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ON FINITE AND INFINITE FORK ALGEBRAS AND THEIR RELATIONAL REDUCTS

Paulo A. S. VELOSO

Abstract

A fork algebra is a relational algebra enriched with a new binary operation, called fork. Such algebras have been introduced because their equational calculus has applications in program construction. They also have some interesting connections with algebraic logic. We examine the finite and infinite fork algebras and their relational reducts. The aim is twofold: contrasting finite and infinite fork algebras as well as comparing relational and fork algebras. First, we show that the finite fork algebras are essentially Boolean algebras, being somewhat uninteresting. Then, we argue that this is not the case with the infinite fork algebras: they display a large diversity of behaviours even if their relational reducts are kept fixed.

1. *Introduction*

A fork algebra is a relational algebra enriched with a new binary operation, called fork. Such algebras have been introduced because their equational calculus has applications in program construction. They also have some interesting connections with algebraic logic.

In this paper we examine the finite and infinite fork algebras and their relational reducts. The aim is twofold: contrasting finite and infinite fork algebras as well as comparing relational and fork algebras. First, we show that the finite fork algebras are essentially Boolean algebras, being somewhat uninteresting. Then, we argue that this is not the case with the infinite fork algebras: they display a large diversity of behaviours even when their relational reducts are kept fixed.

Algebraic properties concerning simplicity and subdirect decompositions, of fork algebras closely parallel their analogues for relational algebras.

Thus, a complete description of the finite (simple) fork algebras is not difficult to obtain. The finite fork algebras are completely described as the finite direct powers of the two-element fork algebra, showing that they are essentially Boolean algebras. The spectrum of the finite fork algebras cannot be covered by the simple ones, in contrast with the infinite ones: there are simple fork algebras with each infinite cardinality.

In the finite case. the relational reduct of a fork algebra already determines its (unique) fork expansion. As a measure of to what extent a relational algebra constrains its fork expansions, we introduce the fork index.

We also introduce a simple tool for the analysis of fork algebras, which is connected to the set of fixpoints of the underlying coding. Then, a set-theoretical construction is provided for producing special codings with prescribed sets of fixpoints.

These constructions are then brought together to exhibit infinite relational algebras with many fork expansions. These elastic relational algebras display a broad diversity of possible fork behaviours.

2. *Preliminaries*

The purpose of this preliminary section is fixing some general algebraic terminology and notations and recalling some concepts about algebras of relations.

We shall call an algebra *trivial* when its carrier has a single element. As usual, an algebra is *simple* iff it has no proper homomorphic images. We shall call an algebra *prime* iff it is simple and non-trivial.

Consider a class **A** of algebras with the same signature. We will use the notation $|\mathbf{A}|$ for the (cardinal) number of pairwise non-isomorphic algebras in **A**, i. e. $|\mathbf{A}|$ is the cardinality of $\mathbf{A}/\cong$, where $\cong$ is the relation of being isomorphic algebras. Given a cardinal κ, we use $\mathbf{A}[\kappa]$ for the class of algebras in **A** with cardinality κ; so $|\mathbf{A}[\kappa]|$ gives the (cardinal) number of pairwise non-isomorphic algebras in **A** with cardinality κ.

A fork algebra is a relational algebra enriched with a new binary operation, called fork. A relational algebra is an expansion of a Boolean algebra with some Peircean operations and constant.

We now briefly recall some concepts pertaining to algebras of relations on sets and their abstract versions [Jónsson & Tarski '52; Maddux '91].

We first recall the proper, set-based, versions of algebras of relations.

A *proper algebra of relations* (*PAR*, for short) on set U is an algebra $\mathcal{P} = < P, \cup, \cap, \sim, \varnothing, V, |, {}^T, 1_U >$, such that

- its reduct $< Q, \cup, \cap, \sim, \varnothing, V >$ is a field of subsets of $V \subseteq U^2$ (where $U^2 := U \times U$;
- operation ${}^T: Q \to Q$ is relation transposition;
- operation $|: Q \times Q \to Q$ is relation composition;
- $1_U \in R$ is the identity (diagonal) relation on $U: 1_U = \{< u, v > \in U^2 \,/\, u = v\}$.

We recall [Jónsson & Tarski '52, Theorem 4.24, p. 140] that, in a *PAR* $\mathcal{P}$ on set U, universal V is an equivalence relation on U.

Given an equivalence relation V on set U, the powerset $\wp(V)$ is closed under the Boolean and Peircean operations and constants; we thus have the *PAR* $\mathcal{P}(V) := <\wp(V), \cup, \cap, \sim, \emptyset, V, |, {}^T, 1_U>$, called the *powerset PAR of V*.

The *full PAR on* set U is the *PAR* $\mathcal{P}(U^2) := <\wp(U^2) \cup, \cap, \sim, \emptyset, U^2, |, {}^T, 1_U>$. The full *PAR* $\wp(U^2)$ and its subalgebras are simple. (Because they have $V=U^2$ [Jónsson & Tarski '52, Theorem 4.28, p. 142].)

We now briefly examine the abstract relational algebras (ARA's for short). Much as Boolean algebras arise as abstractions from fields of sets, these relational algebras are abstractions from their set-based versions.

An *abstract relational algebra* is an algebra $\mathfrak{R}=<R, +, \bullet, {}^-, 0, \infty, ;, {}^\dagger, 1>$, satisfying familiar equations, to the effect that

- its reduct $<R, +, \bullet, {}^-, 0, \infty>$ is a Boolean algebra with Boolean ordering $\leq$;
- its Peircean reduct $<R, ;, {}^\dagger, 1>$ is a semigroup with identity $1 \in R$ and involution ${}^\dagger : R \to R$, so $1^\dagger = 1$, $(r^\dagger)^\dagger = r$ and $(r;s)^\dagger = (s^\dagger);(r^\dagger)$,
- for all $r, s \in$ R: $(r^\dagger);(r;s)^- \leq s^-$, i. e. $(r^\dagger);(r;s)^- + s^- = s^-$.

The class of *ARA*'s has an equational characterisation [Chin & Tarski '50; Jónsson & Tarski '52], forming a variety which we will denote by **ARA**.

Recall that the simple *ARA*'s are those satisfying Tarski's rule: $\infty;r;\infty=\infty$ whenever $r\neq 0$ [Jónsson & Tarski '52, Theorem 4.10, p. 132, 133].

In contrast with Boolean algebras, not every *ARA* can be represented as some proper algebra of relations (see e. g. [Maddux '91]).

A pair of (*conjugated*) *quasiprojections* for *ARA* $\mathfrak{R}=<R, +, \bullet, {}^-, 0, \infty, ;, {}^\dagger, 1>$ amounts to elements $\gamma, \delta \in R$ such that $\gamma^\dagger;\gamma \leq 1$, $\delta^\dagger;\delta \leq 1$, and $\gamma^\dagger;\delta = \infty$ [Tarski & Givant '87, p. 96]. A *quasiprojective ARA* (a *QRA*, for short) is an *ARA* that has a pair of quasiprojections. Quasiprojective *ARA*'s are representable by proper algebras of relations: each *QRA* is isomorphic to a *PAR* on a set [Tarski & Givant '87, p. 242].

3. *Fork Algebras*

A fork algebra is an expansion of a relational algebra by a new binary operation, called fork, with certain properties. We shall use λ for the signature of the *ARA*'s, and ϕ for the signature obtaining by adding to λ a new binary operation (symbol) ∇.

Given a ϕ–algebra $\mathfrak{F}=<F, +, \bullet, {}^-, 0, \infty, ;, {}^\dagger, 1, \nabla>$ we can form its ∇-*reduct* $\mathfrak{F}_\nabla$: the λ-reduct $<F, +, \bullet, {}^-, 0, \infty, ;, {}^\dagger, 1>$. Conversely, consider an algebra $\mathfrak{R}=<R, +, \bullet, {}^-, 0, \infty, ;, {}^\dagger, 1>$ of signature λ; by adding a binary operation $\nabla : R\times R \to R$, we obtain a ϕ–algebra $\mathfrak{R}^\nabla := (\mathfrak{R}, \nabla)$, called its ∇-*expansion*.

Note that in any ϕ-algebra $\mathfrak{F}$=<F, +, •, $^-$, 0, ∞, ;, †, 1, ∇>, we have the elements π:=$(1\nabla\infty)^\dagger$ and ρ:=$(\infty\nabla 1)^\dagger$, called its (defined) *projections*.

3.1 *Abstract Fork Algebras*

The new binary operation (symbol) ∇ and the (defined) projections exhibit proper behaviour if one imposes some constraints on them. We now consider an equational formulation for these constraints and introduce the abstract fork algebras (*AFA*'s, for short) [Frias et al. '95].

An *abstract fork algebra* is a ϕ-algebra $\mathfrak{F}$=<F, +, •, $^-$, 0, ∞, ;, †, 1, ∇>, such that

- its ∇-reduct $\mathfrak{F}_\nabla$=<F, +, •, $^-$, 0, ∞, ;, †, 1> is an ARA with ordering ≤;
- algebra $\mathfrak{F}$ satisfies the following three fork properties:
  for every $r, s, p, q\in F$: $(r\nabla s);(p\nabla q)^\dagger=(r;p^\dagger)\bullet(s;q^\dagger)$ (∇&•),
  for every $r,s\in F$: $r\nabla s=(r;\pi^\dagger)\bullet(s;\rho^\dagger)$ (∇&θ),
  $\pi\nabla\rho\leq 1$ (∇θ1)
  (with π:=$(1\nabla\infty)^\dagger$ and ρ:=$(\infty\nabla 1)^\dagger$ as above).

Equation (∇&•) gives a property of ∇ with respect to relational operations, equation (∇&θ) connects ∇ to the defined projections, and inequation (∇θ1) states a property required of the defined projections.

Notice that, in view of equation (∇&θ), fork operation ∇ and the defined projections π and ρ become interdefinable. This simple observation leads to some important properties of *AFA*'s.

Since the class of *ARA*'s has an equational characterisation, so does the class of *AFA*'s. We use **AFA** for the variety of the abstract fork algebras.

It is not difficult to see that, in any *AFA* $\mathfrak{F}$ the defined elements π:=$(1\nabla\infty)^\dagger$ and ρ:=$(\infty\nabla 1)^\dagger$ form a pair of quasiprojections (such that $(\pi;\pi^\dagger)\bullet(\rho;\rho^\dagger)\leq 1$) [Frias et al. '95]. Thus, the *ARA* reducts of *AFA*'s are *QRA*'s.

We now examine some simple results connecting the algebraic structures of fork and relational algebras [Veloso '96a].

These algebraic structures are very tightly connected because of the following expansion construction.

Consider an *AFA* $\mathfrak{F}$ and an *ARA* $\mathfrak{R}$. Every surjective λ-homomorphism $h:F\to R$ from ∇-reduct $\mathfrak{F}_\nabla$ onto $\mathfrak{R}$ is a ϕ-homomorphism from *AFA* $\mathfrak{F}$ onto an *AFA* $(\mathfrak{R},\nabla^h)$ expanding $\mathfrak{R}$.

We use the projections to define $\nabla^h:R\times R\to R$ by $r\nabla^h s:=[r;h(\pi^\dagger)]\bullet[s;h(\rho^\dagger)]$.

Thus, we can characterise the simple *AFA*'s as those with simple relational reducts.

- An *AFA* $\mathfrak{F}$ is simple iff its ∇-reduct $\mathfrak{F}_\nabla$ is a simple *ARA*.

We can also see that the non-simple *AFA*'s are those with non-trivial direct factorisations, just like *ARA*'s [Jónsson & Tarski '52, Theorem 4.13, p. 134].

- An *AFA* $\mathfrak{F}$ is non-simple iff $\mathfrak{F} \cong \mathfrak{G} \times \mathcal{H}$ for some non-trivial *AFA*'s $\mathfrak{G}$ and $\mathcal{H}$.

In a similar manner, we can see that we have a correspondence between subdirect decompositions for *AFA*'s and their relational reducts.

- Given an *AFA* $\mathfrak{F}$, each subdirect decomposition of ∇-reduct $\mathfrak{F}_\nabla$ into simple *ARA*'s can be expanded to a subdirect decomposition of $\mathfrak{F}$.

Thus, the subdirect decompositions for *ARA*'s Jónsson & Tarski '52, Theorem 4.15, p. 135] carry over to *AFA*'s.

- Every *AFA* $\mathfrak{F}$ is isomorphic to some subdirect product of simple homomorphic images of $\mathfrak{F}$:

3.2 *Proper Fork Algebras (of Relations)*

We now examine the set-based versions of fork algebras. Algebras of relations involve relations on a set (of points), whereas fork algebras of relations deal with relations involving structured objects. Such structured objects present a behaviour akin to that of pairs. To have a set-based version, we use a fork operation on relations induced by a coding on the underlying set.

A natural way of combining two relations amounts to feeding a common input to both of them. This idea produces a binary operation on relations: the cartesian fork $\underline{©}$, defined by $r\underline{©}s := \{<u, <v,w>> \in U \times U^2 / <u,v> \in r \;\&\; <u,w> \in s\}$.

Now, the cartesian fork of relations r and s on U is a relation $r\underline{©}s$ between U and U^2. If set U happens to be closed under cartesian product, then the cartesian fork of relations r and s on U is again a relation $r\underline{©}s$ on U.

In general, we may resort to a coding function (or relation). Notice that, if function $*: U^2 \rightarrow U$ is injective, one can recover v and w from $v*w$. For instance, for the universe $U=\mathbf{N}$ of natural numbers, one might consider a Gödel-like coding $*:\mathbf{N} \rightarrow \mathbf{N}$, given by , say, $m*n := 2^m.(2n+1)$, coding pair $<m,n>$ of naturals by the single natural $m*n \in \mathbf{N}$.

By a *pairing relation* for set U we mean a relation $*$ from U^2 to U. A pairing relation for set U induces a binary operation $\underline{*}$ on relations on U, defined by $r\underline{*}s := (r\underline{©}s)|*$. So, $r\underline{©}s = \{<u,z> \in U^2 / \exists <x,y> \in U^2 : <u,x> \in r \;\&\; <u,y> \in s \;\&\; <<x,y>,z> \in *\}$.

A simple property of the induced fork is its monotonicity with respect to inclusion: if $r \subseteq p$ and $s \subseteq q$ then $r\underline{*}s \subseteq p\underline{*}q$.

Given a universal equivalence relation $V \subseteq U^2$, we wish to have the fork of any two relations included in V. In view of monotonicity, it suffices to guarantee that $V\underline{*}V \subseteq V$. For this purpose, it is sufficient (and necessary) to require V to be closed under $*$: $<x,z> \in V$ whenever $<x,y> \in V$ and $<<x,y>,z> \in *$. For such a closed equivalence $V \subseteq U^2$, the powerset $\wp(V)$ is closed under operation $\underline{*}$, and we can expand the powerset *PAR* $\mathcal{P}(V)$ to the ϕ-algebra $(\mathcal{P}(V),\underline{*})$, which we denote simply by $\mathcal{P}^*(V)$. To guarantee that

$\mathcal{P}^*(V)$ satisfies the fork equations, it is sufficient (and necessary) to require the restriction of pairing relation * to V to be an injective function $_{V|}*:V \to U$.

By imposing these constraints on the underlying pairing relation *, we arrive at the concept of proper fork algebras of relations on a set.

A *proper fork algebra* (of relations) (*PFA*, for short) on set U is a ϕ-algebra $\mathfrak{Q}=\langle Q, \cup, \cap, \tilde{}, \Delta, V, |, ^T, 1_U, \angle\rangle$, such that

- its $\angle$-reduct $\mathfrak{Q}_{\angle}=\langle Q, \cup, \cap, \tilde{}, \varnothing, V, |, ^T, 1_U\rangle$ is a *PAR* on set U;
- operation $\angle: Q \times Q \to Q$ is induced by a pairing relation * whose restriction to V is an injective function $_{V|}*:V \to U$, i. e.
  $r \angle s = \{\langle u, x*y\rangle \in U^2 / \exists \langle x,y\rangle \in V: \langle u,x\rangle \in r \,\&\, \langle u,y\rangle \in s\}$.

Notice that the underlying pairing relation is hidden: its exact definition is not important, as long as one can recover the appropriate arguments.

In particular, each injective $*:U^2 \to U$ gives rise to the *full PFA* $\mathcal{P}^*(U^2):=(\mathcal{P}(U^2),\underline{*})$. The full *PFA*'s and their subalgebras are simple.

Like Boolean algebras and contrasting with *ARA*'s, every abstract fork algebra can be represented as some proper fork algebra of relations on a set [Frias et al. '95]. Here we shall make only limited use of this result.

4. *Finite and Infinite Fork Algebras*

We now wish to compare the finite and infinite *AFA*'s. For this purpose, we shall examine the finite *AFA*'s and show that they are essentially Boolean algebras, being uninteresting.

We begin by examining some simple results concerning the Boolean *AFA*'s (where fork is Boolean meet) and the finite *AFA*'s [Veloso '96b].

4.1 *Boolean Relational and Fork Algebras*

Two very simple finite *AFA*'s are those with one and two elements. It is not difficult to show that these are the only finite simple *AFA*'s (see 4.2).

The trivial one-element *AFA* *1* has single element $0=1=\infty$. It is isomorphic to the full *PFA* $\mathcal{P}^*(\varnothing)$, with carrier $\wp(\varnothing)=\{\varnothing\}$, and underlying pairing $*=\varnothing$.

The two-element *AFA* *2* has two elements 0 and $1=\infty$ with $\infty \nabla \infty = \infty$. It is isomorphic to the full *PFA* $\mathcal{P}^*(\{\langle u,u\rangle\})$ over a singleton $\{\langle u,u\rangle\}$, with underlying pairing function * given by $u*u=u$.

The (simple) *AFA*'s *1* and *2* are the only *AFA*'s, up to isomorhism, with respectively *1* and 2 elements (since $0 \nabla 0=0$ and $0 \leq 1 \leq \infty$).

The *ARA* reducts of the simple *AFA*'s *1* and *2* are *QRA*'s, with $1=\infty$ as both quasiprojections. They are Boolean *ARA*'s as well.

Recall that a *Boolean ARA* is one where ; is •, † is the identity function, and 1=∞ [Jónsson & Tarski '52, p. 151]. . Thus, a Boolean *ARA* is a somewhat uninteresting expansion of a Boolean algebra to an *ARA*.

Clearly, every Boolean *ARA* is a *QRA*, with 1=∞ as both quasiprojections. Also, a direct product $\times_{i\in I}\mathcal{R}_i$ of *ARA*'s is Boolean iff every $\mathcal{R}_i$, i∈I, is Boolean.

By analogy with the Boolean *ARA*'s, let us call an *AFA* $\mathfrak{F}$=<*F*, +, •, $^-$, 0, ∞, ;, †, 1, ∇> *Boolean* iff its fork ∇ is •. Clearly, an *AFA* is Boolean iff its projections are the identity ($\pi = 1 = \rho$).

It is not difficult to see that the relational reduct of a Boolean *AFA* is a Boolean *ARA*. The converse being clear, we have:

- An *ARA* $\mathcal{R}$ is a Boolean *ARA* iff ($\mathcal{R}$,•) is a Boolean *AFA*.

So, Boolean *AFA*'s, like Boolean *ARA*'s, are essentially Boolean algebras.
Thus, the Boolean *AFA*'s can be characterised as the subalgebras of direct powers of the two-element *AFA* *2*.

- A ϕ-algebra $\mathfrak{F}$ is a Boolean *AFA* iff $\mathfrak{F}$ can be embedded into some direct power 2^I of the two-element *AFA* *2*.

Also, a Boolean *ARA* has a single *AFA* expansion: the Boolean one.

- For a Boolean *ARA* $\mathcal{R}$, ($\mathcal{R}$,∇) is an *AFA* iff ∇ is •.

4.2 *Finite Fork Algebras*

We now describe the finite *AFA*'s as the finite direct powers of the simple ones, the latter being those with one and two elements.
By a property of Boolean algebras, a finite *AFA* must have cardinality 2^n, for some $n \geq 0$. The direct power 2^n, provides an (uninteresting) example of a (Boolean) *AFA* cardinality 2^n, for each $n \geq 0$. Thus, the finite spectrum of the *AFA*'s is the set $\{2^n \mid n \in \mathrm{N}\}$.

It is not difficult to see (by examining the iterates $\gamma^k = \gamma;\ldots;\gamma$ (*k* times) of the quasiprojections) that finite simple *QRA*'s must have at most two elements. (For a proper *QRA* $\mathcal{P}$ over a finite set *U*, $|U| \leq 1$ [Tarski & Givant '87, p. 96].) We thus have the following upper bound on finite simple *QRA*'s:

- There exists no finite simple *QRA* with more than two elements.

Since *AFA*'s have (quasi)projections, we can see that, up to isomorphism, *1* and *2* are all the finite simple *AFA*'s. We thus have the following description of the finite simple *AFA*'s.

- A ϕ-algebra $\mathfrak{F}$ is a finite simple *AFA* iff $\mathfrak{F} \cong 1$ or $\mathfrak{F} \cong 2$.

In view of the subdirect decomposition of *AFA*'s, we can now see that the finite *AFA*'s are Boolean.

- Every finite *AFA* is a Boolean *AFA*.

By induction on their sizes, we now have a complete description of the finite *AFA*'s as the finite direct powers of the two-element prime *AFA* *2*.

- A ϕ-algebra $\mathfrak{F}$ is a finite *AFA* iff $\mathfrak{F}$ is isomorphic to some finite direct power $\mathit{2}^n$ of the two-element *AFA* *2*.

Thus, there exists exactly one, up to isomorphism, *AFA* of each finite cardinality 2^n for $n \geq 0$. In particular, the *ARA* reduct of a finite *ARA* has exactly one, up to isomorphism, expansion to a fork algebra.

This description of the finite *AFA*'s is summarised in the following table:

Cardinality 2^n,	*AFA*	Simplicity
$n=0$	*1*	trivial
$n=1$	*2*	prime
$n>1$	$\mathit{2}^n$	non-simple

Summing up, the finite *AFA*'s have the following properties:
- every finite *AFA* is a Boolean *AFA*;
- for each finite $n \geq 0$: $\mathfrak{F} \in \mathbf{AFA}[2^n]$ iff $\mathfrak{F} \cong \mathit{2}^n$ (so $|\mathbf{AFA}[2^n]|=1$);
- a finite simple *AFA* with *n* elements is simple iff $n \leq 2$.

4.3 *Infinite Non-Boolean Fork Algebras*

As we have seen, the finite *AFA*'s are Boolean, so the simple finite *AFA*'s have at most two elements. Thus, the finite spectrum of the *AFA*'s is not covered by the simple ones. We shall now show that the situation is entirely different in the case of the infinite *AFA*'s.

Every Boolean algebra can be expanded to a (Boolean) *AFA*. Thus, there exists an *AFA* of each infinite cardinality.

We shall now establish the existence of simple non-Boolean *PFA*'s at each given infinite cardinality.

- For each infinite set *U*, there exists a simple non-Boolean *PFA* with cardinality $|U|$.

Since set *U* is infinite, there exists an injective and non-surjective function $*:U^2 \to U$. The full *PFA* $\mathcal{P}^*(U^2)$ over *U* is simple *PFA* with $V=U^2$ and $V \underline{*} V \neq V$ (as $*:U^2 \to U$ is not surjective). Thus, the subalgebra of $\mathcal{P}^*(U^2)$ generated by the set $\wp_\omega(U^2)$ of the finite subsets of U^2 is a *PFA* as asserted.

We thus see that the simple infinite *PFA*'s already suffice to cover the infinite spectrum of the *AFA*'s, in contrast with the finite case.

5. *Relational and Fork Algebras*

We have seen that finite and infinite *AFA*'s differ in one aspect: the breadth of the simple algebras. We now wish to contrast them even further, to see that the infinite *AFA*'s are much more interesting than the finite ones.

We have already seen another aspect that render the finite *AFA*'s uninteresting. The finite *AFA*'s, being Boolean, are uniquely characterised by their relational reducts, which does not happen with the infinite *AFA*'s, as we will shortly see. So, this contrast between finite and infinite *AFA*'s is connected to a comparison between *ARA*'s and *AFA*'s.

The close parallel between the algebraic structures of *ARA*'s and *AFA*'s indicates a similarity in the behaviour of *ARA*'s and *AFA*'s. But, representability, as mentioned previously, is already a clear difference (expressivity is another one [Veloso & Haeberer '91]). Some further distinctions, indicating that they are quite different, will be examined in the sequel.

For the purpose of comparing (mainly infinite) *ARA*'s and *AFA*'s, we now examine some considerations and introduce some terminology.

5.1 *Fork Expansions of Relational Algebras*

What *AFA*'s have more than *ARA*'s is a fork operation. This difference vanishes in the Boolean *AFA*'s, when fork is $\bullet$. In the case of finite *ARA*'s, as we have seen, the Boolean fork is the only possibility of expansion to an *AFA*. We will shortly see that this is not so for the infinite *ARA*'s. For this purpose, we examine possible expansions of an *ARA* by a fork operation.

Consider an *ARA* $\mathfrak{R}$ with carrier $R \subseteq F$. We naturally call *ARA* $\mathfrak{R}$ *expandable by* binary operation $\nabla : F \times F \to F$ iff R is closed under ∇.

Expandability of subalgebras of reducts is easily characterised.

- Given an *AFA* $\mathfrak{F}$, a λ-subalgebra $\mathfrak{R}$ of its ∇-reduct $\mathfrak{F}_{\nabla}$. is expandable by $\nabla : F \times F \to F$ iff the projections $\pi = (1 \nabla \infty)^{\dagger}$ and $\rho = (\infty \nabla 1)^{\dagger}$ are in R.

As a tool for comparing *ARA*'s and *AFA*'s, we introduce the concept of fork index. By giving an indication of how much freedom is available in expanding a given *ARA* to an *AFA*, it serves as an answer to the question: "How much information about a fork algebra is already given by its relational reduct?"

The *fork index* of *ARA* $\mathfrak{R}$ is the cardinality $\varphi(\mathfrak{R}) := |\mathbf{FRK}(\mathfrak{R})|$ of the set $\mathbf{FRK}(\mathfrak{R}) := \{ \mathfrak{F} \in \mathbf{AFA} / \mathfrak{F}_{\nabla} = \mathfrak{R} \}$ of fork expansions of $\mathfrak{R}$.

Since fork operations define projections, an upper bound on fork indices of *ARA*'s can be easily obtained.

- For every *ARA* $\mathfrak{R}$: $\varphi(\mathfrak{R}) \trianglelefteq |R|^2$.

We shall call an *ARA* $\mathfrak{R}$ *explosive* iff it has fork index $\varphi(\mathfrak{R}) = |R|^2$.

Not surprisingly, some *ARA*'s (for instance, those that are not QRA's) have null fork indices; let us call them *non-forkable*.

Also, let us call an *ARA* $\mathfrak{R}$ *rigid* iff it has exactly one, up to isomorphism, *AFA* expansion: $\varphi(\mathfrak{R}) = 1$. As we have seen, the Boolean *ARA*'s (in particular the finite ones) are all rigid.

At the other extreme, an *ARA* may have many non-isomorphic *AFA* expansions. *ARA*'s with high fork indices will be of special interest, since they exhibit quite clearly the diversity of possible fork operations.

The *elastic ARA*'s will be those with infinite fork indices. Notice that elastic *ARA*'s cannot be Boolean.

In the sequel we will construct examples of *ARA*'s by combining given *ARA*'s. We shall then have occasion to use some bounds on the fork indices of some direct powers and products of *ARA*'s.

Each fork expansion of a direct product of *ARA*'s gives rise to fork expansions of the components. Thus, the following upper bound is clear.

- For a direct product $\mathcal{R}\times S$ of *ARA*'s: $\varphi(\mathcal{R}\times\mathbf{S})\leq\varphi(\mathcal{R}).\varphi(\mathbf{S})$.

Conversely, fork expansions of components produce fork expansions of their direct product. But, to obtain lower bounds, we must know that certain fork algebras among these are not isomorphic.

We shall consider two special cases involving direct-product factorisations.

Let us call *ARA* $\mathcal{P}$ a *factor* of *ARA* $\mathcal{R}$ iff $\mathcal{R}\cong\mathcal{P}\times\mathfrak{Q}$ for some non-trivial *ARA* $\mathfrak{Q}$. The prime factors of a direct product $\times_{i=I}\mathcal{R}_i$ of prime *ARA*'s are the components $\mathcal{R}_i$ $i\in I$. (This can be seen by examining the corresponding ideal elements [Jónsson & Tarski '52, p. 129-135].)

We can now establish some lower bounds.

- For each prime *ARA* $\mathcal{R}$, if $I\neq\varnothing$ then $\varphi(\mathcal{R}^I)\geq\varphi((\mathcal{R})$.
- Consider a prime *ARA* $\mathcal{R}$ with cardinality $|\mathcal{R}|=\mu$. For every set of prime *AFA*'s $\mathfrak{Q}_i$ with cardinality $|\mathfrak{Q}_i|=n_i$ with *ARA* reducts $\mathcal{P}_i$, $i\in I$, such that $\mu\notin\{v_i\ /\ i\in I\}$: $\varphi[(\times_{i\in I}\mathcal{P}_i)\times\mathfrak{R}]\geq\varphi(\mathfrak{R})$.

5.2 *Analysis of (Proper) Fork Algebras*

We now introduce a tool for the analysis of fork algebras.

Given a ϕ-algebra $\mathfrak{F}$, let $2:=1\nabla 1$, and consider the set of its *sub-identities of* 2: $SI2(\mathfrak{F}):=\{f\in F/f\leq 2\bullet 1\}$.

Any ϕ-isomorphism between ϕ-algebras $\mathfrak{F}$ and $\mathfrak{G}$ provides a bijection between $SI2(\mathfrak{F})$ and $SI2(\mathfrak{G})$, so $|SI2(\mathfrak{F})|=|SI2(\mathfrak{G})|$.

A proper fork algebra $\mathfrak{Q}$ has set of sub-identities of 2 $SI2(\mathfrak{Q})=\{q\in Q/q\subseteq 2\cap 1_U\}$. For a simple PFA, its set of sub-identities of 2 is connected to the set of fixpoints of its underlying coding.

Given a function $*:U^2\to U$, consider its *set of fixpoints* $fxpt(*):=\{u\in U/u*u=u\}$. This set of fixpoints can also be conveniently represented by its identity $1_{fxpt(*)}:=\{<u,u>\in U^2/u*u=u\}$. Notice that $2\cap 1_U=1_{fxpt(*)}$.

We can see the following connection between the set of sub-identities of 2 of a simple PFA and the set of fixpoints of its underlying coding.

- For a simple PFA $\mathfrak{Q}=<Q,\cup,\cap,\sim,\varnothing,U^2,|,^T,1_U,\angle>$ with fork $\angle$ induced by coding $*:U^2\to U$, we have $SI2(\mathfrak{Q})=\wp[1_{fxpt(*)}]\cap Q$.

6. *Constructions for Infinite Algebras of Relations*

In the sequel, we shall construct some infinite (simple) proper algebra of relations with given infinite cardinalities and prescribed fork indices. We shall control the cardinalities of the PAR's by means of the sizes of their sets of generators. We shall control their fork expansions by means of the sizes of the sets of sub-identities of 2.

6.1 *Special Codings and Sets of Fixpoints*

To control the set of fixpoints of its underlying coding, and so the set of sub-identities of 2 of simple PFA's, we employ special codings with given sets of fixpoints.

We now present a set-theoretical construction for a special coding on an infinite set U with prescribed set of fixpoints.

- Consider an infinite set U. For each subset $T \subseteq U$ with cardinality $|T|=|U|$, there exists a bijection $*^S: U^2 \to U$ with $fxpt(*^S)=S$, where $S:=U-T$.

Indeed, we can partition T into disjoint subsets A and B of U, both with cardinality $|U|$. So we have a bijection $f: 1_U{}^{\sim} \to A$. We also have a bijection $g: T \to B$ without fixpoints (obtained by partitioning B as $B=\cup_{n \in N} B_n$).

We now define $*^S: U^2 \to U$ as follows:

for $u \in S$ we set $u *^S u := u$ (notice that $u \notin A \cup B$);
for $u \in T$ we set $u *^S u := g(u)$ (notice that $g(u) \in B$);
for $<v,w> \in 1_U{}^{\sim}$ we set $v *^S w := f(v,w)$ (notice that $f(v,w) \in A$).

So, $*^S: U^2 \to U$ is a bijection, from $U^2 = 1_S \cup 1_T \cup 1_U{}^{\sim}$ onto $U = S \cup B \cup A$, since it is the disjoint union of bijections with pairwise disjoint domains and images.

Also, $u *^S u = u$ iff $u \in S$, because for $u \notin S$ $u *^S u = g(u) \neq u$. Thus $fxpt(*^S)=S$.

On an infinite set we have many special codings.

- Given an infinite set U, for each subset $S \subseteq U$ with cardinality $|S|<|U|$, there exists a bijection $*^S: U^2 \to U$ with $fxpt(*^S)=S$.

6.2 *Infinite Explosive Algebras of Relations*

We will now construct some infinite (simple) non-Boolean proper algebra of relations that have infinitely many fork expansions. We control their fork expansions by means of the sizes of the sets of sub-identities of 2.

For this purpose, we now examine fork expansions of simple PFA's.

Each injective function $*: U^2 \to U$ induces a fork operation $\underline{*}: \wp(U^2) \times \wp(U^2) \to \wp(U^2)$, which gives rise to *induced projections* $p^* := (1_U \underline{*}\, U^2)^T$ and $q^* := (U^2 \underline{*}\, 1_U)^T$. Now, consider a (simple) subalgebra $\mathcal{P}$ of the full PAR $\mathcal{P}(U^2)$, and notice that

- if induced projections p^* and q^* are in the carrier P, then PAR $\mathcal{P}$ is expandable by the induced fork operation $\underline{*}$ to PFA $\mathcal{P}^*:=(\mathcal{P},\underline{*})$;
- if $\wp[1_{fxpt(*)}]\subseteq P$, then PFA $\mathcal{P}^*=(\mathcal{P},\underline{*})$ has set of sub-identities of 2 $SI2(\mathcal{P}^*)=\wp[1_{fxpt(*)}]$.

By putting together these constructions and considerations, we can now construct a prime explosive proper algebra of relations of each infinite cardinality.

Proposition Large prime explosive PAR's

For every infinite cardinal $\kappa\geq\aleph_0$, there exists a prime explosive PAR $\mathcal{P}_\kappa$ with cardinality $|\mathcal{P}_\kappa|=\kappa$: PAR $\mathcal{P}_\kappa$ has κ, pairwise non-isomorphic, fork-expansions $\mathfrak{Q}_\gamma$ for each smaller cardinal $\gamma<\kappa$.

Proof outline

Consider the set $U:=\kappa$. For each cardinal $\gamma<\kappa$, γ is a subset $\gamma\subseteq U$ with $|\gamma|=\gamma<\kappa$; so, we have a bijection $*^\gamma:U^2\to U$, with $fxpt(*^\gamma)=\gamma$, inducing fork $\underline{*}^\gamma$ and projections p^γ and q^γ.

Set $H:=\wp_\omega(U^2)\cup\bigcup_{\gamma<\kappa}[\{p^\gamma,q^\gamma\}\cup\wp(1_\gamma)]$ (notice that $|H|=\kappa$) and consider the subalgebra $\mathcal{P}_\kappa$ of the full PAR $\mathcal{P}(U^2)$ generated by H.

Then, $\mathcal{P}_\kappa$ is a simple PAR of cardinality $|\mathcal{P}_\kappa|=\kappa\geq\aleph_0$ (so prime), which has a fork expansion $Q_\gamma=(\mathcal{P}_\kappa,\underline{*}^\gamma)$ with $|SI2(Q_\gamma)|=2^\gamma$, for each $\gamma<\kappa$.

QED

We thus see that at each infinite cardinality there exist PAR's that convey a very modest amount of information concerning their possible fork expansions.

This results shows that the class **EXP** of explosive PAR's is populated at each infinite cardinality with simple PAR's. We now wish to see that it is not sparsely populated.

Towards this goal, we combine prime PAR's. We can guarantee that direct products of *ARA*'s are not isomorphic by controlling the numbers of ideal elements of their factors.

- Given an infinite cardinal $\kappa\geq\aleph_0$, for every cardinal $\zeta<\kappa$ the direct product $2^\zeta\times\mathcal{P}_\kappa$ is a representable explosive *ARA* with cardinality κ and $2^{\zeta+1}$ ideal elements.

Since $2^\zeta\leq\kappa$, the direct product $2^\zeta\times\mathcal{P}_\kappa$ is a representable *ARA* with cardinality κ. Its prime factors are rigid 2 and explosive $\mathcal{P}_\kappa$, so $\varphi(2^\zeta\times\mathcal{P}_\kappa)=\kappa$. Clearly, $2^\zeta\times\mathcal{P}_\kappa$ has $2^\zeta.2$ ideal elements

We have thus achieved our aim of exhibiting many elastic PAR's with prescribed infinite cardinalities: at each infinite cardinality $\kappa\geq\aleph_0$ there are at least κ pairwise non-isomorphic explosive PAR's.

Theorem Many large explosive PAR's
For every infinite cardinal $\kappa \geq \aleph_0$, there exist at least κ pairwise non-isomorphic explosive (so elastic) PAR's with cardinality κ: $|\mathbf{EXP}[\kappa]| \geq \kappa$.

We thus see that at each infinite cardinality there are many PAR's that constrain very little their possible fork expansions.

6.3 *Other Infinite Algebras of Relations*
The preceding results show that there are many pairwise non-isomorphic elastic PAR's at each infinite cardinality. But such elastic *ARA*'s do not exhaust the infinite PAR's.

We now examine infinite non-forkable and rigid proper algebra of relations. We will show that the classes **NFK**, of non-forkable PAR's, and **RGD**, of rigid PAR's, are populated at each infinite cardinality with simple PAR's.

We first consider (simple) infinite proper algebra of relations with null fork indices. The non-representable relational algebras are non-forkable, but we wish non-forkable proper algebra of relations.

For this purpose, we make the following considerations.
We first present a denumerably infinite non-forkable simple PAR.

- The algebra $\mathcal{S}_\omega$ of definable binary relations in the structure $\mathfrak{N}$=<**N**,S,0> is a simple non-forkable PAR with cardinality $|\mathcal{S}_\omega|=\aleph_0$.

Clearly, $\mathcal{S}_\omega$ is a simple PAR with cardinality $|\mathcal{S}_\omega|=\aleph_0$. Assume PAR $\mathcal{S}_\omega$ forkable. Then, the binary relation $2=1_\mathbf{N}\nabla 1_\mathbf{N}$ is definable, but its image Im[2] is neither finite (because 2 is total, functional, and injective) nor cofinite (since $\mathrm{Im}[2]\cap\mathrm{Im}[1_\mathbf{N}{}^{\sim}\nabla 1_\mathbf{N}]=\varnothing$).

We now know that there exist (simple) non-forkable proper algebra of relations at each infinite cardinality.

- For every infinite cardinal $\kappa \geq \aleph_0$, there exists a simple non-forkable PAR $\mathcal{S}_\kappa$ with cardinality $|\mathcal{S}_\kappa|=\kappa$.

The representable *ARA*'s form an elementary class in the wide sense [Tarski '55]. Thus the class **SRQ** of simple representable *ARA*'s that are not QRA's is EC_Δ as well. Hence, for each infinite cardinality $\kappa \geq \aleph_0$ we have some $\mathcal{S}_\kappa \in$**SRQ** (whence a simple non-forkable PAR) with cardinality $|\mathcal{S}_\kappa|=|\kappa|$.

We now consider (simple) infinite rigid proper algebras of relations.

Rigid proper algebras of relations are quite easy to construct. For each infinite cardinal $\kappa \geq \aleph_0$, one can easily construct a Boolean, so rigid, simple PAR of identities I_κ with cardinality $|I_\kappa|=\kappa$.

- For each infinite set U, there exists a simple rigid Boolean PAR of identities I_U with cardinality $|I_U|=|U|$

The powerset PAR $\mathcal{P}(1_U)$ of 1_U is a simple PAR $<\wp(1_U),\cup,\cap,{}^{\sim},\varnothing,V,|,{}^{T},1_U>$ with cardinality $|\wp(1_U)|>|U|$. We take I_U as its subalgebra generated by the set $\wp_{\omega}(1_U) \subseteq \wp(1_U)$ of finite subsets of 1_U (notice that $|\wp_{\omega}(1_U)|=|U|$).

We can now use direct products to show that the classes of rigid and of non-forkable proper algebra of relations are not sparsely populated.

Proposition Many large non-forkable and rigid PAR's

For every infinite cardinal $\kappa \geq \aleph_0$: $|\mathbf{NFK}[\kappa]| \geq \kappa$ and $|\mathbf{RGD}[\kappa]| \geq \kappa$.

For every cardinal $\zeta < \kappa$:

a) the direct product $2^{\zeta} \times S_{\kappa}$ is a representable non-forkable *ARA* with cardinality κ and $2^{\zeta+1}$ ideal elements;
b) the direct product $2^{\zeta} \times I_{\kappa}$ is a representable rigid *ARA* with cardinality κ and $2^{\zeta+1}$ ideal elements.

We can also use direct products to construct some new elastic (and non-explosive) infinite representable *ARA*'s. We can exhibit many infinite representable *ARA*'s with prescribed infinite fork indices.

Consider infinite cardinals $\kappa \geq \aleph_0$ and $\aleph_0 \leq \gamma < \kappa$:

- the direct product $\mathcal{P}_{\gamma} \times I_{\kappa}$ is a representable *ARA* with cardinality κ, fork index γ and 4 ideal elements;
- for every cardinal $\zeta < \kappa$, the direct product $2^{\zeta} \times \mathcal{P}_{\gamma} \times I_{\kappa}$ is a representable *ARA* with cardinality κ, fork index γ and $2^{\zeta+2}$ ideal elements.

The following table gives an overview of the examples of elastic infinite proper algebra of relations presented. For each infinite cardinal $\kappa \geq \aleph_0$ the following representable algebras of relations are pairwise non-isomorphic elastic (so, non-Boolean) *ARA*'s with cardinality κ:

Fork index $\aleph_0 \leq \gamma \leq \kappa$	$\aleph_0 \leq \gamma < \kappa$ (non-explosive)	$\gamma = \kappa$ (explosive)
Cardinal $\zeta < \kappa$	$2^{\zeta} \times \mathcal{P}_{\gamma} \times I_{\kappa}$	$2^{\zeta} \times \mathcal{P}_{\kappa}$

7. *Conclusion*

Fork algebras have been introduced because their equational calculus has applications in program construction. They also have some interesting connections with algebraic logic.

We have examined the finite and infinite fork algebras and their relational reducts. The finite fork algebras are completely described as the finite direct powers of the two-element fork algebra, showing that they are essentially Boolean algebras. In contrast, the infinite fork algebras display a large diversity of behaviours even when their relational reducts are kept fixed.

To examine the contrast between fork algebras and their relational reducts that occurs in the infinite case, we have introduced some novel concepts (such as fork index) and techniques (for the analysis and construction of fork algebras).

This comparison between relational and fork algebras is based on the concept of fork index, which indicates how much information about a fork algebra is already given by its relational reduct. This immediately suggests classifying relational algebras according to their fork indices as non-forkable, rigid, elastic, explosive, etc.

As a technique for the analysis of fork algebras we have used the set of sub-identities of $2=1\nabla 1$, because of its close connections with the set of fixpoints of the underlying coding. A set-theoretical construction is provided to produce codings with prescribed sets of fixpoints, thus enabling the control of the sets of sub-identities of 2.

These ideas are used to exhibit examples of infinite relational algebras with a wide diversity of fork behaviour, ranging from none or a single one to high number of fork expansions.

We should mention that we have provided complete descriptions only for the finite (and for the Boolean) fork algebras. For the infinite fork algebras, we have concentrated on showing the diversity of their fork behaviour beyond the relational reduct.

The concept of fork index appears to capture an important aspect of the interplay between fork algebras and their reducts. The fork index appears to provide a useful manner for a more detailed classification of the spectrum. In this connection, our examples suggest some questions:

- are there infinite relational algebras with other fork indices?
- are there simple elastic, non-explosive relational algebras?

Other useful tools for the analysis of fork algebras, and particularly their relational reducts, are connected to the iterates of the projections and indices of their idempotency or nilpotency. These also appear to be controllable by variations of our set-theoretical construction.

Institute of Mathematics and COPPE
Federal University of Rio de Janeiro

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