calculus derivations are disguised natural deduction derivations, and that the cut-rule could produce good natural deduction derivations out of "bad" sequent calculus derivations.

THE SYSTEM LKP

$$\frac{}{A \vdash A}\,Axiom$$

$$\frac{\Gamma; A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}(Exchange_{\mathfrak{L}}) \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}(Exchange_{\mathfrak{R}})$$

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}(Weakening_{\mathfrak{L}}) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}(Weakening_{\mathfrak{R}})$$

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}(Contraction_{\mathfrak{L}}) \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}(Contraction_{\mathfrak{R}})$$

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,Cut$$

$$\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}(\neg_{\mathfrak{R}}) \qquad \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}(\neg_{\mathfrak{L}})$$

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{(A \vee B), \Gamma \vdash \Delta}(\vee_{\mathfrak{L}}) \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}(\vee_{\mathfrak{R}}) \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}(\vee_{\mathfrak{R}})$$

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}(\wedge_{\mathfrak{L}}) \qquad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}(\wedge_{\mathfrak{L}}) \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}(\wedge_{\mathfrak{R}})$$

$$\frac{\Gamma \vdash \Delta, A \qquad B, \Gamma' \vdash \Delta'}{A \to B, \Gamma, \Gamma' \vdash \Delta, \Delta'}(\to_{\mathfrak{L}}) \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}(\to_{\mathfrak{R}})$$

Figure 1: Sequent Calculus LKP

THE SYSTEM LJP

$$\frac{}{A \vdash A}\,Axiom$$

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}(Exchange_{\mathfrak{L}})$$

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}(Weakening_{\mathfrak{L}})$$

$$\frac{\Gamma \vdash}{\Gamma \vdash A}(Weakening_{\mathfrak{R}})$$

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}(Contraction_{\mathfrak{L}})$$

$$\frac{\Gamma \vdash A \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta'}\,Cut$$

$$\frac{A, \Gamma \vdash}{\Gamma \vdash \neg A}(\neg_{\mathfrak{R}}) \qquad \frac{\Gamma \vdash A}{\neg A, \Gamma \vdash}(\neg_{\mathfrak{L}})$$

$$\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{(A \vee B), \Gamma \vdash \Delta}(\vee_{\mathfrak{L}}) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}(\vee_{\mathfrak{R}}) \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}(\vee_{\mathfrak{R}})$$

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}(\wedge_{\mathfrak{L}}) \quad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}(\wedge_{\mathfrak{L}}) \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}(\wedge_{\mathfrak{R}})$$

$$\frac{\Gamma \vdash A \qquad B, \Gamma' \vdash \Delta'}{A \to B, \Gamma, \Gamma' \vdash \Delta'}(\to_{\mathfrak{L}}) \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \to B}(\to_{\mathfrak{R}})$$

Figure 2: Sequent Calculus LJP

THE SYSTEM FIL

$$\frac{}{A(n)\vdash A\{n\}}\,Axiom$$

$$\frac{\Gamma\vdash A/S,\Delta \qquad A(n),\Gamma'\vdash\Delta'}{\Gamma,\Gamma'\vdash\Delta,\Delta^{*}}\,Cut*1$$

$$\frac{}{\bot(n)\vdash A_1/\{n\},\ldots,A_k/\{n\}}\,(\bot_{\mathfrak{L}})$$

$$\frac{\Gamma,A(n),B(m),\Gamma'\vdash\Delta}{\Gamma,B(m),A(n),\Gamma'\vdash\Delta}\,(Exchange_{\mathfrak{L}}) \qquad \frac{\Gamma\vdash A/S,B/S',\Delta}{\Gamma\vdash B/S',A/S,\Delta}\,(Exchange_{\mathfrak{R}})$$

$$\frac{\Gamma\vdash\Delta}{\Gamma,A(n)\vdash\Delta^{*}}\,*2(Weakening_{\mathfrak{L}}) \qquad \frac{\Gamma\vdash\Delta}{\Gamma\vdash A/\{\},\Delta}\,(Weakening_{\mathfrak{R}})$$

$$\frac{\Gamma,A(n),A(m)\vdash\Delta}{\Gamma,A(k)\vdash\Delta^{*}}\,*3(Contraction_{\mathfrak{L}}) \qquad \frac{\Gamma\vdash A/S,A/S',\Delta}{\Gamma\vdash A/S\cup S',\Delta}\,(Contraction_{\mathfrak{R}})$$

$$\frac{\Gamma,A(n)\vdash\Delta \qquad \Gamma',B(m)\vdash\Delta'}{\Gamma,\Delta,(A\vee B)(k)\vdash\Delta^{*},\Delta'^{*}}\,*4(\vee_{\mathfrak{L}})$$

$$\frac{\Gamma\vdash A/S,\Delta}{\Gamma\vdash A\vee B/S,\Delta}\,(\vee_{\mathfrak{R}}) \qquad \frac{\Gamma\vdash B/S,\Delta}{\Gamma\vdash A\vee B/S,\Delta}\,(\vee_{\mathfrak{R}})$$

$$\frac{\Gamma,A(n),B(m)\vdash\Delta}{\Gamma,A\wedge B(k)\vdash\Delta'}\,*7(\wedge_{\mathfrak{L}}) \qquad \frac{\Gamma\vdash A/S,\Delta \qquad \Gamma'\vdash B/S',\Delta'}{\Gamma,\Gamma'\vdash A\wedge B/S\cup S',\Delta,\Delta'}\,(\wedge_{\mathfrak{R}})$$

$$\frac{\Gamma\vdash A/S,\Delta \qquad \Gamma',B(n)\vdash\Delta'}{\Gamma,\Gamma',A\to B(n)\vdash\Delta,\Delta'^{*}}\,*5(\to_{\mathfrak{L}}) \qquad \frac{\Gamma,A(n)\vdash B/S,\Delta}{\Gamma\vdash A\to B/S-\{n\},\Delta}\,*6(\to_{\mathfrak{R}})$$

(*1) : $\Delta^{*} = \Delta'\,(n|S)$.

(*2) : n is new and Δ^{*} is obtained from Δ through the introduction of n in some S in Δ.

(*3) : $k = min\,(n, m)$ and $\Delta^{*} = \Delta\,(max\,(n, m)|k)$.

(*4) : k is new, $\Delta^{*} = \Delta\,(n|k)$ and $\Delta'^{*} = \Delta'\,(m|k)$.

$(*5) : \Delta'^{*} = \Delta\ (n \Leftarrow S)$.
$(*6) : n \in S$ and $n \notin S'_j$, for each $S_j \in \Delta$.
$(*7) : k = min\ (n, m)$ and $\Delta^{*} = \Delta\ (max\ (n, m)|k)$.

Figure 3: Sequent Calculus formulation of *FIL*

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