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AXIOMS FOR COLLECTIONS OF INDISTINGUISHABLE OBJECTS

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Abstract

The search for axioms like those of set theories for dealing with collections of indistinguishable elementary particles was posed by Yu. I. Manin, in 1974, as one of the important problems of present day researches on the foundations of mathematics. In this paper we presented a *quasi-set theory* which stands for a mathematical framework for dealing with collections of indistinguishable objects, whose 'intended interpretation' is precisely the behaviour of elementary particles as described by non-relativistic quantum mechanics. A sketch of the proof that this theory and ZFC are equiconsistent is also presented and the relationship with the case of quantum particles is mentioned throughout the paper.

1. *Introduction*

Mathematical frameworks for dealing with collections of objects such as elementary particles have two independent but related origins. In 1983, M. L. Dalla Chiara and G. Toraldo di Francia proposed a *quaset theory* to provide mathematical tools for semantical analyses of the languages of microphysics [10]. According to this approach, standard set theories are not adequate to represent microphysical phenomena, since the ontology of physics apparently does not reduce to that of usual sets. One of the basic motivations underlying such a supposition is that collections of objects like elementary particles, do not obey the axioms of set theories like Zermelo-Fraenkel due to the indistinguishability of their elements. In addition, they have suggested that identity questions from quantum theories demand a kind of *intensional semantics*, for which quaset theory may provide the (meta)mathematical framework (see also [12]).

The basic idea of a *quaset* is that of a collection of objects which have a well-defined cardinal, but in such a way that there is no way to tell (with certainty) which are the elements that belong to the quaset. This is achieved by distinguishing between the primitive predicates $\in$ and $\notin$ (which is not the negation of the former), meaning 'certainly belongs to' and 'certainly does not belong to' respectively. The postulates imply that $z \in y$ entails $\neg z$

$\notin y$, but not the converse. So, it may be the case that it is false that z certainly does not belong to y, but this does not entail that z (certainly) belongs to y. The elements z to which it may be said that 'it is false that they certainly do not belong to y' might act as members *in potentia* of y.[1] Since the cardinal of the quaset is fixed, then there is a kind of 'epistemic' indeterminacy with respect to the elements of a quaset. Convenient postulates provide the grounds for the whole theory, and a semantical analysis of microphysics has also been sketched.[2]

Starting from a distinct but related motivation, N. C. A. da Costa discussed in his book [1] the possibility of presenting logical systems in which some form of the Principle of Identity could be violated.[3] Based on Schrödinger's ideas concerning the fact that the concept of identity, or sameness, lacks sense with respect to the elementary particles [30, pp. 17-18], da Costa defined a two-sorted first order logic in which identity statements $a = b$ make sense only with respect to the objects of one of the considered sort; to the others (which should be regarded as denoting elementary particles), the expression $x = y$ simply is not a formula. Hence, for these last objects, it is not possible to say either that they are identical or that they are distinct from one another.

Da Costa realized further that a complete semantics could be found for such 'Schrödinger Logics',[4] but he noted that such a semantics, grounded in the standard set theories, was not adequate to express the intuitive idea of collections of objects for which the concept of identity should lack sense. Then he proposed that a kind of *theory of quasi-sets* should be developed, a theory in which standard sets were particular cases, and then, it was suggested, in such a theory a more adequate semantics for his logics could be achieved.

In [17] (see [19]), a *quasi-set theory* (called S^*) in this sense was proposed. The main motivation was not only to obtain a mathematical framework to provide semantics for Schrödinger logics, but also to pursue Schrödinger's intuitions and to explore the mathematical counterpart of a theory which admits collections of objects for which identity and diversity

[1] The authors did not use this terminology.

[2] Other papers of related interest are [8], [9] and [11].

[3] His motivations were essentially philosophical, in trying to show that the laws of classical logic are not so secure that cannot be violated [1, p. 102]. Concerning da Costa's book, see [5], and on some other ways to violate the Principle of Identity, see [23].

[4] These systems were extended to higher-order logics in [17] (see [2]) and to a higher-order intensional system in [3].

are meaningless concepts, but in such a way that, taking into account the motivation provided by the quantum mechanical treatment of the elementary particles, a weaker concept of 'indistinguishability' could be considered. In this way, quasi-set theory might be viewed as a mathematical device for dealing with collections of indistinguishable objects [20].

The idea of S^* is to allow the presence of atoms (*Urelemente*) of two sorts, and to restrict to just one of these species the applicability of the concept of identity. To the others, a weaker 'relation of indistinguishability' (which has only the properties of an equivalence relation) is used instead of identity. Since the identity relation (that is, the predicate of equality) cannot be applied to this last kind of objects, there is a precise sense in saying that they can be indistinguishable without been identical. As a consequence, Leibniz Law, which (roughly speaking) asserts that there can be no 'distinct' indistinguishable entities, is violated. So, contrary to the case of quasets, the lack of sense in applying the concept of identity produces in quasi-set theory a kind of 'ontic' indeterminacy among (some of) the elements of a quasi-set.[5]

Subsequently, taking into account the concept of 'non-individuals' in H. Post's sense [28], it was suggested in [22] a rather distinct theory encompassing a weaker axiom of extensionality.[6] This new quasi-set theory, which was called S^{**}, was developed in details in [4] and in [13]. As a consequence of the improvement achieved with the weak axiom of extensionality, in that theory we were able to derive a theorem which expresses (in a sense) the 'unobservability of permutations' of certain objects, which is one of the most basic pressupositions of quantum mechanics, expressed by the so-called Indistinguishability Postulate. This theorem is presented below in this paper (Theorem 2.8).

Notwithstanding the improvement achieved with the substitution of the 'old' extensionality axiom by the 'new' one, the idea of indistinguishable quasi-sets was still restrictive, since the 'old' axiom allowed (intuitively speaking) that only quasi-sets with the same quantity of elements of the same sort could be indistinguishable.[7]

In this work, a new formulation of a quasi-set theory is presented. The basic idea is to keep the axiom of extensionality still more flexible. Some other improvements are also introduced. A sketch of the proof that $\mathfrak{Q}$ is

[5]The use of quasi-set theories in the discussion on 'vagueness' in quantum mechanics was presented in [14], [15].

[6]In S^* the axiom of extensionality was used as in the standard formulations of ZF (see [33].

[7]See the next section, where all these points are explained in details.

equiconsistent with ZFC is presented, and some further developments are suggested. A kind of 'comparison' between quasets and quasi-sets is provided with more details in [13].

Interesting to recall that the development of the afore mentioned theories is closely related to the problem posed to the mathematical community by Yu. I. Manin in 1974, during the Congress on the Hilbert Problems. As he said,

"We should consider possibilities of developing a totally new language to speak about infinity [that is, axioms for set theory]. Classical critics of Cantor (Brouwer *et al.*) argued that, say, the general choice is an illicit extrapolation of the finite case.

I would like to point out that it is rather an extrapolation of commonplace physics, were we can distinguish things, count them, put them in some order, etc.. New quantum physics has shown us models of entities with quite different behaviour. Even *sets* of photons in a looking-glass box, or of electrons in a nickel piece are much less Cantorian that the *sets* of grains of sand.

The twentieth century return to Middle Age scholastics taught us a lot about formalisms. Probably it is time to look outside again. Meaning is what really matters. [24]"

2. *The quasi-theory* $\mathfrak{Q}$

The language of $\mathfrak{Q}$ is that of the first order predicate calculus *without* identity. The intuitive idea is to allow the existence of *Urelemente* of two kinds, which are called m-atoms and M-atoms. The latter act as atoms of ZFU (Zermelo-Fraenkel with *Urelemente*), while the former are supposed to be objects to which the concept of identity cannot be applied in a sense to be explained below. This intuitive motivation conforms itself with E. Schrödinger dictum that the concept of identity does not make sense with respect to the elementary particles of modern physics, as we have mentioned in the Introduction (see also [2]). Then, with regard to the m-atoms it should be meaningless to talk about either their identity or about their diversity.

The specific symbols of $\mathfrak{Q}$ are three unary predicates m, M and Z, two binary predicates $\equiv$ and $\in$ and an unary functional symbol qc. Terms and (well-formed) formulas are defined in the standard way, so as are the concepts of free and bound variables, etc. We use x, y, z, u, v, w and t to denote individual variables, which range over quasi-sets (henceforth, qsets) and *Urelemente*. Intuitively, $m(x)$ says that 'x is a microobject' (m-atom), $M(x)$ says that 'x is a macroobject' (M-atom) while $Z(x)$ says that 'x is a set'. The term $qc(x)$ stands for 'the quasi-cardinal of (the qset) x'. The *sets* will be characterized as exact copies of the sets in ZFU.

The formulas $\forall_P x\,(...)$ and $\exists_P x\,(...)$ abbreviate $\forall x\,(P(x) \rightarrow (...))$ and $\exists x\,(P(x) \wedge (...)$ respectively, where P is a predicate.

Definition 2.1

1. $Q(x) := \neg\,(m(x) \vee M(x))$ *(x is a* quasi-set*)*
2. $P(x) := Q(x) \wedge \forall y\,(y \in x \rightarrow m(y))$ *(x is a* 'pure' quasi-set*, that is, a quasi-set whose elements are m-atoms only).*
3. $D(x) := M(x) \vee Z(x)$ *(x is a* classical object*, or* Dinge*, in Zermelo's original sense, that is, x is a (classical) Urelement or a set).*
4. $E(x) := Q(x) \wedge \forall y\,(y \in x \rightarrow Q(y))$
5. [Extensional Equality] *For all x and y, if they are not m-atoms, then:*

   $$x =_E y := (Q(x) \wedge Q(y) \wedge \forall z\,(z \in x \leftrightarrow z \in y)) \vee (M(x) \wedge M(y) \wedge x \equiv y))$$

6. [Subquasi-set] *For all x and y, if they are not atoms, then:*

   $$x \subseteq y := \forall z\,(z \in x \rightarrow z \in y)$$

If $x \neq_E y$, that is, $\neg\,(x =_E y)$, we say that x and y are *extensionally distincts*. As is usual, $x \subset y$, means $x \subseteq y \wedge x \neq_E y$. It is immediate that $x \subseteq y \wedge y \subseteq x \rightarrow x =_E y$.

The first four axioms of $\mathfrak{Q}$ are *The Axioms of Indistinguishability:*

(Q1) $\forall x\,(x \equiv x)$
(Q2) $\forall x\,\forall y\,(x \equiv y \rightarrow y \equiv x)$
(Q3) $\forall x\,\forall y\,\forall z\,(x \equiv y \wedge y \equiv z \rightarrow x \equiv z)$
(Q4) $\forall x\,\forall y\,(\neg m(x) \wedge \neg m(y) \rightarrow (x \equiv y \rightarrow (\mathrm{A}(x,x) \rightarrow \mathrm{A}(x,y))))$,
with the usual syntactic restrictions.

We will prove below (Theorem 2.7) that the extensional equality has all the properties of classical equality.

Axiom Q4 excludes m-atoms from the substitutivity law since if substitutivity is postulated to include them as well, then Q1 plus Q4 turn to be exactly the axioms usually used for the predicate of identity [25, pp. 74ff] and no syntactical difference between identity and indistinguishability could be achieved. By using Q4 as above, we preserve Leibniz Law for the 'macroscopic' (that it, those which are not m-atoms) indistinguishable entities (including qsets) and this procedure does not cause problems regarding the m-atoms, since there is a theorem (Theorem 2.8) which allows a kind of 'substitutivity' among them.

Other axioms of $\mathfrak{Q}$ are the following:

(Q5) No *Urelemente* is at the same time an m-atom and an M-atom:

$$\forall x\,(\neg\,(m\,(x) \wedge M\,(x)))$$

(Q6) If x has an element, then x is a qset. In other words, the atoms are empty:[8]

$$\forall x\,\forall y\,(x \in y \rightarrow Q\,(y))$$

(Q7) Every set is a qset:

$$\forall x\,(Z\,(x) \rightarrow Q\,(x))$$

(Q8) No set contains m-atoms as elements:

$$\forall_Q x\,(\exists_m\, y\,(y \in x) \rightarrow \neg Z\,(x))$$

(Q9) Qsets whose elements are 'classical objects' are sets and conversely:

$$\forall_Q x\,(\forall y\,(y \in x \rightarrow D\,(y)) \leftrightarrow Z\,(x))$$

(Q10) Objects which are indistinguishable from m-atoms are also m-atoms:

$$\forall x\,(m\,(x) \wedge x \equiv y \rightarrow m\,(y))$$

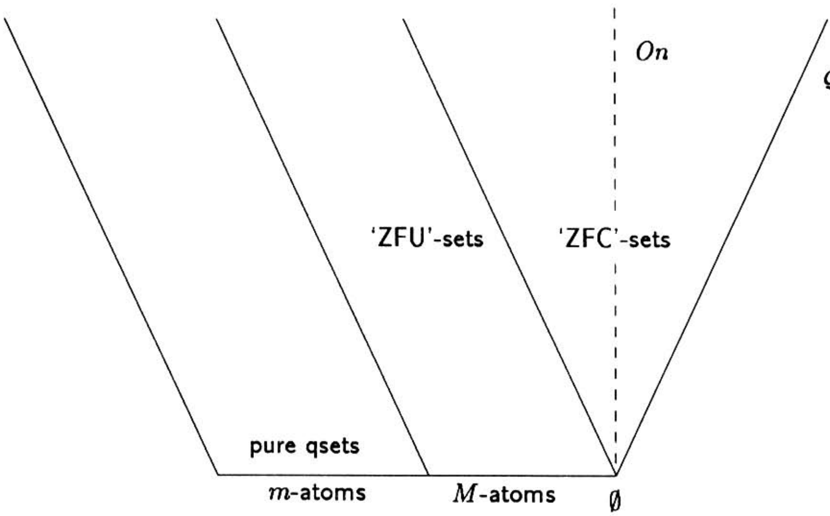
Theorem 2.1 If x is an M-atom (respectively, a qset) and $x \equiv y$, then y is also an M-atom (respect., a qset).

Proof: If $M\,(x)$ and $y \equiv x$, then $M\,(y)$ by $Q4$, for otherwise we could derive a contradiction. In short, if $M\,(x)$ and $m\,(y)$, then if $x \equiv y$, from $Q10$ we derive $m\,(x)$, hence $\neg M\,(x)$ by $Q5$, which contradicts the hypothesis that

[8]This raises an interesting question: in what sense may we say that a macroscopic object is 'composed' of microscopic elements? If the microobjects should be taken as 'non-individuals', as some like Post and Schrödinger have suggested [32], [28], how does a macroscopic object acquire its individuality? We comment in brief on this important point in the last section.

$M(x)$. Then, from $Q4$ with $M(x)$ instead $A(x,x)$, we obtain $M(y)$. Similary we prove the case concerning qsets. ⊣

From the above axioms, it follows that *sets* cannot have m-atoms as elements and, in order its elements be also'classical', they also cannot have m-atoms as elements, and so on. Hence this idea pervades the 'interior' of the elements of a qset, and this implies that a qset is a set iff its transitive closure (this concept can be defined in the standard way —see below) does not contain m-atoms. In other words, *sets* are those qsets obtained in 'stages' (see [33]) in the 'classical' part of the theory $\mathfrak{Q}$ (see the Section 2.2). They turn out to be exact copies of the sets in ZFU, as we will also emphasize below. The intuitive idea behind the 'quasi-set universe' is ower by the following picture, in which the 'ZFC-sets' (respect., the 'ZFU-sets') are those qsets that are 'copies' of the sets in ZFC (respect., in ZFU).



The Quasi-Set Universe

(Q11) [*The empty set*] There exists a *qset* (denoted '$\emptyset$') which is a set and which does not have elements:

$$\exists_Z x \, \forall y \, (\neg \, (y \in x))$$

Definition 2.2 [*Similar quasi-sets*] *For all non empty quasi-sets x and y,*

$$Sim\,(x,y) := \forall z \, \forall t \, (z \in x \wedge t \in y \rightarrow z \equiv t)$$

Intuitively, similar qsets have as elements objects 'of the same sort'. The idea of 'objects of the same sort' can be realized by passing the quotient by the relation of indistinguishability (cf. Section 2.1). This procedure defines equivalence classes of indistinguishable objects and, if they are 'classical', the classes turn to be unitary sets, since the indistinguishability relation coincides with equality in this case (as it results from Q1 plus Q4 for entities that are not *m*-atoms).

(Q12) Indistinguishable sets are extensionally identical:

$$\forall_Z x \forall_Z y (x \equiv y \rightarrow x =_E y)$$

Q12 imposes the requirement that the usual extensional properties of the sets of ZFU are valid for the sets of $\mathfrak{Q}$. Further explanations regarding this axiom are presented after the axiom Q27.[9]

(Q13) [*'Weak-Pair'*] For all x and y, there exists a qset whose elements are the indistinguishable objects from either x or y:

$$\forall x \forall y \exists_Q z \forall t (t \in z \leftrightarrow t \equiv x \vee t \equiv y)$$

The weak-pair of x and y is denoted $[x, y]$ and in the case when x and y are both classical objects, we may use the standard notation $\{x, y\}$, since in this case the only things indistinguishable from x and y will be respectively x and y themselves. If $x \equiv y$, we denote the weak-pair by $[x]$, called the *weak-singleton* of x, which is the qset of that which is indistinguishable from x. It is important to realize, as it will be clear below, that it is consistent with the theory to admit that the weak-singleton of x may have quasi-cardinal greater than one. In this sense, $\mathfrak{Q}$ *allows* the existence of indistinguishable objects which cannot be said to be identical.

(Q14) [*The Separation Schema*] By considering the usual syntactical restrictions on the formula $A(t)$, we have:

$$\forall_Q x \exists_Q y \forall t (t \in y \leftrightarrow t \in x \wedge A(t))$$

This qset will be written $[t \in x : A(t)]$. The separation axiom allows us to form subquasi-sets of a quasi-set x by considering those elements of x that satisfy a certain property expressed (in the language of $\mathfrak{Q}$) by a formula $A(t)$. This idea conforms itself with our intended interpretation of the *m*-

[9]We owe the necessity of Q12 to M. L. Dalla Chiara (private communication).

atoms as elementary particles, since in ordinary physics it is possible to 'select', from a certain collection of elementary particles, a certain number of them that satisfy a particular condition.[10]

(Q15) [*Union*] $\forall_Q x\, (E(x) \rightarrow \exists_Q y\, (\forall z\, (z \in y) \leftrightarrow \exists t\, (z \in t \wedge t \in x)))$

As usual, this qset is written

$$\bigcup_{t \in x} t$$

and we still write $x \cup y$ in the same sense as in the standard set theories.

(Q16) [*Power-qset*] $\forall_Q x\, \exists_Q y\, \forall t\, (t \in y \leftrightarrow t \subseteq x)$

The power quasi-set of x is denoted by $\mathcal{P}(x)$. Among other concepts, the following can be introduced in $\mathfrak{Q}$.

Definition 2.3
1. $\bar{x} := [y \in x : m(y)]$
2. $\langle x, y \rangle := [[x],[x, y]]$ *(the generalized ordered pair)*
3. For every quasi-sets x and y, $x \times y := [\langle z,u \rangle \in \mathcal{P}\mathcal{P}(x \cup y) : z \in x \wedge u \in y]$
4. *The intersection $x \cap y$ of two quasi-sets can be defined so that $t \in x \cap y$ iff $t \in x \wedge t \in y$ as usual. This concept can of course be generalized.*

Infinity and regularity may be introduced as follows:

(Q17) [*Infinity*] $\exists_Q x\, (\emptyset \in x \wedge \forall y\, (y \in x \wedge Q(y) \rightarrow y \cup [y] \in x))$

(Q18) [*Regularity*] Quasi-sets are well-founded, that is, for every qset x, there are no infinite chains $\ldots \in x_2 \in x_1 \in x$:

$$\forall_Q x\, (E(x) \wedge x \neq_E \emptyset \rightarrow \exists_Q\, (y \in x \wedge y \cap x =_E \emptyset))$$

We could avoid the introduction of *Q18* in order to allow qsets which are not well founded. This point will be mentioned in the last section.

[10]See [10], where examples are given.

2.1 *Relations*
The concept of *relation* and in particular that of *equivalence relation* is like the standard one: w is a relation between two quasi-sets x and y if w satisfies the following predicate R:

$$R(w) := Q(w) \wedge \forall z (z \in w \rightarrow \exists u \, \exists v (u \in x \wedge v \in y \wedge z =_E \langle u,v \rangle))$$

As in the classical case, $R \in \mathcal{PPP}(x \cup y)$. Furthermore, as usual, if $x =_E y$, we say that R is a relation *on* x. We denote by $Dom(R)$ (the *domain* of R) the quasi-set $[u \in x : \langle u,v \rangle \in R]$ and by $Rang(R)$ (the *range* of R) the quasi-set $[v \in y : \langle u,v \rangle \in R]$.

A particular interesting case of an equivalence relation on a qset x is the indistinguishability relation, which satisfies the predicate R above and, due to the axioms Q1-Q3, has the required properties. In this case, if x is a pure qset, then the 'quotient qset' $x/\equiv$ stands for a collection of equivalence classes of indistinguishable objects. This qset has similarities with H. Weyl's concept of aggregates of individuals [37, App. B], as was pointed out in [18].

With respect to order relations, we can state the following result:

Theorem 2.2 No partial, total or strict order relation can be defined on a pure qset whose elements are indistinguishable from one another.

Proof: (Sketch) Partial and total orders require antisymmetry, and this property cannot be stated without identity. Asymmetry also cannot be supposed. In fact, if $x \equiv y$, then for every R such that $\langle x, y \rangle \in R$, it follows that $\langle x, y \rangle =_E [[x]] =_E \langle y, x \rangle \in R$. ⊣

2.2 *The 'classical' counterpart of* $\mathfrak{Q}$
Based on what was presented above, it is possible to define a translation from the language of ZFU (Zermelo-Fraenkel with *Urelemente*) into the language of $\mathfrak{Q}$ as follows: firstly we admit that the language of ZFU contains an unary predicate S such that $S(x)$ says intuitively that x is a set. By simplicity, we will admit that the primitive logical symbols of ZFU are $\neg$, $\vee$, $\forall$ and equality. So, if A is a formula of ZFU, then its *translation* A^q in $\mathfrak{Q}$ is defined inductively as follows:

1. If A is $S(x)$, then A^q is $Z(x)$
2. If A is $x = y$, then A^q is $((M(x) \wedge M(y)) \vee (Z(y) \wedge Z(y)) \wedge x \equiv y)$
3. If A is $x \in y$, then A^q is $((M(x) \vee Z(x)) \wedge Z(y)) \wedge x \in y$
4. If A is $\neg B$, then A^q is $\neg B^q$
5. If A is $B \vee C$, then A^q is $B^q \vee C^q$
6. If A is $\forall x\, B$, then A^q is $\forall x (M(x) \vee Z(x) \rightarrow B)$

It is immediate that for every formula A of ZFU, $\vdash_{ZFU} A$ iff $\vdash_{\mathsf{Q}} A^q$. In other words, there is a 'copy' of ZFU in $\mathfrak{Q}$ and, in this copy, we can define the analogues of all classical set-theoretical concepts, such as for instance those of *cardinal, finite set* and *natural number*. The order relations $\leq$ and $<$ among cardinals defined in such a copy are also introduced in the usual way by using the extensional equality (hence the order symbols above should be written —as we will do below— $\leq_E$ and $<_E$ respectively). It is also evident that these relations have the standard properties, since cardinals are sets in $\mathfrak{Q}$. We use the following additional terminology: $Cd\ (x)$ stands for 'x is a cardinal'; $card\ (x)$ denotes 'the cardinal of x' and $Fin\ (x)$ says that 'x is a finite quasi-set'. Furthermore, the *transitive closure* of the quasi-set x is denoted by $TC\ (x)$, and it is defined in the standard way. As we mentioned above, it follows from the above axioms and definitions that a quasi-set x is a *set* if and only if $TC\ (x)$ does not contain m-atoms, as is easy to show.

In particular, the above translation shows that if $\mathfrak{Q}$ is consistent, so is ZFU, hence so is ZFC. The converse of this result demands more detailed considerations, which will be mentioned below.[11]

2.3 *Axioms for Quasi-Cardinals*

Although in standard set theories the concept of cardinal can be defined independently of that of ordinal, according to Weyl "the concept of ordinal is the primary one" [37, pp.34-35]. But quantum mechanics has presented the grounds for questioning this idea, since apparently there are collections of entities which have a cardinal but not an ordinal [10], [19], [15]. So, taking into account the intended interpretation of the m-atoms, we have taken the concept of quasi-cardinal as primitive, subjected to the following axioms:

(Q19) Every object which is not a qset (that is, every *Urelement*) has quasi-cardinal zero:

$$\forall x\, (\neg\, Q\,(x) \rightarrow qc\,(x) =_E 0)$$

(Q20) Every qset has an unique quasi-cardinal which is a cardinal (as defined in the 'copy' of ZFU) and, if the qset is in particular a set, then this quasi-cardinal is its cardinal stricto sensu:[12]

[11] See also [19], [21], [4] for similar discussion on the previous versions of quasi-set theory.

[12] Then, every quasi-cardinal is a cardinal and the above expression 'there is an unique' makes sense. Furthermore, from the fact that $\varnothing$ is a set, it follows that its quasi-cardinal is 0.

$$\forall_Q x\, \exists! y\, (Cd\,(y) \wedge y =_E qc\,(x) \wedge (Z\,(x) \rightarrow y =_E card\,(x)))$$

(Q21) Every non-empty qset has a non null quasi-cardinal:

$$\forall_Q x\, (x \neq_E \varnothing \rightarrow qc\,(x) \neq_E 0)$$

The next axiom says that if the quasi-cardinal of a qset x is α, then for every quasi-cardinal $\beta \leq \alpha$, there is a a subquasi-set of x whose quasi-cardinal is β.

(Q22) $$\forall_Q x\, (qc\,(x) =_E \alpha \rightarrow \forall \beta\, (\beta \leq_E \alpha \rightarrow \exists_Q y\, (y \subseteq x \wedge qc\,(y) =_E \beta))$$

(Q23) The quasi-cardinal of a subquasi-set of x is not greater than the quasi-cardinal of x:

$$\forall_Q x\, \forall_Q y\, (y \subseteq x \rightarrow qc\,(y) \leq_E qc\,(x))$$

The next two axioms have obvious meaning.

(Q24) $$\forall_Q x\, \forall_Q y\, (Fin\,(x) \wedge x \subset y \rightarrow qc\,(x) <_E qc\,(y))$$

(Q25) $$\forall_Q x\, \forall_Q y\, (\forall w\, \neg\, (w \in x \wedge w \in y) \rightarrow qc\,(x \cup y) =_E qc\,(x) + qc\,(y))$$

In the next axiom, $2^{qc(x)}$ denotes (intuitively) the quantity of subquasi-sets of x. Then,

(Q26) $$\forall_Q x\, (qc\,(\mathcal{P}\,(x)) =_E 2^{qc(x)})$$

This last axiom needs explanation, which we do by using an example. Let us suppose that we are considering the electrons of the level 2p of a sodium atom. Physics teaches us that there are six absolutely indiscernible electrons in that level. Despite our incapacity of distinguish them, we still reason *as if* there are six entities there. Then, we might suggest that (in set-theoretical terms) there are six subcollections of that collection which are 'singletons', 15 subcollections with two elements, and so on. To assume this fact is equivalent to claiming that the electrons might be thought as 'distinct' entities, despite being 'non-individuals' in some sense.[13] Then, the theory makes sense of the idea that it is by mentioning the differences among the

[13]In the sense of [32] or [28], for instance.

(quasi)cardinals of the collections that we might think of indistinguishable *m*-atoms as not being the *same* entity. In other words, $\mathfrak{Q}$ allows the existence of objects that can be merely aggregated into certain quantities, but objects that cannot be counted or ordered, as has been claimed occur with *quanta* [34, p. 12]. In fact, concerning the subcollections of the above aggregate of electrons, the only distinction among them is with respect to their quasi-cardinal.[14]

2.4 *The notion of 'weak' extensionality*

The next axiom is one of the most peculiar of the theory $\mathfrak{Q}$. The version we use here is more general than that one presented in previous papers, as we have indicated. As we will see below, the 'old' version can now be proven as a theorem of $\mathfrak{Q}$ (Theorem 2.3). Furthermore, the version of the axiom we use here is more suitable for expressing indistinguishability of qsets.

We begin by recalling that the quasi-sets x and y are similar, ($Sim\ (x,y)$) (cf. Definition 2.2) if the elements are indistinguishable. Then, we define:

Definition 2.4 The quasi-sets x and y are Q-Similar (*$QSim\ (x,y)$*) *if they are similar and have the same quasi-cardinality.*

By observing that the quotient quasi-set $x/\equiv$ may be regarded as a collection of equivalence classes of indistinguishable objects, the weak axiom of extensionality stated as:

(Q27) [*Weak Extensionality*]

$$\forall_Q x\, \forall_Q y\, (((x \neq_E \phi \wedge y \neq_E \phi \rightarrow (\forall z\, (z \in x/_{\equiv} \rightarrow \exists t\, (t \in y/_{\equiv} \wedge QSim\ (z,t)))) \wedge \forall t\, (t \in y/_{\equiv} \rightarrow \exists z\, (z \in x/_{\equiv} \wedge QSim\ (t,z))))) \rightarrow x \equiv y) \wedge (x \equiv \phi \leftrightarrow x =_E \phi))$$

The axiom simply says that those quasi-sets that have the 'the same quantity of elements of the same sort'[15] are indistinguishable.

An alternative way of formulating the intuitive idea of the above axiom should be as follows (both formulations are equivalent, as it is straightforward to prove):

[14]Dalla Chiara and Toraldo di Francia have called *objectuation* this propensity we have to reason on 'distinct' things. According to them, it is an innate activity that precedes all discursive activities [35, p. 222], [36], [11]. Then, $\mathfrak{Q}$ protects objectuation in a certain sense.

[15]In the sense that they belong to the same equivalence classes of indistinguishable objects.

[*Weak Extensionality* — Alternative Form]

$$\forall_Q x \forall_Q y \; (x \neq_E \phi \wedge y \neq_E \phi \wedge (\forall t\,(t \in x \rightarrow \exists t'\,(t' \in y \wedge t \equiv t') \wedge qc\,([t] \cap x) =_E qc\,([t'] \cap y)) \wedge qc\,(x) =_E qc\,(y)) \rightarrow x \equiv y) \wedge (x \equiv \phi \leftrightarrow x =_E \phi))$$

In a previous version of the theory (see [4], [13]), we postulated that only similar qsets (cf. Definition 2.2) with the same quasi-cardinality were indistinguishable qsets (let us call that version the 'old' version of the axiom of extensionality) But such hypothesis is so restrictive, since it deals with qsets that contain elements of the same sort only. Axiom Q27, instead, permit us to consider more general qsets as indistinguishable. In certain sense, this is more in conformity with the usual idea of supposing that, say, two molecules of a certain material are 'indistinguishable' in the sense that they encompasse 'the same quantity of components of the same sort' (electrons, protons, ...). Perhaps a nice intuitive example of indistinguishable qsets in this sense might be the polymers.

Then the 'old' version of the axiom is now a theorem of $\mathfrak{Q}$:

Theorem 2.3 $\forall_Q x \forall_Q y\,(Sim\,(x,y) \wedge qc\,(x) =_E qc\,(y) \rightarrow x \equiv y)$

Proof: Immediate consequence of Q27. ⊣

As a corollary, it follows that $x =_E y \rightarrow x \equiv y$. By considering this, we can explain with more details the axiom Q12. In fact, if both x and y are sets and $x \equiv y$, then by Q8 we conclude that x and y are either both empty or that they are similar and have the same quasi-cardinal, but this does not entail that they have 'the same' elements (that is, we cannot infer that $x =_E y$). So, since it seems intuitive that $x \equiv y \rightarrow x =_E y$ if both x and y are sets, Q12 was introduced as stating this fact. It helps us in characterizing the concept of 'set' within the scope of $\mathfrak{Q}$.

Theorem 2.4 $\forall_Q x \forall_Q y\,(\forall z\,(z \in x \leftrightarrow z \in y) \rightarrow x \equiv y)$

Proof: (Sketch) If x and y have the same elements, then they have the 'same quantity' of elements of the same sort and hence the hypothesis of the axiom Q27 is fulfilled. So, x and y are indistinguishable. ⊣

Theorem 2.5 $x \equiv y \wedge qc\,([x]) =_E qc\,([y]) \leftrightarrow [x] \equiv [y]$

Proof: Immediate consequence of Q27 and of the axioms of indistinguishability. ⊣

2.5 *Quasi-functions*

With respect to the concept of function, we note that functions, as usually conceived, cannot distinguish between its arguments and values if there were m-atoms involved. So, we introduce the more general concept of a *q*-function (quasi-function) as a relation which maps indistinguishable objects into indistinguishable objects:

*Definition 2.5 Let x and y be quasi-sets. Then we say that f is a q-*function *from x to y iff f is such that (R is the predicate for 'relation' defined in the Section 2.1):*

$$R(f) \wedge \forall u\, (u \in x \rightarrow \exists v\, (v \in y \wedge \langle u,v\rangle \in f)) \wedge$$
$$\forall u\, \forall u'\, \forall v\, \forall v'\, (\langle u,v\rangle \in f \wedge \langle u',v'\rangle \in f \wedge u \equiv u' \rightarrow v \equiv v')$$

If f is a q-function from x to y and satisfies the additional condition:

$$\forall u\, \forall u'\, \forall v\, \forall v'\, (\langle u,v\rangle \in f \wedge \langle u',v'\rangle \in f \wedge v \equiv v' \rightarrow u \equiv u')$$
$$\wedge\, qc\,(Dom\,(f)) \leq_E qc\,(Rang\,(f))$$

*then f is a q-*injection, *and f is a q-*surjection *if it is a function from x to y such that*

$$\forall v\, (v \in y \rightarrow \exists u\, (u \in x \wedge \langle u,v\rangle \in f)) \wedge qc\,(Dom\,(f)) \geq_E qc\,(Rang\,(f)).$$

*An f which is both a q-injection and a q-surjection is said to be a q-*bijection. *In this case,* $qc\,(Dom\,(f)) =_E qc\,(Rang\,(f))$.

In the general case there is no criterion to check if two quasi-sets have the same quasi-cardinal or not, since there is no 'counting process' if they have *m*-atoms as elements. This means, for instance, that if (say) x has five elements (formally: its quasi-cardinal is 5), then we cannot define a bijection from 5 = {0, 1, 2, 3, 4} to x, since we would not be able to define without ambiguity the images of $f(0) \ldots f(4)$.[16]

If $A(x,y)$ is a formula in which x and y are free variables, we say that $A(x,y)$ defines a *y-(q-functional)* condition on the quasi-set t if $\forall w\, (w \in t \rightarrow \exists s\, A(w,s) \wedge \forall w\, \forall w'\, (w \in t \wedge w' \in t \rightarrow \forall s\, \forall s'\, (A(w,s) \wedge A(w',s') \wedge w \equiv w' \rightarrow s \equiv s'))$ (this is abbreviated by $\forall x\, \exists! y\, A(x,y)$). Then, we have:

[16] Recall that the natural numbers (in fact, the $\mathfrak{C}$-version of them) were introduced in the 'copy' of ZFU defined above.

(Q28) [*Replacement*]

$$\forall x\, \exists! y\, A(x,y) \rightarrow \forall_Q u\, \exists_Q v\, (\forall z\, (z \in v \rightarrow \exists w\, (w \in u \wedge A(w,z)))$$

Intuitively, the replacement schema says that the images of qsets by q-functions are also qsets. It is easy to see that if there are no *m*-atoms involved, that is, if the qsets are *sets*, then the above axiom is exactly that of ZFC (or of ZFU).

Now we turn to the axiom of choice (new formulation).[17] In order to state the axiom, we need to introduce an important concept by the following definition.

Definition 2.6 A strong singleton *of x is a quasi-set x' which satisfies the following predicate St:*

$$St(x') \Leftrightarrow x' \subseteq [x] \wedge qc(x') =_E 1.$$

That is, x' is a subquasi-set of $[x]$ that has just 'one element' which is indistinguishable from x.

Theorem 2.6 For all x, there exists a strong singleton of x.

Proof: Firstly we note that $[x]$ exists by force of the axiom of the weak-pair Q13. Axiom Q20 says that every qset has a quasi-cardinal and, if it is not empty, its quasi-cardinal is 1 by Q21, and this occurs with $[x]$, since $x \in [x]$ due to the fact that $\equiv$ is reflexive (Q1). Hence, $qc([x]) \geq_E 1$, but then, by Q22, there exists a subqset of $[x]$ which has quasi-cardinal 1. By the separation axiom, we obtain x'. ⊣

Let us comment more about this result. The above theorem has shown that since $qc([x]) \geq_E 1$, then there exists what was called the 'strong singleton'

[17]The 'old' version we used ([19]) was $\forall_Q x\, (E(x) \wedge \forall y\, \forall z\, (y \in x \wedge z \in x \rightarrow y \cap z = \emptyset \wedge y \neq \emptyset) \rightarrow \exists_Q u\, \forall y\, \exists v\, \exists_Q w\, (y \in x \wedge v \in y \rightarrow y \cap u = w \wedge S_v(w) \wedge \forall t\, (t \in w \rightarrow t \in y)))$, where $S_v(w)$ says that w is a subquasi-set of $[v]$ and the symbol of equality '=' must be understood in the sense of our extensional equality. In words, this version takes as elements of the choice quasi-set *u all* the elements indistinguishable from elements $v \in y \in x$ which belong to y. In $\mathfrak{Q}$, by using the concept of 'strong singleton', a more adequate formulation can be achieved by taking just *one* element indistinguishable from v. From the point of view of the development of $\mathfrak{Q}$, the 'old' axiom could be maintained but, as was claimed in [26], the 'old' formulation be improved. Now, by using the concept of strong singleton, Q29 takes to be elements of u 'just one' element of every element $y \in x$.

of x, namely, the qset x' which satisfies the predicate $x' \subseteq [x] \wedge qc(x') =_E 1$. The problem is that *all* the strong singletons of x are indistinguishable in the sense of axiom Q27. If x is not an m-atom, then of course $x' =_E [x] =_E \{x\}$ which is unique by theorem 2.4 —in this case $\equiv$ turns to be $=_E$). Since the strong singletons x'_1 and x'_2 of x are not m-atoms (since a qset is not an *Urelement*), then Q4 holds and x'_1 and x'_2 can be interchanged *salva veritate* in any context. This is a particular case of the theorem 2.8 below. Now we turn to the formulation of the axiom of choice.

(Q29) [*The Axiom of Choice*]

$$\forall_Q x (E(x) \wedge \forall y \forall z (y \in x \wedge z \in x \rightarrow y \cap z =_E \emptyset \wedge y \neq_E \emptyset) \rightarrow \\ \exists_Q u \forall y \forall v (y \in x \wedge v \in y \rightarrow \\ \exists_Q w (w \subseteq [v] \wedge qc(w) =_E 1 \wedge w \cap y \equiv w \cap u)))$$

Some other basic facts can be proven in $\mathfrak{Q}$ (the proofs do not use the axiom of choice).

Theorem 2.7 The extensional equality has all the properties of the usual equality.

Proof: In fact, if x and y are both indistinguishable M-atoms then they are extensionally identicals by definition and in this case axioms Q1 and Q4 provide the basic properties of classical equality (reflexivity and substitutivity). If x and y are both qsets, then by the Theorem 2.3, in order for x and y be indistinguishable, it is sufficient that they are similar and have the same quasi-cardinality. But this occurs if they are extensionally identicals. Hence, axiom Q4 holds again and once more the classical properties of equality are obtained. ⊣

Theorem 2.8 [*Unobservability of Permutations*] *Let x be a qset such that $x \neq_E [z]$ and z an m-atom such that $z \in x$. If $w \equiv z$ and $w \notin x$, then there exists w' such that*

$$(x - z') \cup w' \equiv x$$

The operation of difference between qsets is defined as in standard set-theories. The theorem is an immediate consequence of Q27.

We recall that z' (respect. w') denotes the strong singleton of z (respect., of w). Furthermore, it may be the case that $w \notin x$, and this motivates the interpretation according to which the theorem is saying that we have 'exchanged' an element of x by an indistinguishable one, and the resulting fact is that 'nothing has occurred at all'. In other words, the resulting qset is indistinguishable from the original one. The theorem is the quasi-set theoretical version of the quantum mechanical fact which expresses that permutations of indistinguishable particles are not regarded as observable, as expressed by the so called Indistinguishability Postulate in quantum mechanics.[18]

Further technical results should be mentioned, but we will do not develop the 'quasi-set mathematics' here. The more important fact we would like to mention is about the equiconsistency of this theory with ZFC, whose proof will be sketched in the next section.

3. *Quasi-set theory and ZFC are equiconsistent*

The translation from the language of ZFU to the language of $\mathfrak{Q}$ has shown that if $\mathfrak{Q}$ is consistent, so is ZFU (and, hence, so is ZFC). In this section we outline the converse result. We will adapt to the case under study the proof presented in [4] for a previous version of quasi-set theory. All the mathematical constructions of this section are performed in ZFC.

Definition 3.1 Let m be a non empty set and R an equivalence relation on m. The equivalence classes of the quotient set m/R are denoted C_1, C_2 .... If $x \in m$, then we define:[19]

$$\hat{x} := \langle x, C_x \rangle$$

where C_x is the equivalence class to which x belongs. Furthermore, we still define:

$$\hat{m} := \{ \hat{x} : x \in m \}$$

Let X be the set $X = \hat{m} \cup M$, where $\hat{m}$ is as above and M is a set such that $\hat{m} \cap M = \varnothing$. Then we define a superstructure $\mathfrak{Q}$ over the set X, called *the Quasi-set Universe*.[20] As we will see, $\mathfrak{Q}$ is a 'model' for the quasi-set theory $\mathfrak{Q}$. The definition is as follows, where On is the class of the ordinals:

[18] The relationship between quasi-sets and quantum objects is explained in more detail in [22], [15].

[19] The sets $\hat{x}$ are exactly H. Weyl's aggregates of individuals, that is, an 'individual' and an equivalence class to which the individual belongs. See [37, App. B].

[20] We are using the same notation as in the Section 2.2, but of course the superstructure is not the same entity as the 'copy' of ZFU presented in that section.

Definition 3.2

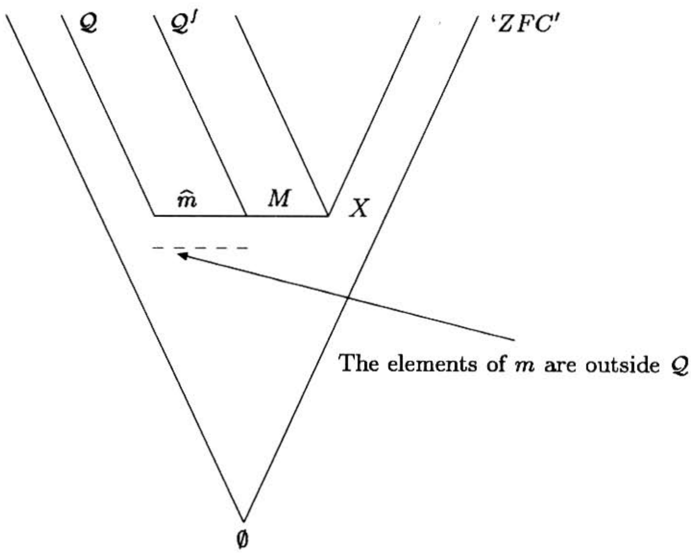
$Q_0 = X$

$Q_1 = X \cup \mathcal{P}(X)$

$\vdots$

$Q_\lambda = \bigcup_{\beta<\lambda} Q_\beta$ *if λ is a limit ordinal*

$\mathfrak{Q} = \bigcup_{\alpha \in On} Q_\alpha$



The superstructure Q in ZFC

In accordance with the terminology of $\mathfrak{O}$, we call M-atoms, M-elements or M-objects the elements of M, while the elements of $\hat{m}$ are called m-atoms, m-elements or m-objects. The final goal is to interpret the basic elements of $\mathfrak{O}$ in the objects of $\mathfrak{Q}$ with the same name.

For the sake of simplicity, we introduce another superstructure which we will call $\mathfrak{Q}^s$, constructed in a similar way as $\mathfrak{Q}$ above but having the set M only in its 'ground' basis. The idea is that the sets of $\mathfrak{O}$ (that is, those x that satisfy the predicate $Z(x)$) are interpreted as elements of $\mathfrak{Q}^s$.

Now we define a translation from the language of $\mathfrak{O}$ into the the language of ZFC. But firstly let us define on the set $\hat{m}/R$ the following relation, which turns out to be an equivalence relation, as is easy to see:

Definition 3.3 If x and y are elements of m, then:

$$\hat{x} \sim \hat{y} \text{ iff } C_x = C_y$$

If $\hat{x} \sim \hat{y}$, we say that x and y are *indistinguishable*. We note that in this way we are identifying x and y by the class (or 'state', or 'sort') they are in, represented by the equivalence class to which they belong to, and this is done without direct reference to the objects themselves.[21]

Let us turn to the translation. Suppose that A is a term or an atomic formula of $\mathfrak{Q}$; let us call A' its translation into the language of ZFC.[22] We still suppose that all the sets (of ZFC) involved in the definition below belong to $\mathfrak{Q}$ and that the quantifiers are restricted to the sets in this class. Then,

Definition 3.4
1. If A is $m(x)$, then A' is $x \in \hat{m}$.
2. If A is $M(x)$, then A' is $x \in M$.
3. If A is $Z(x)$, then A' is $x \in \mathfrak{Q}^s \wedge x \notin M$.
4. If A is $qc(x)$, then A' is $card(x)$, the cardinal of the set x.
5. If A is $x \equiv y$, then A' is $(x \in \hat{m} \wedge y \in \hat{m} \wedge x \sim y) \vee x = y)$.
6. If A is $x \in y$, then A' is $x \in y$.

The other formulas are translated in a usual way. By means of this definition, some of the definitions given in $\mathfrak{Q}$ can now be translated to ZFC. Let us mention some examples:

1. In $\mathfrak{Q}$, a quasi-set is an object which is neither an m-atom nor an M-atom. The formal definition, let us recall, is $Q(x) := \neg (m(x) \vee M(x))$. Due to the translation, in ZFC this simply means that $x \in \mathfrak{Q}$ but neither $x \in \hat{m}$ nor $x \in M$. That is, a set which in ZFC 'represents' a quasi-set is a set in $\mathfrak{Q}$ that neither belongs to M nor is an ordered pair of the form $\langle x, C_x \rangle$.

2. In $\mathfrak{Q}$, the 'pure' quasi-sets are those quasi-sets whose elements are m-atoms only. In the present case, they are interpreted (in ZFC) as subsets of $\hat{m}$. Furthermore, in $\mathfrak{Q}$ we definde $D(x) := M(x) \vee Z(x)$, what simply means that x is either an element of M or of $\mathfrak{Q}^s$.

3. The Extensional Equality is defined in $\mathfrak{Q}$ as $x =_E y := (Q(x) \wedge Q(y) \wedge \forall z (z \in x \leftrightarrow z \in y)) \vee (M(x) \wedge M(y) \wedge x \equiv y)$, for all x and y. The translation of this formula expresses (in ZFC) the usual identity be-

[21]We note that by the definition of $\mathfrak{Q}$, the elements x and y are 'outside' the superstructure, since one of the ground sets is $\hat{m}$ and not m itself. Technically, for all $x \in m$, $rank(x) < rank(X)$.

[22]We use $x, y, \ldots$ as individual variables in both theories.

tween sets given by the extensionality axiom (of ZFC) (in particular, it may expresses the identity between elements of M).

4. If x and y are similar quasi-sets (in $\mathfrak{Q}$), then in ZFC this means that their elements are indistinguishable elements of $\hat{m}$ or that they are identical. In fact, the translation of $z \in x \wedge t \in y \rightarrow z \equiv t$ (from the *definiens* of *Sim* (x,y)) may be written $z \in x \wedge t \in y \rightarrow (z \in \hat{m} \wedge t \in \hat{m} \wedge z \sim t) \vee z = t$, which is obviously true in $\mathfrak{Q}$.

Now let us comment on the translations of the axioms of $\mathfrak{Q}$. We will state only informally how the translations of the 'very peculiar' axioms of $\mathfrak{Q}$ can be proved as theorems of ZFC. Those axioms of $\mathfrak{Q}$ which are ZFU-like axioms will be mentioned only in brief. All the details can be performed without difficulty.[23]

In $\mathfrak{Q}$, the Axioms of Indistinguishability state that $\equiv$ has the properties of an equivalence relation and that the substitutivity law holds for those indistinguishable objects which are not m-atoms. If we consider item 5 of the above definition of the translation, it is easy to see that the images of the pairs $\langle x,y \rangle$ such that $x \equiv y$ define an equivalence relation in ZFC. But for those indistinguishable objects which are no m-atoms, the formula $x \equiv y$ is translated in the identity (of ZFC) between x and y; thus, the substitutivity law is true in this case and (the translation of) axiom Q4 is also true in ZFC.

Let us consider now axioms Q5-Q10 and their informal translations. The translation of Q5 simply means that x cannot be simultaneously an element of both M and $\hat{m}$, which is a true fact due to the definition of the set X. Q6 says that $\forall x \, \forall y \, (x \in y \rightarrow Q\,(y))$; its translation simply asserts that the objects of $\mathfrak{Q}$ which have elements are not elements of either M or $\hat{m}$, which is true due to the definition of the superstructure. Axiom Q7 states that $\forall x \, (Z\,(x) \rightarrow Q\,(x))$, that is, every set is a quasi-set. The translations of the formulas $Z\,(x)$ and $Q\,(x)$ both say that x is an element of the class $\mathfrak{Q}$ but that it is not an element of either $\hat{m}$ or M and also that the sets are particular quasi-sets since the elements of $\mathfrak{Q}^s$ are also elements of $\mathfrak{Q}$. So, the translation of the axiom is of course true. Axiom Q8 says that no set can have an m-atom as element. In symbols, $\forall_Q x \, (\exists_m y \, (y \in x) \rightarrow \neg Z\,(x))$; the translation, informally stated, is clearly true in ZFC due to the meaning of the word 'set' given by the above definition: a 'set' is an element of $\mathfrak{Q}^s$, and the m-atoms are ruled out of such 'sets' by definition.

[23] In the present case, the elements of X play the role of the *Urelemente* in $\mathfrak{Q}$, while the elements of $\hat{m}$ act as the m-atoms.

Axiom Q9, namely, $\forall_Q x\, (\forall y\, (y \in x \rightarrow D\,(y)) \leftrightarrow Z\,(x))$, says that every quasi-set whose elements are either sets or 'classical' *Urelemente* is a set, and its translation is obviously true in ZFC and it is easy to realize. The converse, namely, that the elements of a set (that is, of an object of $\mathfrak{Q}^s$) are either other sets or elements of M, is an obvious consequence of the definition of $\mathfrak{Q}^s$. Q10 is $\forall x\, (m\,(x) \wedge x \equiv y \rightarrow m\,(y))$. In words, the translation says that only elements of $\hat{m}$ must be in the relation $\sim$ with elements of $\hat{m}$, which is a direct consequence of the definition of $\sim$.

The axioms Q11-Q18, except Q12, are adaptations of the axioms of ZFU and it is easy to verify that their translations are true in ZFC. Axiom Q12 says that for every 'sets' x and y, if $x \equiv y$, then they are extensionally identical, what is true in $\mathfrak{Q}$ taking into account the translation defined above. The axioms for the concept of quasi-cardinal (Q19-Q26) do not present problems since their translations simply state basic properties of cardinals of the sets in $\mathfrak{Q}$. Axioms Q28 and Q29 (Replacement and Choice), when translated, state the usual replacement and choice arguments concerning the elements of $\mathfrak{Q}$, and they also hold in ZFC.

The translation of the weak extensionality axiom Q27 informally says that those sets of $\mathfrak{Q}$ that have the same quantity of elements are identical, which is a consequence of the axiom of extensionality of ZFC.

Thus, although roughly stated as above, it is easy to verify that we have defined a way to read quasi-set theory into the scope of ZFC.

4. *Further topics*

Despite the improvements achieved by the theory $\mathfrak{Q}$, quasi-set theory still demands further investigation and mathematical development. The conceptual problems caused by the lack of sense in applying the concept of identity to some objects are subtle, and of course $\mathfrak{Q}$ cannot be taken as the definitive theory of collections of elementary particles. Even so, it has been applied to some questions involving quantum objects [22], [14], [15], [3] and in certain sense it might help us in clarifying some of the conceptual difficulties of quantum theory.

Finally, let us mention a question remarked in a previous section concerning how a macroscopic object is composed of microscopic elementary particles. Perhaps this question can be translated into $\mathfrak{Q}$ by asking how an M-atom could be 'composed' by m-atoms. Since the m-atoms lack identity, how could such an M-atom then be individualized? With respect to macroscopic objects, Schrödinger suggested that there is a kind of *Gestalt* involved [30]. In our case, of course $\mathfrak{Q}$ does not answer this question, since both the m and the M-atoms were taken as primitive concepts without any kind of relationship among them. We suggest that perhaps one might add to the axiomatics of $\mathfrak{Q}$ some kind of mereological axioms in such a way that certain M-atoms could be viewed as objects whose 'parts' (in the mereological sense) were m-atoms.

We do not know any elaboration of a 'quantum mereology' in this respect, that is, a mereology suitable for quantum theory. In this sense, quasi-set theory, due to the formalization of the notion of indistinguishable but not identical objects, might play an important role in its development.

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