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A PROOF SYSTEM FOR FORK ALGEBRAS AND ITS APPLICATIONS TO REASONING IN LOGICS BASED ON INTUITIONISM

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Abstract

Relational proof systems have been already proposed for certain modal, relevant and substructural logics. In this paper we present a general method for constructing Rasiowa-Sikorski-style deduction systems for nonclassical logics within the powerful relational framework of fork algebras. We apply the method to intuitionistic logic and a wide class of intermediate logics. The method consists in establishing interpretability of these logics in relational logics based on fork algebras (referred to as fork logics) and in developing a Rasiowa-Sikorski-like calculus for the respective fork logics. We prove soundness and completeness of the presented proof systems.

1. *Introduction*

Since the introduction of the formalism of relation algebras by Tarski [34], relational methods are increasingly applied in a variety of fields of logic, algebra and computer science. A sample of recent developments can be found in Brink et al. [1], Jaoua, A. [17], MacCaull [22], Lambek [20], Vakarelov [38], and Venema [41]. In this paper a relational method is proposed for reasoning in intuitionistic logic and some intermediate logics. The paradigm of relational formalization of logical systems is based on the principle of replacing any logic by a theory of a suitable class of algebras of relations [22, 27, 28, 29, 30]. In order to define such a theory for a given logic, the language of the logic is to be translated into a sufficiently expressive language of relational terms in a validity preserving manner, i.e. a

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logical formula α is valid iff its translation $T(\alpha)$ is a term such that $T(\alpha)=1$holds in every algebra of relations from the underlying class of algebras, where 1 is the Boolean unit of the algebra. In this paper we show that the class of fork algebras provides an adequate basis for intuitionistic reasoning. The paper is a continuation of our work presented in Frias and Orlowska [13], where modal and relevant logics have been represented in a fork algebra formalism referred to as fork logic. Fork logic is a logical formalism based on fork algebras. Fork algebras were extensively investigated in[5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 40]. In this paper, first, we define a Rasiowa-Sikorski style deduction system for fork logic. Second, we define a validity preserving translation from the language of intuitionistic logic and minimal intuitionistic logic into the language of fork logic. Next, we discuss three methods of intuitionistic reasoning within the framework of fork logic. The first method consists in extending the Rasiowa-Sikorski proof system of fork logic with some specific rules that reflect properties of the accessibility relation from Kripke models of logics based on intuitionism. The second method is based on a kind of relational deduction theorem that enables us to express derivability in fork logic of a term (representing a formula of a logic) from a finite number of terms (representing conditions on the accessibility relation). In this case the plain proof system for fork logic is an adequate deduction tool. The third method employs the equational theory of fork algebras. We extend this equational theory with equations that represent the required properties of the accessibility relation and treat them as specific axioms.

2. *Preliminaries*

In this section we define elementary concepts to be used in further sections. We introduce the notion of algebra of binary relations as well a their abstract counterpart, relation algebras. We will also present arithmetical properties of relation algebras that will be used without making explicit mention in the proofs of the theorems to come.

Given a binary relation X in a set A, and $a, b \in A$, we will denote the fact that a and b are related via the relation X by $\langle a, b\rangle \in X$ or aXb.

Definition 2.1 Let E be a binary relation on a set A, and let R be a set of binary relations satisfying:

1. $\bigcup R \subseteq E$,
2. Let Id denote the identity relation on the set A. Then $\emptyset$, E and Id belong to R,
3. R is closed under set union, intersection and complement relative to E,

4. R is closed under relational composition (denoted by $|$) and converse (denoted by $\breve{}$). These two operations are defined by
   $X|Y = \{\langle a, b\rangle : \exists c \text{ such that } aXc \wedge cYb\}$
   $\breve{X} = \{\langle a, b\rangle : bXa\}$

Then, the structure $\langle R, \cup, \cap, \bar{}, \varnothing, E, |, Id, \breve{}\rangle$ is called an *algebra of binary relations* (*ABR* for short).

Definition 2.2 A *relation algebra* is an algebraic structure

$$\mathfrak{R} = \langle R, +, \cdot, \bar{}, 0, 1, ;, 1', \breve{}\rangle$$

satisfying the following set of axioms
1. Axioms stating that $\langle R, +, \cdot, \bar{}, 0, 1,\rangle$ is a Boolean algebra.
2. Axioms stating that $\langle R, ;, 1'\rangle$ is a semigroup, in which ; is called the *relative product*, and 1' is called the *identity*.
3. The formula $(r;s)\cdot t = 0 \Leftrightarrow (\breve{r};t)\cdot s = 0 \Leftrightarrow (t;\breve{s})\cdot r = 0$ holds in $\mathfrak{R}$ for any $r, s, t \in R$. The operator $\breve{}$ is called *relational converse*.

As an immediate consequence of Defs. 2.1 and 2.2 we obtain the following theorem.

Theorem 2.3 *Every algebra of binary relations is a relation algebra.*

Notice that in the similarity type of relation algebras we have three constants 0, 1 and 1'. Since 0 and 1 are the least and the greatest elements in a Boolean algebra respectively, they can be defined as $0 = 1'\cdot\overline{1'}$ and $1 = 1'+\overline{1'}$. We will use these definitions in further sections. We will also define $0' = \overline{1'}$. The relation 0' is called the *diversity* relation.

Notation 2.1 • In general we will denote algebras and structures by capital german letters ($\mathfrak{A}$, $\mathfrak{B}$ ...), and their universes by the associated roman letter ($A, B, \ldots$).

- Given algebras $\mathfrak{A}$ and $\mathfrak{B}$, by $\mathfrak{A} \preceq \mathfrak{B}$ we denote the fact that $\mathfrak{A}$ is embeddable in $\mathfrak{B}$.
- Given a set S, by $\mathcal{P}(S)$ we denote the power set of S.
- Given a relation algebra $\mathfrak{R}$ and $r \in R$, by *Dom* (r) and *Ran* (r) we denote the relations $(r;\breve{r})\cdot 1'$ and $(\breve{r};r)\cdot 1'$, respectively. When interpreted in an algebra of binary relations, these relational terms yield partial identities over the domain and range of r, respectively.

- Given a concrete relation r (a set of pairs), by dom (r) and ran (r) we denote the sets $\{x : \exists y(\langle x, y\rangle \in r)\}$ and $\{y : \exists x(\langle x, y\rangle \in r)\}$, respectively.
- A relation F is called *functional* if $\breve{F}; F \leq 1'$.
- A relation C is called *constant* if it satisfies the following properties:
  1. C is functional,
  2. C=1;C,
  3. C;1=1

  When viewed as concrete relations, constant relations have in their range one unique element (the constant being represented), and all the objects are related with it. Condition 3 is necessary in order to guarantee that C is nonempty.
- A relation r is said to be a *right-ideal* if $r = r;1$.

3. *Fork Algebras*

The class of proper fork algebras (*PFA* for short) is a extension of the class *ABR* [19, 34] with a new operator called *fork*, and denoted by $\underline{\nabla}$. This new operator induces a structure on the underlying domain of algebras from *PFA*. The objects, instead of being binary relations on a plain set, are binary relations on a structured domain $\langle A, \star\rangle$, where $\star$ fulfills some simple conditions. Fig. 1 shows the relationship existing between fork and $\star$, namely, that fork is defined in terms of $\star$.

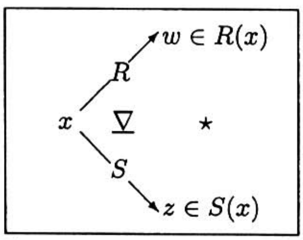

Figure 1: The operator *fork*

As a particular instance of the application of the operator fork, we have the relation $1' \,\underline{\nabla}\, 1'$. When this relation is viewed in a proper fork algebra, it can be understood as a copying operation, that produces two copies of a given element. Fig. 2 illustrates its definition. We denote this relation by *2*.

3.1 *Proper Fork Algebras*

In order to define *PFA*, we will first define the class of powerset $\star$*PFA* by

Definition 3.1 A powerset proper fork algebra (denoted by $\star PFA$) is a two sorted structure with domains $\mathcal{P}(V)$ and U

$$\langle \mathcal{P}(V), U, \cup, \cap, ', \emptyset, V, \mid, Id, \breve{\ }, \underline{\nabla}, \star \rangle$$

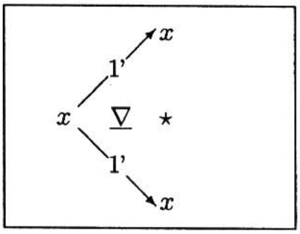

Figure 2: The relation $\mathcal{Q}$.

such that

1. V is an equivalence relation with domain U,
2. $\mid$, Id, $\breve{\ }$ and $'$ stand for composition between binary relations, the diagonal relation on U, the converse of binary relations, and set complementation with respect to V, respectively. Thus, the relation algebraic reduct $\langle \mathcal{P}(V), \cup, \cap, ', \emptyset, V, \mid, Id, \breve{\ } \rangle$ is an algebra of binary relations,
3. $\star : U \times U \to U$ is an injective function when its domain is restricted to V,
4. whenever xVy and xVz, also $xV\star(y, z)$,
5. $R\underline{\nabla}S = \{\langle x, *(y, z)\rangle : xRy \text{ and } xSz\}$.

Definition 3.2 The class PFA is defined as $S\ Rd[\star PFA]$ where Rd takes reducts to the similarity type $\langle \cup, \cap, ', \emptyset, V, \mid, Id, \breve{\ }, \underline{\nabla} \rangle$ and S takes subalgebras.

Definition 3.3 Notice that according to Def. 3.1 each algebra $\mathfrak{A} \in PFA$ contains a set U on which the binary relations are defined. This set will be called the *underlying domain* of $\mathfrak{A}$, and will be denoted by $U_{\mathfrak{A}}$.

In Defs. 3.1 and 3.2, the function $\star$ performs the rôle of pairing, encoding pairs of objects into single objects. It is important to notice that there are $\star$ functions which are distinct from set-theoretical pair formation, i.e., $\star(x, y)$ differs from $\{x, \{x, y\}\}$.

In the proof of several theorems it will be necessary to explicitly construct proper fork algebras. Let us consider the following definition.

Definition 3.4 A $PFA\ \langle \mathcal{P}(V), \cup, \cap, ', \emptyset, V, \mid, Id, \breve{\ }, \underline{\nabla} \rangle$ is called a full proper fork algebra (FullPFA) if $V = A \times A$ for some set A.

Notice that in order to define a Full*PFA* it suffices to provide the set A and an injective mapping $\star : A \times A \to A$.

If by P we denote the closure operator that closes classes of algebras under direct products, from Defs. 3.2 and 3.4 we obtain the following result.

Theorem 3.5 $PFA = S\ P$ FullPFA

Given a pair of binary relations, the operation called *cross* performs a kind of parallel product. A graphic representation of cross is given in Fig. 3. Its set theoretical definition is given by

$$R \otimes S = \{\langle \star(x,y), \star(w,z) \rangle : xRw \wedge ySz\}.$$

It is not difficult to check that cross is definable from the other relational operators with the use of fork. It is a simple exercise to show that

$$R \otimes S = ((Id \underline{\nabla} V)^{\smile} | R) \underline{\nabla} ((V \underline{\nabla} Id)^{\smile} | S).$$

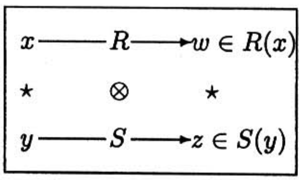

Figure 3: The operator *cross.*

3.2 *Abstract Fork Algebras*

Much the same as relation algebras [19, 23, 34] are an abstract version of algebras of binary relations, *PFA* has also its abstract counterpart. The class of abstract fork algebras (*AFA* for short) is a finitely based variety, i.e., its axiomatization is given by a finite set of equations.

Definition 3.6 An abstract fork algebra is an algebraic structure

$$\left\langle R, +, \cdot, {}^{-}, 0, 1, ;, 1', {}^{\smile}, \nabla \right\rangle$$

satisfying the following set of axioms

1. Axioms stating that $\langle R, +, \cdot, {}^{-}, 0, 1, ;, 1', {}^{\smile} \rangle$ is a relation algebra[1] in which the structure $\langle R, +, \cdot, {}^{-}, 0, 1 \rangle$ is a Boolean algebra, $\langle R, ;, 1' \rangle$ is a monoid, and ${}^{\smile}$ stands for the relational converse,

[1] Equational axiomatic systems for relation algebras are given in [2, 33, 35].

2. $r\nabla s = (r;(1'\nabla 1))\cdot(s;(1\nabla 1'))$,
3. $(r\nabla s);(t\nabla q)^{\smile} = (r;\breve{t})\cdot(s;\breve{q})$,
4. $(1'\nabla 1)^{\smile}\nabla(1\nabla 1')^{\smile} \leq 1'$, where $\leq$ is the partial ordering induced by the Boolean part.

From the abstract definition of fork induced by the axioms in Def. 3.6, it is possible to define cross by the equation

$$R\otimes S = ((1'\nabla 1)^{\smile};R)\nabla((1\nabla 1')^{\smile};S).$$

Once *AFA* is defined, a question is immediately raised: What is the relationship between *PFA* and *AFA*? It is clear that the axiomatization of *AFA* is valid in *PFA*, but the inverse is not obvious. This problem was already posed for relation algebras by Tarski in [34], when he was trying to establish the relationship between algebras of binary relations and models of the calculus of relations. In [21], Lyndon shows with an example that some relation algebras cannot be viewed as algebras of binary relations. The next theorem will show that the representability problem has a positive answer for fork algebras. The complete proof is given in [11].

Theorem 3.7 *Every abstract fork algebra is representable, i.e., given an abstract fork algebra* $\mathfrak{A}$, *there exists a proper fork algebra* $\mathfrak{B}$ *and an isomorphism* $h: A \to B$.

The relations $(1'\nabla 1)^{\smile}$ and $(1\nabla 1')^{\smile}$ behave as projections, projecting components not necessarily of set-theoretical pairs, but of pairs constructed with the injective function $\star$. We call them respectively π and ρ. They allow to cope with the lack of variables over individuals of fork algebras. Fig. 4 illustrates the meaning of these relations. We introduce the constant ${}_U1$ *by* ${}_U1 = \overline{(1\nabla 1)^{\smile}}$. This constant is a right-ideal relation whose domain has only *urelements,* i.e., primitive objects whose construction does not involve the pairing function $\star$. We also define the constant 1_U by $1_U = \overline{(1\nabla 1)}$ and $1_{\overline{U}} = \overline{1}_U . 1_U$ has urelements in the range. Also a partial identity ranging only over urelements is defined by $1'_{U} = ({}_U1;1_U)\cdot 1'$.

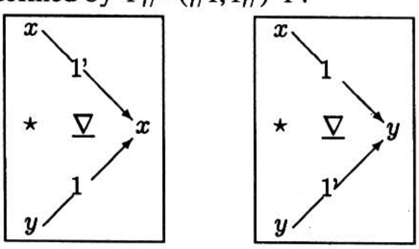

Figure 4: The projections π and ρ.

Definition 3.8 The class of fork algebras *with urelements* (*FAU* for short) is the subclass of *AFA* satisfying the equation $1;1'_{\cup};1=1$. This equation is denoted by *Ur*.

Notice that when a proper fork algebra $\mathfrak{A}$ is being considered in Def. 3.8, the condition *Ur* is equivalent to the existence of a subset of $U_{\mathfrak{A}}$ (denoted by $Urel_{\mathfrak{A}}$) disjoint from the range of the pairing function $\star$, i.e., the set $Urel_{\mathfrak{A}}$ contains those elements that are not pairs. The class of proper fork algebras with urelements will be denoted by *PFAU*.

Definition 3.9 We define the class of simple proper fork algebras with urelements (denoted by *SPFAU*) as the subclass of *PFAU* that satisfies the condition

$$\forall R(R \neq 0 \Rightarrow 1;R;1=1). \tag{1}$$

It was shown in [19] that in the case of relation algebras (and the same is true for fork algebras), condition (1) is equivalent to the standard notion of simple algebra.

Theorem 3.10 *Let* $\mathcal{V}$ *be the variety generated by SPFAU. Then* $\mathcal{V}$ = *FAU*.

Proof. It is clear that $\mathcal{V} \subseteq FAU$. Let us prove the other inclusion. Let $\mathfrak{A} \in FAU$ Since in fork algebras, as well as in relation algebras, every algebra can be decomposed as a subdirect product of simple algebras, $\mathfrak{A} \preceq \prod_{i \in I} \mathfrak{A}_i$ with $\mathfrak{A}_i$ simple for all $i \in I$. By the representation theorem (Thm. 3.7), $\mathfrak{A} \preceq \prod_{i \in I} \mathfrak{B}_i$ with $\mathfrak{B}_i \in SPFAU$ for all $i \in I$. Therefore, by definition of $\mathcal{V}$, $\mathfrak{A} \in \mathcal{V}$. ■

Regarding the expressiveness of fork algebras, it was proved [5, 39, 40, 13] that first-order theories can be interpreted as equational theories in fork algebras. Formally, let L be a first-order language. We assume variables constitute an infinite countable set and are denoted by $v_n (n \leq \omega)$, constant symbols in L are denoted by $c_i (i \in I)$, function symbols are denoted by $f_j (j \in J)$, and predicate symbols by $p_k (k \in K)$. Let us denote by $\langle A, L' \rangle$ the extension of the similarity type of abstract fork algebras (A) with a sequence of constant symbols L' containing symbols $C_i (i \in I)$, $F_j (j \in J)$ and $P_k (k \in K)$. These new symbols will stand for binary versions of constants, functions and predicates, and therefore satisfy the following conditions.

1. C_i is a constant relation for all $i \in I$,
2. F_j is a functional relation for all $j \in J$, and
3. P_k is a right ideal relation for all $k \in K$.

In what follows, $t^{;n}$ is an abbreviation for $(t; ...;t)$ n times. For the sake of completeness, $t^{;0}$ is defined as 1'.

The following mapping translates first-order formulas into fork algebra terms. The definition proceeds in two steps, since also terms (not only formulas) need to be translated.

For the following definitions, σ will be a sequence of numbers increasingly ordered. Intuitively, the sequence will contain the indices of those variables that appear free in the formula (or term) being translated. By $Ord(n,\sigma)$ we will denote the position of the index n in the sequence σ, by $[\sigma \oplus n]$ we denote the extension of the sequence σ with the index n, and by $\sigma(k)$ we denote the element in the k-th position of σ.

Definition 3.11 The mapping δ_σ translating first-order terms into fork algebraic ones, is defined inductively by the conditions:

1. $\delta_\sigma(v_i) = \begin{cases} \rho^{;Ord(i,\sigma)-1};\pi & \text{if } i \text{ is not the last index in } \sigma, \\ \rho^{;Length(\sigma)-1} & \text{if } i \text{ is the last index in } \sigma. \end{cases}$
2. $\delta_\sigma(c_i) = C_i$
3. $\delta_\sigma(f_i(t_1, \ldots, t_m)) = (\delta_\sigma(t_1)\nabla \ldots \nabla \delta_\sigma(t_m)); F_i$

Before defining the mapping T_σ translating first-order formulas, we need to define some auxiliary terms. Given a sequence σ such that $Length(\sigma) = l$, we define the term $\Delta_{\sigma,n}(n < \omega)$ by the condition

$$\Delta_{\sigma,n} = \begin{cases} \delta_\sigma(v_{\sigma(1)})\nabla \ldots \nabla \delta_\sigma(v_{\sigma(k-1)})\nabla 1_U \nabla \delta_\sigma(v_{\sigma(k+1)})\nabla \ldots \nabla \delta_\sigma(v_{\sigma(l)}) \\ \qquad \text{if } k = Ord(n,[\sigma \oplus n]) < l, \\ \delta_\sigma(v_{\sigma(1)})\nabla \ldots \nabla \delta_\sigma(v_{\sigma(l-1)})\nabla 1_U \\ \qquad \text{if } Ord(n,[\sigma \oplus n]) = l. \end{cases}$$

The term $\Delta_{\sigma,n}$ can be understood as a cylindrification [16] in the k-th. coordinate of a l-dimensional space.

Definition 3.12 Once the mapping δ_σ is defined, we are ready for defining the mapping T_σ, translating first-order formulas onto fork algebraic terms.

1. $T_\sigma(t_1 \equiv t_2) = (\delta_\sigma(t_1)\nabla \delta_\sigma(t_2)); \breve{2}; 1,$
2. $T_\sigma(p_n(t_1, \ldots, t_m)) = (\delta_\sigma(t_1)\nabla \ldots \nabla \delta_\sigma(t_m)); P_n,$
3. $T_\sigma(\neg\alpha) = \overline{T_\sigma(\alpha)},$
4. $T_\sigma(\alpha \vee \beta) = T_\sigma(\alpha) + T_\sigma(\beta),$
5. $T_\sigma(\alpha \wedge \beta) = T_\sigma(\alpha) \cdot T_\sigma(\beta),$
6. $T_\sigma(\exists v_n(\alpha)) = \Delta_{\sigma,n}; T_{[\sigma \oplus n]}(\alpha),$
7. $T_\sigma(\forall v_n(\alpha)) = T_\sigma(\neg\exists v_n \neg\alpha).$

We will present the expressiveness theorem without proof. The complete proof follows the lines of the proof given in [13]. We will denote the empty sequence of indices by $\langle\ \rangle$.

Theorem 3.13 *For any set of first-order sentences* Γ *and any first-order sentence* α *we have*

$$\Gamma \vdash \alpha \Leftrightarrow \{T_{\langle\ \rangle}(\gamma)=1 : \gamma \in \Gamma\} \vdash_{\nabla, Ur} T_{\langle\ \rangle}(\alpha)=1.$$

The symbol $\vdash_{\nabla, Ur}$ in Thm. 3.13 is to be understood as provability in equational logic under the theory of abstract fork algebras with urelements.

4. *The Logic FL*

In this section we will present the logic *FL*, also called the logic of fork algebras, or more succinctly, fork logic.

4.1 *Syntax of FL*

Definition 4.1 Let *IndVar* and *RelVar* be two disjoint, infinite countable sets, and let *RelConst* be a finite set such that $1' \notin$ *RelConst*, also disjoint from *IndVar* and *RelVar*. The set *IndTerm* of terms over individuals is the smallest set satisfying:

1. *IndVar* $\subseteq$ *IndTerm*,
2. If $t_1, t_2 \in$ *IndTerm*, then the expression $t_1 \star t_2 \in$ *IndTerm*,

We will also assume that there are infinite disjoint sets *UreVar* and *CompVar* such that *IndVar* = *UreVar* $\cup$ *CompVar*.

The set *RelTerm* of terms over relations is the smallest set satisfying conditions:

1. *RelVar* $\cup$ *RelConst* $\subseteq$ *RelTerm*,
2. If $t_1, t_2 \in$ *RelTerm*, then $\{t_1 + t_2, t_1 \cdot t_2, t_1 ; t_2, \overline{t_1}, \breve{t_1}, 1'\} \subseteq$ *RelTerm*,

We define the set of formulas of the logic *FL* (denoted by *ForkFor*) as the set $\{ t_1 R t_2 : t_1, t_2 \in$ *IndTerm* and $R \in$ *RelTerm*$\}$.

Notice that any subset *X* of *RelConst* determines a language. These languages will be referred to as fork languages and denoted by $\mathfrak{L}(X)$.

4.2 *Semantics of FL*

Definition 4.2 Given a fork language $\mathfrak{L}(R_1, \ldots, R_k)$ an adequate structure for $\mathfrak{L}$ is a tuple $\langle \mathfrak{A}, R_1', \ldots, R_k' \rangle$ such that

1. $\mathfrak{A}$ is a *PFAU*,
2. $R_1' \in A$ for all $i, 1 \leq i \leq k$.

Definition 4.3 A fork model for a language $\mathfrak{L}(R_1, \ldots, R_k)$ is a tuple $\mathfrak{M}_{\mathfrak{A}} = \langle \mathfrak{U}, m \rangle$ such that
1. $\mathfrak{U} = \langle \mathfrak{A}, R_1', \ldots, R_k' \rangle$ is an adequate structure for $\mathfrak{L}$,
2. $m(R) \in A$ for all $R \in RelVar$,
3. $m(R_i) = R_i'$ for all $i, 1 \leq i \leq k$,
4. $m(1') = Id \in \mathrm{A}$.

Clearly m extends homomorphically to a function m': $RelTerm \to A$. For the sake of simplicity we will denote both m and m' by m. Notice that in particular $m(0) = \emptyset$ and $m(1) = V$, the greatest relation of the fork algebra $\mathfrak{A}$.

Definition 4.4 Given a fork model $\mathfrak{M}_{\mathfrak{A}}$, a valuation over $\mathfrak{M}_{\mathfrak{A}}$ is a mapping $v : IndVar \to U_{\mathfrak{A}}$ satisfying
1. $v(x) \in Urel_{\mathfrak{A}}$ if $x \in UreVar$,
2. $v(x) \in U_{\mathfrak{A}}$ if $x \in CompVar$.

Every valuation v extends homomorphically to a mapping v': $IndTerm \to U_{\mathfrak{A}}$. We will denote both v and v' by v.

Definition 4.5 A fork formula $t_1 R t_2$ is satisfied in a model $\mathfrak{M}$ by a valuation v (denoted by $\mathfrak{M}, v \models_{FL} t_1 R t_2$) if $\langle v(t_1), v(t_2) \rangle \in m(R)$.

Definition 4.6 A fork formula $t_1 R t_2$ is true in a model $\mathfrak{M}$ (denoted by $\mathfrak{M} \models_{FL} t_1 R t_2$) if for every valuation v, $\mathfrak{M}, v \models_{FL} t_1 R t_2$.

Definition 4.7 A fork formula $t_1 R t_2$ is valid in *FL* (denoted by $\models_{FL} t_1 R t_2$) if it is true in all fork models.

This notion of validity extends in a natural way to sequences of formulas $\gamma_1, \gamma_2, \ldots, \gamma_k$.

Definition 4.8 A sequence of formulas $\gamma_1, \gamma_2, \ldots, \gamma_k$ is valid if for any fork model $\mathfrak{M}$ and any valuations v over $\mathfrak{M}$, there exists $i, 1 \leq i \leq k$, such that $\mathfrak{M}, v \models_{FL} \gamma_i$.

Finally, given sequences of formulas $\Gamma_1, \ldots, \Gamma_n$, we define:

Definition 4.9 The family of sequences of formulas $(\Gamma_i)_{1 \leq i \leq n}$ is valid if for all $i, 1 \leq i \leq n$, the sequence Γ_i is valid.

5. *A Rasiowa-Sikorski Calculus for Fork Logic*

The original Rasiowa-Sikorski proof system presented in [32] refers to the classical predicate logic. The system is designed for verification of validity of formulas of this logic. It consists of a pair of rules for each propositional connective and each quantifier. Every pair of rules, in turn, consists of a "positive" rule and a "negative" rule. A positive (resp. negative) rule exhibits the logical behaviour of the underlying connective or quantifier (negated connective or negated quantifier). For example, the rules for conjunction are the following

$$\frac{\Gamma, \alpha \wedge \beta, \Delta}{\Gamma, \alpha, \Delta \quad \Gamma, \beta, \Delta} \quad (P\wedge)$$

$$\frac{\Gamma, \neg(\alpha \wedge \beta), \Delta}{\Gamma, \neg\alpha, \neg\beta, \Delta} \quad (N\wedge)$$

The system operates in a top-down manner. Application of a rule results in a decomposition of a given formula into the formulas that are the arguments of a respective connective or quantifier. In general, the rules apply to finite sequences of formulas. To apply a rule we choose a formula in a sequence that is to be decomposed and we replace it by its components, thus obtaining either a single new sequence (for 'or'-like connectives) or a pair of sequences (for 'and'-like connectives). In the process of decomposition we form a tree whose nodes consist of finite sequences of formulas. We stop applying rules to the formulas in a node after obtaining an axiom sequence (appropriately defined) or when none of the rules is applicable to the formulas in this node. If a decomposition tree of a given formula is finite, then its validity can be syntactically recognized from the form of the sequences appearing in the leaves of the tree. In the present section we define a Rasiowa-Sikorski style system for the fork logic *FL*. The system is an extension of the proof system presented in Orlowska [27, 30]. The system consists of a positive and a negative decomposition rule for each relational operation from the language of fork logic, and moreover of the specific rules that reflect properties of the function $\star$ and the relational constant 1'.

5.1 *The Deduction System for Fork Logic*

In this subsection we will present the rules of the sequent calculus *FLC* for fork logic. Since we are dealing with fork algebras with urelements (required in order to interpret first-order theories), the calculus we present is more involved than a calculus for fork algebras when no assumption is done on the existence of urelements.

$$\frac{\Gamma,\ xR+Sy,\ \Delta}{\Gamma,\ xRy,\ xSy,\ \Delta}\ (P+) \qquad \frac{\Gamma,\ x\overline{R+S}y,\ \Delta}{\Gamma,\ x\overline{R}y,\ \Delta \quad \Gamma,\ x\overline{S}y,\ \Delta}\ (N+)$$

$$\frac{\Gamma,\ xR\cdot Sy,\ \Delta}{\Gamma,\ xRy,\ \Delta \quad \Gamma,\ xSy,\ \Delta}\ (P\cdot) \qquad \frac{\Gamma,\ x\overline{R\cdot S}y,\ \Delta}{\Gamma,\ x\overline{R}y,\ x\overline{S}y,\ \Delta}\ (N\cdot)$$

$$\frac{\Gamma,\ xR;Sy,\ \Delta}{\Gamma,\ xRz,\ \Delta,\ xR;Sy \quad \Gamma,\ zSy,\ \Delta,\ xR;Sy}\ (P;) \qquad \frac{\Gamma,\ x\overline{R;S}y,\ \Delta}{\Gamma,\ x\overline{R}z_1,\ z_1\overline{S}y,\ \Delta \quad \Gamma,\ x\overline{R}z_2,\ z_2\overline{S}y,\ \Delta}\ (N;)$$

$$\frac{\Gamma,\ x\overline{\overline{R}}y,\ \Delta}{\Gamma,\ xRy,\ \Delta}\ (N^-)$$

$$\frac{\Gamma,\ x\breve{R}y,\ \Delta}{\Gamma,\ yRx,\ \Delta}\ (P\breve{\ }) \qquad \frac{\Gamma,\ x\overline{\breve{R}}y,\ \Delta}{\Gamma,\ y\overline{R}x,\ \Delta}\ (N\breve{\ })$$

$$\frac{\Gamma,\ xR\nabla Sy,\ \Delta}{\Gamma,\ y1'u\star v,\ \Delta,\ xR\nabla Sy \quad \Gamma,\ xRu,\ \Delta,\ xR\nabla Sy \quad \Gamma,\ xSv,\ \Delta,\ xR\nabla Sy}\ (P\nabla)$$

$$\frac{\Gamma,\ x\overline{R\nabla S}y,\ \Delta \quad (N\nabla)}{\begin{array}{ll}\Gamma,\ y0'u_1\star v_1,\ x\overline{R}u_1,\ x\overline{S}v_1,\ \Delta & \Gamma,\ y0'u_2\star v_2,\ x\overline{R}u_2,\ x\overline{S}v_2,\ \Delta \\ \Gamma,\ y0'u_3\star v_3,\ x\overline{R}u_3,\ x\overline{S}v_3,\ \Delta & \Gamma,\ y0'u_4\star v_4,\ x\overline{R}u_4,\ x\overline{S}v_4,\ \Delta\end{array}}$$

$$\frac{\Gamma,\ x_1\star x_2\,1'\,y_1\star y_2,\ \Delta}{\Gamma,\ x_1 1' y_1,\ \Delta \quad \Gamma,\ x_2 1' y_2,\ \Delta}\ (P1') \qquad \frac{\Gamma,\ x_1\star x_2\,0'\,y_1\star y_2,\ \Delta}{\Gamma,\ x_1 0' y_1,\ \ x_2 0' y_2,\ \Delta}\ (N1')$$

$$\frac{\Gamma,\ xRy,\ \Delta}{\Gamma,\ x1'z,\ xRy,\ \Delta \quad \Gamma,\ xRy,\ xRy,\ \Delta}\ (1'_a)$$

$$\frac{\Gamma,\ xRy,\ \Delta}{\Gamma,\ xRz,\ xRy,\ \Delta \quad \Gamma,\ z1'y,\ xRy,\ \Delta}\ (1'_b)$$

$$\frac{\Gamma,\ x1'y,\ \Delta}{\Gamma,\ y1'x,\ \Delta,\ x1'y}\ (Sym)$$

$$\frac{\Gamma, x1\text{'}y, \Delta}{\Gamma, x1\text{'}z, \Delta, x1\text{'}y \quad \Gamma, z1\text{'}y, \Delta, x1\text{'}y} \quad (Trans)$$

$$\frac{\Gamma, x1\text{'}y, \Delta}{\Gamma, x\star u1\text{'}y\star v, \Delta, x1\text{'}y \quad \Gamma, x\star u0\text{'}y\star v, \Delta, x1\text{'}y} \quad (Cut)$$

$$\frac{\Gamma}{\Gamma, x1\text{'}y} \quad (U)$$

In rule (*P*;), $z \in IndTerm$ is arbitrary. In rule (*N*;), $z_1 \in UreVar$ and $z_2 \in CompVar$. In rule $(P\nabla)$, $u, v \in IndTerm$ are arbitrary. In rule $(N\nabla\ u_1,)$, $u_2, v_1, v_3 \in UreVar$ and $u_3, u_4, v_2, v_4 \in CompVar$. In rules (1'_a) and (1'_b), $z \in IndVar$ is arbitrary. In rule (*Trans*), $z \in IndTerm$ is arbitrary. In rule (*Cut*), $u, v \in IndTerm$ are arbitrary. Finally, in rule (*U*), $x \in UreVar$ and $y \in IndTerm \setminus UreVar$.

Definition 5.1 A formula t_1Rt_2 is called *indecomposable* if it satisfies either of the following conditions.
1. $R \in RelVar \cup RelConst$,
2. $R = \overline{S}$ and $S \in RelVar \cup RelConst$,
3. $R \in \{1\text{'}, 0\text{'}\}$

Definition 5.2 A sequence of formulas Γ is called *indecomposable* if all the formulas in Γ are indecomposable.

Definition 5.3 A sequence of formulas Γ is called *fundamental* if either of the following is true.
1. Γ contains simultaneously the formulas t_1Rt_2 and $t_1\overline{R}t_2$, for some $t_1, t_2 \in IndTerm$ and $R \in RelTerm$.
2. Γ contains the formula $t1\text{'}t$ for some $t \in IndTerm$.

Definition 5.4 Let *T* be a tree satisfying:
1. Each node contains a finite sequence of fork formulas.
2. If the sequences of fork formulas $\Delta_1, \ldots, \Delta_2$ are the immediate successors of the sequence of fork formulas Γ, then there exists an instance of a rule from *FLC* of form

$$\frac{\Gamma}{\Delta_1 \quad \Delta_2 \ldots \Delta_k}.$$

Then *T* is a *proof tree*

A branch in a proof tree is called *closed* if it ends in a fundamental sequence.

Definition 5.5 A formula $t_1 R t_2$ is provable in the calculus *FLC* iff there exists a proof tree T satisfying:
1. T is finite,
2. $t_1 R t_2$ is the root of T,
3. Each leaf of T contains a fundamental sequence.

5.2 *Soundness and Completeness of the Calculus FLC*

Theorem 5.6 *The calculus FLC is sound w.r.t. FL.*

Proof. The proof proceeds in two steps. First, we prove that for any rule the upper sequence of the rule is valid if and only if all the lower sequences are valid. This property of the rules will be referred to as their admissibility. Once the first step is established, the second step is an induction on the structure of the proof tree as follows:
1. If the tree has height 1 (i.e., the root is a fundamental sequence), then it is trivially valid.
2. Assume that if the tree has height less or equal than n, then the fact that all leaves contain fundamental sequences implies that the sequence in the root is valid.
3. Let T be a tree with height $n+1$. If the transition from the root to the nodes in the first level was obtained applying a rule $\mathfrak{R}$ of the form

$$\frac{\Gamma}{\Gamma_1 \quad \Gamma_2 \ldots \Gamma_k},$$

let us call $T_i (1 \leq i \leq k)$ the subtree of T with root T_i. Since for all i the height of T_i is less or equal than n and all the leaves contain fundamental sequences, the root of each T_i must contain a valid sequence. Since rules preserve validity in both directions, then the sequence Γ must be valid, as was to be proved.

Let us show as an example that the rule ($P\nabla$) is admissible. That the remaining rules are admissible is proved in a similar way.

Let us consider a sequence of fork formulas $\Gamma, t_1 R \nabla S t_2, \Delta$ from a language $\mathfrak{L}(R_1, \ldots, R_k)$. Let $\mathfrak{M} = \langle\langle \mathfrak{A}, R_1', \ldots, R_{k'}' \rangle, m\rangle$ be a fork model, and let v be a valuation over $\mathfrak{M}$.

If $\mathfrak{M}, v \models_{FL} \Gamma, t_1 R \nabla S t_2, \Delta$ then the following three possibilities arise.
1. $\mathfrak{M}, v \models_{FL} \gamma$, with $\gamma \in \Gamma$

2. $\mathfrak{M}, v \models_{FL} \delta$, with $\delta \in \Delta$
3. $\mathfrak{M}, v \models_{FL} t_1 R\nabla S t_2$

If 1or 2 are true, then it is immediate that the three sequences in the lower part of the rule ($P\nabla$) are satisfied in the fork model $\mathfrak{M}$ by the valuation v. If 3 is true, then, since the fork formula $t_1 R\nabla S t_2$ is repeated in the three sequences in the lower part of the rule, then these sequences are also satisfied in the fork model $\mathfrak{M}$ by the valuation v.

On the other hand, if

$$\mathfrak{M}, v \models_{FL} \Gamma, t_2 1' u{\star}v, \Delta, t_1 R\nabla S t_2,$$
$$\mathfrak{M}, v \models_{FL} \Gamma, t_1 Ru, \Delta, t_1 R\nabla S t_2, \text{ and}$$
$$\mathfrak{M}, v \models_{FL} \Gamma, t_1 Sv, \Delta, t_1 R\nabla S t_2,$$

then the following four possibilities arise:

1. $\mathfrak{M}, v \models_{FL} \gamma$ with $\gamma \in \Gamma$,
2. $\mathfrak{M}, v \models_{FL} \delta$ with $\delta \in \Delta$,
3. $\mathfrak{M}, v \models_{FL} t_1 R\nabla S t_2$,
4. $\mathfrak{M}, v \models_{FL} t_2 1' u{\star}v$, $\mathfrak{M}, v \models_{FL} t_1 Ru$, and $\mathfrak{M}, v \models_{FL} t_1 Sv$.

If 1, 2 or 3 are true then clearly the sequence of fork formulas $\Gamma, t_1 R\nabla S t_2, \Delta$ is satisfied in the fork model $\mathfrak{M}$ by the valuation v. If 4 is true, then, by definition of fork (Defs. 3.1 and 3.2) $\mathfrak{M}, v \models_{FL} t_1 R\nabla S t_2$ and thus $\mathfrak{M}, v \models_{FL} \Gamma, t_1 R\nabla S t_2, \Delta$

Definition 5.7 A proof tree T of a sequence of formulas Γ is called *saturated* if, intuitively, all the applicable rules were applied in the open branches. Formally speaking, a proof tree of Γ is called saturated if for all open branches B, the following conditions are satisfied.

1. If $xR+Sy \in B$, then both $xRy \in B$ and $xSy \in B$ by an application of rule (P+).
2. If $x\overline{R+S}y \in B$, then either $x\overline{R}y \in B$ or $x\overline{S}y \in B$ by an application of rule (N+).
3. If $xR \cdot Sy \in B$, then either $xRy \in B$ or $xSy \in B$ by an application of rule ($P\cdot$).
4. If $x\overline{R \cdot S}y \in B$, then both $x\overline{R}y \in B$ and $x\overline{S}y \in B$ by an application of rule ($N\cdot$).
5. If $x\overline{\overline{R}}y \in B$, then $xRy \in B$, by an application of rule (N^-).
6. If $xR;Sy \in B$, then for all $t \in IndTerm$, either $xRt \in B$ or $tSy \in B$ by an application of rule (P;)

7. If $x\overline{R;S}y \in B$, then for some $z \in IndVar$ both $x\overline{R}z \in B$ and $z\overline{S}y \in B$ by an application of rule $(N;)$.
8. If $x\breve{R}y \in B$, then $yRx \in B$, by an application of rule $(P\breve{\ })$.
9. If $x\overline{\breve{R}}y \in B$, then $y\overline{R}x \in B$ by an application of rule $(N\breve{\ })$.
10. If $x{\star}y1'u{\star}v \in B$, then either $x1'u \in B$ or $y1'v \in B$ by an application of rule $(P1')$.
11. If $x{\star}y0'u{\star}v \in B$, then both $x0'u \in B$ and $y0'v \in B$ by an application of rule $(N1')$.
12 If $xRy \in B$, then for all $z \in IndVar$ either $x1'z \in B$ or $zRy \in B$ by an application of rule $(1'_a)$.
13. If $xRy \in B$, then for all $z \in IndVar$ either $xRz \in B$ or $z1'y \in B$ by an application of rule $(1'_b)$.
14. If $x1'y \in B$, then $y1'x \in B$ by an application of rule *Sym*.
15. If $x1'y \in B$, then for all $z \in IndTerm$ either $x1'z \in B$ or $z1'y \in B$ by an application of rule (*Trans*).
16. If $x1'y \in B$, then for all $u, v \in IndTerm$ either $x{\star}u1'y{\star}v \in B$ or $x{\star}u0y{\star}v \in B$ by an application of rule (*Cut*).
17. If $xR\nabla Sy \in B$, then for all $u, v \in IndTerm$ one of the formulas $y1'u{\star}v$, xRu or xSv is in B by an application of rule $(P\nabla)$.
18. If $x\overline{R\nabla S}y \in B$, then there are $u, v \in IndVar$ such that the formulas $y0'u{\star}v$, $x\overline{R}u$ and $x\overline{S}v$ are in B by an application of rule $(N\nabla)$.
19. For all $x \in UreVar$ and $y \in IndTerm \setminus UreVar$, $x1'y \in B$ by an application of rule (U).

Definition 5.8 We define the order of $R \in RelTerm$ (denoted by $o(R)$) by the conditions:
1. $o(R) = 1$ if $R \in RelConst \cup RelVar \cup \{1'\}$,
2. $o(R) = o(S) + 1$ if $R = \overline{S}$ or $R = \breve{S}$,
3. $o(R) = \max\{o(S), o(T)\} + 1$ if $R = S+T$, $R = S \cdot T$, $R = S;T$, or $R = S \nabla T$.

Theorem 5.9 *The calculus FLC is complete w.r.t. FL, i.e., if a formula tRt' is valid in FL, then it is provable in FLC.*

Proof. Assume *tRt'* is not provable in *FLC*. Then no proof tree exists that provides a proof for *tRt'*. In particular, no saturated tree with root *tRt'* provides a proof. Therefore, if T is a saturated tree, there must exist an infinite branch B in T.

Let $\equiv$ be the binary relation on *IndTerm* defined by
$x \equiv y \Leftrightarrow x1'y \notin B$.

Let us prove that $\equiv$ is an equivalence relation.

Since for all $T \in IndTerm$ $t1't \notin B$ (otherwise B would contain a fundamental sequence), $\equiv$ is reflexive.

If $t_1, t_2 \in IndTerm$ satisfy $t_1 \equiv t_2$ (or equivalently $t_1 1' t_2 \notin B$), then $t_2 \equiv t_1$. Otherwise, if $t_2 1' t_1 \in B$, then, by application of the rule (*Sym*) $t_1 1' t_2 \in B$, which is a contradiction.

If $t_1 \equiv t_2$ and $t_2 \equiv t_3$ ($t_1 1' t_2 \notin B$ and $t_2 1' t_3 \notin B$), let us show that $t_1 \equiv t_3$. If $t_1 \not\equiv t_3$, then $t_1 1' t_3 \in B$. Thus, by one application of the rule (*Trans*) either $t_1 1' t_2 \in B$ or $t_2 1' t_3 \in B$, which is a contradiction.

Let $\mathfrak{A}$ be the Full*PFA* with underlying domain $\{|x| : x \in IndTerm\}$ and pairing function $\star$ defined by $|x| \star |y| = |x \star y|$. If $|x_1| = |x_2|$ and $|y_1| = |y_2|$ then must be $|x_1 \star y_1| = |x_2 \star y_2|$. Otherwise, if $|x_1 \star y_1| \neq |x_2 \star y_2|$, then $x_1 \star y_1 1'x_2 \star y_2 \in B$. Applying rule (*P*1') either $x_1 1'x_2 \in B$ or $y_1 1'y_2 \in B$, which is a contradiction.

Let us check that $\star$ is injective. If $|t_1| \star |t_2| = |t_3| \star |t_4|$ then, by definition of $\star$, $|t_1 \star t_2| = |t_3 \star t_4|$. Thus, $t_1 \star t_2 1't_3 \star t_4 \notin B$. If $|t_1| \neq |t_3|$ then $t_1 1't_3 \in B$. Applying the rule (*Cut*) either $t_1 \star t_2 1't_3 \star t_4 \in B$ or $t_1 \star t_2\ 0't_3 \star t_4 \in B$. Since $t_1 \star t_2 1't_3 \star t_4 \notin B$, then $t_1 \star t_2\ 0't_3 \star t_4 \in B$. Applying rule (*N*1') yields that $t_1 0't_3 \in B$, thus B would be a closed branch, which contradicts our assumptions. We then conclude that $|t_1| = |t_3|$. In a similar way we prove that $|t_2| = |t_4|$.

Notice that $Urel_{\mathfrak{A}} = \{|x| : x \in UreVar\}$, because if $|x| = |t_1| * |t_2|$ then $|x| = |t_1 * t_2|$. Then, $x1't_1 * t_2 \notin B$. By applying rule (*U*) we arrive to a contradiction. Thus, $\mathfrak{A} \in PFAU$.

Let us define, for $\in RelVar \cup RelConst$,

$$\langle |t_1|, |t_2| \rangle \in m(R) \Leftrightarrow t_1 R t_2 \notin B$$

Let us check that m is well defined. Let us see that whenever $t_1 \equiv t_3$ and $t_2 \equiv t_4$, if $\langle |t_1|, |t_2| \rangle \in m(R)$ then $\langle |t_3|, |t_4| \rangle \in m(R)$ Since $\langle |t_1|, |t_2| \rangle \in m(R)$, $t_1 R\ t_2 \notin B$. If $\langle |t_3|, |t_4| \rangle \notin m(R)$, then $t_3 R\ t_4 \in B$. Applying rule ($1'_a$) implies that either $t_3 1't_1 \in B$ or $t_1 R t_4 \in B$. If $t_3 1't_1 \in B$, applying rule (*Sym*) implies that $t_1 1't_3 \in B$. Since $t_1 1't_3 \notin B$, then $t_1\ R\ t_4 \in B$. Applying rule ($1'_b$) implies that either $t_1\ R\ t_2 \in B$ or $t_4 1't_2 \in B$ If $t_4 1't_2 \in B$, one application of rule (*Sym*) would imply that $t_2\ 1't_4 \in B$, which is not the case. Thus, $t_1 R\ t_2 \in B$ which also leads to a contradiction. We have then shown that $\langle |t_3|, |t_4| \rangle \in m(R)$.

Therefore the structure $\langle \langle \mathfrak{A}, m(R_1), \ldots, m(R_k) \rangle, m \rangle$ is a fork model provided $RelConst = \{R_1, \ldots, R_k\}$ and $m(1')$ is the identity on A.

Let v be the valuation defined by $v(x) = |x|$, for $x \in IndVar$. Let us show by induction that $v(t) = |t|$ for all $t \in IndTerm$. By definition it is true for variables. If $t = t_1 \star t_2$, $v(t) = v(t_1 \star t_2) = v(t_1) \star v(t_2) = |t_1| \star |t_2| = |t_1 \star t_2|$.

Let us define

$$S = \{\alpha \in ForkFor : \mathfrak{A}, v \models_{FL} \alpha \text{ and } \alpha \in B\}.$$

Notice that since tRt' is valid, $\mathfrak{A}, v \models_{FL} tRt'$, and thus $S \neq \emptyset$. Then, since the set S is well-ordered by o, by Zorn's lemma S has a minimum element α'.

Notice that α' cannot have the shape $t_1 1' t_2$, because since $\alpha' \in S$, then $\mathfrak{A}, v \models_{FL} t_1 1' t_2$. Then must be $|t_1| = |t_2|$, which implies $t_1 1' t_2 \notin B$, a contradiction.

Notice also that α' cannot have any of the following shapes: $t_1 \overline{\overline{R}} t_2$, $t_1 \breve{R} t_2$, $t_1 \overline{\breve{R}} t_2$, $t_1 R + S t_2$, $t_1 \overline{R + S} t_2$, $t_1 R \cdot S t_2$, $t_1 \overline{R \cdot S} t_2$, $t_1 \overline{R ; S} t_2$ or $t_1 \overline{R \nabla S} t_2$ because in any of this cases a formula α'' appears in S satisfying $o(\alpha'') < o(\alpha')$, contradicting the minimality of α'.

If $\alpha' = t_1 R;S t_2$, by definition of the saturated tree there exists a level in B where we have a derivation with shape

$$\frac{\Gamma_1', \alpha', \Gamma_2'}{\Gamma_1', t_1 R z, \Gamma_2', \alpha' \quad \Gamma_1', z S t_2, \Gamma_2', \alpha'} \ (P;)$$

and z satisfies $\mathfrak{A}, v \models_{FL} t_1 R z$ and $\mathfrak{A}, v \models_{FL} z S t_2$. Therefore there exists $\alpha'' \in S$ with $o(\alpha'') < o(\alpha')$.

If $\alpha' = t_1 R \nabla S t_2$ by definition of the saturated tree there exists a level in B where we have a derivation with shape

$$\frac{\Gamma_1', \alpha', \Gamma_2'}{\Gamma_1', t_2 1' u \star v, \Gamma_2', \alpha' \quad \Gamma_1', t_1 R u, \Gamma_2', \alpha' \quad \Gamma_1', t_1 S v, \Gamma_2', \alpha'} \ (P\nabla),$$

and u and v satisfy $\mathfrak{A}, v \models_{FL} t_1 1' u \star v$, $\mathfrak{A}, v \models_{FL} t_1 R u$, and $\mathfrak{A}, v \models_{FL} t_2\ S v$. Therefore there exists $\alpha'' \in S$ with $o(\alpha'') < o(\alpha')$.

From the previous comments, it follows that α' must be indecomposable.

If $\alpha' = t_1 Q t_2$ for some $Q \in RelVar \cup RelConst$, and t_1, $t_2 \in IndTerm$, then, since $\mathfrak{A}, v \models_{FL} \alpha'$, $\langle |t_1|, |t_2| \rangle \in m(Q)$, but this is so if and only if (by definition of m), $t_1 Q t_2 \notin B$, which leads to a contradiction.

If $\alpha' = t_1 \overline{Q} t_2$ for some $Q \in RelVar \cup RelConst$, and t_1, $t_2 \in IndTerm$, then, since $\mathfrak{A}, v \models_{FL} \alpha'$, $\langle |t_1|, |t_2| \rangle \in m(Q)$, or equivalently $t_1 Q t_2 \in B$. Since $t_1 \overline{Q} t_2 \in B$ too, B is closed, which is a contradiction.

If $\alpha' = t_1 0' t_2$ for some $t_1, t_2 \in IndTerm$, then, since $\mathfrak{A}, v \models_{FL} \alpha'$, $\langle |t_1|, |t_2| \rangle \in m(0')$, and thus $|t_1| \neq |t_2|$. This implies that $t_1 1' t_2 \in B$ and also that B is closed which is a contradiction. ■

5.3 *Examples of Proofs in the Calculus FLC*

As an exercise let us show that some valid properties of fork algebras are provable in the calculus *FLC*. As a general practice we will sometimes omit some formulas when passing from a level to the level below, provided the formulas are not required to obtain the fundamental sequences. This will not affect the soundness of the calculus, and will simplify reading the proofs.

Let us prove that $(R\nabla S);(T\nabla Q)^{\smile} \leq R;\breve{T} \cdot S;\breve{Q}$. In order to start the derivation, we need first to convert the formula into an equation of the form $t = 1$. Notice that in general, $R \leq S \Leftrightarrow \overline{R} + S = 1$. Then,

$$\cfrac{\cfrac{x\overline{(R\nabla S);(T\nabla Q)^{\smile}} + R;\breve{T}\cdot S;\breve{Q}y \quad (P+)}{x\overline{(R\nabla S);(T\nabla Q)^{\smile}}y,\ xR;\breve{T}\cdot S;\breve{Q}y} \quad (N;)}{\underbrace{x\overline{R\nabla S}z_1,\ z_1\overline{(T\nabla Q)^{\smile}}y,\ xR;\breve{T}\cdot S;\breve{Q}y}_{\Sigma_1} \quad \underbrace{x\overline{R\nabla S}z_2,\ z_2\overline{(T\nabla Q)^{\smile}}y,\ xR;\breve{T}\cdot S;\breve{Q}y}_{\Sigma_2}}$$

pIn the sequence Σ_1, $z_1 \in UreVar$, while in Σ_2, $z_2 \in CompVar$. Let us analyze each sequence.

If we apply the rule $(N\nabla)$ on sequence Σ_1, then we obtain the following four sequences

1. $z_1 0' u_1 \star v_1,\ x\overline{R}u_1,\ x\overline{S}v_1,\ z_1\overline{(T\nabla Q)^{\smile}}y,\ xR;\breve{T}\cdot S;\breve{Q}y$, with $u_1, v_1 \in UreVar$,
2. $z_1 0' u_2 \star v_2,\ x\overline{R}u_2,\ x\overline{S}v_2,\ z_1\overline{(T\nabla Q)^{\smile}}y,\ xR;\breve{T}\cdot S;\breve{Q}y$, with $u_2 \in UreVar$ and $v_2 \in CompVar$,
3. $z_1 0' u_3 \star v_3,\ x\overline{R}u_3,\ x\overline{S}v_3,\ z_1\overline{(T\nabla Q)^{\smile}}y,\ xR;\breve{T}\cdot S;\breve{Q}y$, with $u_3 \in CompVar$ and $v_3 \in UreVar$,
4. $z_1 0' u_4 \star v_4,\ x\overline{R}u_4,\ x\overline{S}v_4,\ z_1\overline{(T\nabla Q)^{\smile}}y,\ xR;\breve{T}\cdot S;\breve{Q}y$, with $u_4, v_4 \in CompVar$.

Any of the four branches is closed by applying once in each branch the rule (U), adding the fork formula $z_1 1' u_i * v_i$, $1 \leq i \leq 4$.

Regarding branch Σ_2, applying rule $(N\nabla)$ we obtain (as with Σ_1) the following four sequences

1. $z_2\, 0' u_1{\star}v_1,\ x\overline{R}u_1,\ x\overline{S}v_1,\ z_2\overline{(T\nabla Q)\breve{}}\,y,\ xR;\breve{T}\cdot S;\breve{Q}y$, with $u_1, v_1 \in UreVar$,
2. $z_2\, 0' u_2{\star}v_2,\ x\overline{R}u_2,\ x\overline{S}v_2,\ z_2\overline{(T\nabla Q)\breve{}}\,y,\ xR;\breve{T}\cdot S;\breve{Q}y$, with $u_2 \in UreVar$ and $v_2 \in CompVar$,
3. $z_2\, 0' u_3{\star}v_3,\ x\overline{R}u_3,\ x\overline{S}v_3,\ z_2\overline{(T\nabla Q)\breve{}}\,y,\ xR;\breve{T}\cdot S;\breve{Q}y$, with $u_3 \in CompVar$ and $v_3 \in UreVar$,
4. $z_2\, 0' u_4{\star}v_4,\ x\overline{R}u_4,\ x\overline{S}v_4,\ z_2\overline{(T\nabla Q)\breve{}}\,y,\ xR;\breve{T}\cdot S;\breve{Q}y$, with $u_4, v_4 \in CompVar$.

We then proceed in the same way with the four branches, as follows.

$$\dfrac{\dfrac{z_2\, 0'u_i{\star}v_i,\ x\overline{R}u_i,\ x\overline{S}v_i,\ z_2\overline{(T\nabla Q)\breve{}}\,y,\ xR;\breve{T}\cdot S;\breve{Q}y \quad (N\breve{})}{z_2\, 0'u_i{\star}v_i,\ x\overline{R}u_i,\ x\overline{S}v_i,\ y\overline{T\nabla Q}z_2,\ xR;\breve{T}\cdot S;\breve{Q}y} \quad (1'b)}{\underbrace{z_2\, 0'u_i{\star}v_i,\ x\overline{R}u_i,\ x\overline{S}v_i,\ y\overline{T\nabla Q}u_i{\star}v_i,\ y\overline{T\nabla Q}z_2,\ xR;\breve{T}\cdot S;\breve{Q}y}_{\Sigma_3} \quad \underbrace{z_2\, 0'u_i{\star}v_i,\ u_i{\star}v_i\, 1'z_2}_{\Sigma_4}}$$

Regarding branch Σ_4, we have

$$\dfrac{z_2\, 0'u_i{\star}v_i,\ u_i{\star}v_i\, 1'z_2 \quad (Sym)}{z_2\, 0'u_i{\star}v_i,\ z_2\, 1'u_i{\star}v_i}$$

The last sequence is clearly fundamental, and thus the branch is closed.

Regarding branch Σ_3, we proceed as follows.

$$\dfrac{\dfrac{z_2\, 0'u_i{\star}v_i,\ x\overline{R}u_i,\ x\overline{S}v_i,\ y\overline{T\nabla Q}u_i{\star}v_i,\ y\overline{T\nabla Q}z_2,\ xR;\breve{T}\cdot S;\breve{Q}y \quad (N\nabla)}{x\overline{R}u_i,\ x\overline{S}v_i,\ u_i{\star}v_i\, 0'r_j{\star}s_j,\ y\overline{T}r_j,\ y\overline{Q}s_j,\ xR;\breve{T}\cdot S;\breve{Q}y} \quad (P\cdot)}{\underbrace{x\overline{R}u_i,\ u{\star}v_i\, 0'r_j{\star}s_j,\ y\overline{T}r_j,\ xR;\breve{T}y}_{\Sigma_{5,i,j}} \quad \underbrace{x\overline{S}v_i,\ u_i{\star}v_i\, 0'r_j{\star}s_j,\ y\overline{Q}s_j,\ xS;\breve{Q}y}_{\Sigma_{6,i,j}}}$$

Since none of the sequences $\Sigma_{5,i,j}$ or $\Sigma_{6,i,j}$ are closed, we will derive a closed tree for each sequence. For the sequences $\Sigma_{5,i,j}$ we have:

$$\dfrac{\dfrac{x\overline{R}u_i,\ u_i{\star}v_i\, 0'r_j{\star}s_j,\ y\overline{T}r_j,\ xR;\breve{T}y \quad (N1')}{x\overline{R}u_i,\ u_i\, 0'r_j,\ v_i\, 0's_j,\ y\overline{T}r_j,\ xR;\breve{T}y \quad (P;)}}{x\overline{R}u_i,\ xRu_i \qquad \dfrac{\dfrac{u_i\, 0'r_j,\ y\overline{T}r_j,\ u_i\breve{T}y \quad (P\breve{})}{u_i\, 0'r_j,\ y\overline{T}r_j,\ yTu_i} \quad (1'b)}{y\overline{T}u_i,\ yTu_i \qquad u_i\, 0'r_j,\ u_i\, 1'r_j}}$$

Finally, for the sequences $\Sigma_{6,i,j}$ we have:

$$
\dfrac{\dfrac{x\overline{S}v_i,\ u_i\star v_i\,0\text{'}r_j\star s_j,\ y\overline{Q}s_j,\ xS;\breve{Q}y \quad (N1\text{'})}{x\overline{S}v_i,\ u_i\,0\text{'}r_j,\ v_i\,0\text{'}s_j,\ y\overline{Q}s_j,\ xS;\breve{Q}y \quad (P;)}}{x\overline{S}v_i,\ xSv_i \qquad \dfrac{\dfrac{v_i\,0\text{'}s,\ y\overline{Q}s,\ v_i\breve{Q}y \quad (P^{\smile})}{v_i\,0\text{'}s,\ y\overline{Q}s,\ yQv_i \quad (1\text{'}b)}}{y\overline{Q}v_i,\ yQv_i \qquad v_i\,0\text{'}s_j,\ v_i\,1\text{'}s_j}}
$$

Let us prove now the other inclusion, namely, that $R;\breve{T}\cdot S;\breve{Q}\leq(R\nabla S);(T\nabla Q)^{\smile}$.

$$
\dfrac{\dfrac{\dfrac{x\overline{R;\breve{T}\ \cdot\ S;\breve{Q}} + (R\nabla S);(T\nabla Q)^{\smile}y \quad (P+)}{x\overline{R;\breve{T}\ \cdot\ S;\breve{Q}}y,\ x(R\nabla S);(T\nabla Q)^{\smile}y \quad (N\cdot)}}{x\overline{R;\breve{T}}y,\ x\overline{S;\breve{Q}}y,\ x(R\nabla S);(T\nabla Q)^{\smile}y \quad (N;)}}{\underbrace{x\overline{R}u_1,\ u_1\overline{\breve{T}}y,\ x\overline{S;\breve{Q}}y,\ x(R\nabla S);(T\nabla Q)^{\smile}y}_{\Theta_1} \quad \underbrace{x\overline{R}u_2,\ u_2\overline{\breve{T}}y,\ x\overline{S;\breve{Q}}y,\ x(R\nabla S);(T\nabla Q)^{\smile}y}_{\Theta_2}}
$$

In the sequence Θ_1, $u_1\in UreVar$ and $u_2\in CompVar$, We will proceed the derivation with Θ_1, since the same steps can be applied indistinctly to Θ_2.

$$
\dfrac{x\overline{R}u_1,\ u_1\overline{\breve{T}}y,\ x\overline{S;\breve{Q}}y,\ x(R\nabla S);(T\nabla Q)^{\smile}y \quad (N;)}{\underbrace{x\overline{R}u_1,\ u_1\overline{\breve{T}}y,\ x\overline{S}v_1,\ v_1\overline{\breve{Q}}y,\ x(R\nabla S);(T\nabla Q)^{\smile}y}_{\Theta_3} \quad \underbrace{x\overline{R}u_1,\ u_1\overline{\breve{T}}y,\ x\overline{S}v_2,\ v_2\overline{\breve{Q}}y,\ x(R\nabla S);(T\nabla Q)^{\smile}y}_{\Theta_4}}
$$

In sequences Θ_3 and Θ_4 $v_1\in UreVar$ and $v_2\in CompVar$, We will proceed the derivation with Θ_3, although the same steps apply to sequence Θ_4.

$$
\dfrac{\dfrac{\dfrac{x\overline{R}u_1,\ u_1\overline{\breve{T}}y,\ x\overline{S}v_1,\ v_1\overline{\breve{Q}}y,\ x(R\nabla S);(T\nabla Q)^{\smile}y \quad (\breve{N})}{x\overline{R}u_1,\ y\overline{T}u_1,\ x\overline{S}v_1,\ v_1\overline{\breve{Q}}y,\ x(R\nabla S);(T\nabla Q)^{\smile}y \quad (\breve{N})}}{x\overline{R}u_1,\ y\overline{T}u_1,\ x\overline{S}v_1,\ y\overline{Q}v_1,\ x(R\nabla S);(T\nabla Q)^{\smile}y \quad (P;)}}{\underbrace{x\overline{R}u_1,\ x\overline{S}v_1,\ x(R\nabla S)u_1\star v_1}_{\Theta_5} \quad \underbrace{y\overline{T}u_1,\ y\overline{Q}v_1,\ u_1\star v_1;(T\nabla Q)^{\smile}y}_{\Theta_6}}
$$

Since both sequences Θ_5 and Θ_6 are not fundamental, we will proceed with the derivation. For sequence Θ_5 we have:

$$
\dfrac{x\overline{R}u_1,\ x\overline{S}v_1,\ xR\nabla Su_1\star v_1 \quad (P\nabla)}{u_1\star v_1\,1\text{'}u_1\star v_1 \quad x\overline{R}u_1,\ xRu_1 \quad x\overline{S}v_1,\ xSv_1}
$$

The last sequences are all fundamental.

Finally, for sequence Θ_6 we have:

$$\cfrac{\cfrac{y\overline{T}u_1, y\overline{Q}v_1, u_1 * v_1 (T\nabla Q)^{\smile} y \quad (P^{\smile})}{y\overline{T}u_1, y\overline{Q}v_1, yT\nabla Q u_1 * v_1} \quad (P\nabla)}{u_1 * v_1 1' u_1 * v_1 \quad y\overline{T}u_1, yTu_1 \quad y\overline{Q}v_1, yQv_1}$$

6. *A Relational Proof System for Intuitionistic Logic*

In this section we prove interpretability of intuitionistic logic in the fork logic *FL* and extend the proof system *FLC* to a relational proof system for intuitionistic logic.

6.1 *Intuitionistic Logic*

Syntax and semantics of the intuitionistic logic (*Int*) is as follows.

Definition 6.1 The alphabet of *Int* is given by:
1. an infinite countable set of propositional variables denoted by *PropVar*,
2. the set of propositional connectives $\{\neg, \vee, \wedge, \rightarrow\}$, and
3. the set of auxiliary symbols {"(", ", ", ")"}.

Definition 6.2 The set of intuitionistic formulas (denoted by *IntFor*) is the smallest set satisfying
1. $PropVar \subseteq IntFor$,
2. If $\alpha, \beta \in IntFor$, then $\{(\neg\alpha), (\alpha \vee \beta), (\alpha \wedge \beta), (\alpha \rightarrow \beta)\} \subseteq IntFor$.

Definition 6.3 An intuitionistic model is a triple $\langle W, R, m \rangle$ such that
1. $W \neq \emptyset$
2. $R \subseteq W \times W$ is a reflexive and transitive relation,
3. $m : PropVar \rightarrow \mathcal{P}(W)$ satisfies the heredity condition given by:
   If wRw' and $w \in m(p)$, then $w' \in m(p)$.

Definition 6.4 Let $\Im = \langle W, R, m \rangle$ be an intuitionistic model. A formula α is satisfied in a world $w \in W$ (denoted by $\Im, w \models_{Int} \alpha$ if the following conditions are satisfied:
1. $\alpha = p_i \in PropVar$:
   $\Im, w \models_{Int} p_i$ iff $w \in m(p_i)$
2. $\alpha = \neg\beta$:
   $\Im, w \models_{Int} \neg\beta$ iff $(\forall w' \in W)(wRw' \Rightarrow \Im, w' \not\models_{Int} \beta)$.
3. $\alpha = \beta \vee \gamma$:
   $\Im, w \models_{Int} \beta \vee \gamma$ iff $\Im, w \models_{Int} \beta$ or $\Im, w \models_{Int} \gamma$.

4. $\alpha = \beta \wedge \gamma$:
$\mathfrak{I}, w \models_{Int} \beta \wedge \gamma$ iff $\mathfrak{I}, w \models_{Int} \beta$ and $\mathfrak{I}, w \models_{Int} \gamma$.
5. $\alpha = \beta \rightarrow \gamma$:
$\mathfrak{I}, w \models_{Int} \beta \rightarrow \gamma$ iff $(\forall w' \in W)$ (wRw' and $\mathfrak{I}, w' \models_{Int} \beta$ implies $\mathfrak{I}, w' \models_{Int} \gamma$).

Definition 6.5 A formula α is *true in an intuitionistic model* $\mathfrak{I} = \langle W, R, m \rangle$ (denoted by $\mathfrak{I} \models_{Int} \alpha$) if, for all $w \in W$, $\mathfrak{I}, w \models_{Int} \alpha$.

Definition 6.6 A formula is *Int-valid* if it is valid in all intuitionistic models.

6.2 *Interpretability of Intuitionistic Logic in Fork Logic*

In this subsection we will present a mapping $T_I : IntFor \rightarrow RelTerm$ that will allow to interpret the logic *Int* in the logic *FL*.

Definition 6.7 Let us have a fork language with one constant symbol R interpreted as an accessibility relation from intuitionistic models. Let us define the recursive mapping $T_I : IntFor \rightarrow RelTerm$ as follows:
1. $T_I(p_i) = R_i$ with $p_i \in PropVar$ and $R_i \in RelVar$.
2. $T_I(\neg\alpha) = \overline{R;T_I(\alpha)}$.
3. $T_I(\alpha \wedge \beta) = T_I(\alpha) \cdot T_I(\beta)$.
4. $T_I(\alpha \vee \beta) = T_I(\alpha) + T_I(\beta)$.
5. $T_I(\alpha \rightarrow \beta) = \overline{R;(T_I(\alpha) \cdot \overline{T_I(\beta)})}$.

Since the accessibility relation in intuitionistic models satisfies conditions of reflexivity, transitivity and heredity, we will define abstract relational counterparts of these conditions, as follows:

(C1):	$1'_U \leq R$,	(reflexivity)
(C2):	$R;R \leq R$,	(transitivity)
(C3):	$\overline{(R_i \cdot R);\overline{R_i}} = 1$,	(heredity)
(C4):	$R_i \leq {}_U 1$ for all i,	(R_i has urelements in its domain)
(C5):	$R_i;1 = R_i$ for all i,	(R_i is right-ideal)
(C6):	$R \leq {}_U 1;1_U$.	(R is defined in the set of urelements)

Theorem 6.8 *Let* $\mathfrak{I} = \langle W, R, m \rangle$ *be an intuitionistic model. Then there exists a fork model* $\mathfrak{F} = \langle\langle \mathfrak{A}, R' \rangle, m' \rangle$ *constructed from* $\mathfrak{I}$ *satisfying conditions* (C1)-(C6) *such that for all* $w \in W$ *and for all* $\varphi \in IntFor$

$$\mathfrak{I}, w \models_{Int} \varphi \Leftrightarrow w \in \mathrm{dom}\,(m'(T_I(\varphi))).$$

Proof Define $\mathfrak{A}$ as the Full*PFAU* with set of urelements W, let $R' = R$, and for each $R_i \in RelVar$ define $m'(R_i) = \{\langle x, y\rangle : x \in m(p_i)\}$. Conditions (C1)-(C6) hold because of the way R' and R_i are defined.

The remaining part of the proof proceeds by induction on the structure of the formula φ.

$\varphi = p_i \in PropVar$:

$$\begin{aligned}
\mathfrak{I}, w \models_{Int} p_i \quad & \text{iff } w \in m(p_i) \\
& \text{iff } w \in \mathrm{dom}\,(m'(R_i)) \\
& \text{iff } w \in \mathrm{dom}\,(m'(T_I(p_i))).
\end{aligned}$$

$\varphi = \neg\alpha$:

$$\begin{aligned}
\mathfrak{I}, w \models_{Int} \neg\alpha \quad & \text{iff } (\forall w' \in W)\,(wRw' \Rightarrow \mathfrak{I}, w' \not\models_{Int} \alpha) \\
& \text{iff } (\forall w' \in W)\,(wRw' \Rightarrow w' \notin \mathrm{dom}\,(m'(T_I(\alpha)))) \\
& \text{iff } (\nexists w' \in W)\,(wRw' \wedge w' \in \mathrm{dom}\,(m'(T_I(\alpha)))) \\
& \text{iff } (\nexists w' \in W)\,(wR'\,w' \wedge w' \in \mathrm{dom}\,(m'(T_I(\alpha)))) \\
& \text{iff } w \in \mathrm{dom}\,(\overline{R';\overline{m'(T_I(\alpha))}}) \\
& \text{iff } w \in \mathrm{dom}\,(m'(\overline{R';\overline{(T_I(\alpha)}})) \\
& \text{iff } w \in \mathrm{dom}\,(m'(T_I(\neg\alpha))).
\end{aligned}$$

$\varphi = \alpha \vee \beta$:

$$\begin{aligned}
\mathfrak{I}, w \models_{Int} \alpha \vee \beta \quad & \text{iff } \mathfrak{I}, w \models_{Int} \alpha \text{ or } \mathfrak{I}, w \models_{Int} \beta \\
& \text{iff } w \in \mathrm{dom}\,(m'(T_I(\alpha))) \text{ or } w \in \mathrm{dom}\,(m'(T_I(\beta))) \\
& \text{iff } w \in \mathrm{dom}\,(m'(T_I(\alpha)) + m'(T_I(\beta))) \\
& \text{iff } w \in \mathrm{dom}\,(m'(T_I(\alpha \vee \beta))).
\end{aligned}$$

$\varphi = \alpha \wedge \beta$

$$\begin{aligned}
\mathfrak{I}, w \models_{Int} \alpha \wedge \beta \quad & \text{iff } \mathfrak{I}, w \models_{Int} \alpha \text{ and } \mathfrak{I}, w \models_{Int} \beta \\
& \text{iff } w \in \mathrm{dom}\,(m'(T_I(\alpha))) \text{ and } w \in \mathrm{dom}\,(m'(T_I(\beta))) \\
& \text{iff } w \in \mathrm{dom}\,(m'(T_I(\alpha)) \cdot m'(T_I(\beta))) \\
& \text{iff } w \in \mathrm{dom}\,(m'(T_I(\alpha \wedge \beta))).
\end{aligned}$$

$\varphi = \alpha \rightarrow \beta$

$$\begin{aligned}
& \mathfrak{I}, w \models_{Int} \alpha \rightarrow \beta \\
& \text{iff } (\forall w' \in W)\,(wRw' \wedge \mathfrak{I}, w' \models_{Int} \beta \Rightarrow \mathfrak{I}, w' \models_{Int} \gamma) \\
& \text{iff } (\nexists w' \in W)\,(wRw' \wedge \mathfrak{I}, w' \models_{Int} \beta \wedge \mathfrak{I}, w' \not\models_{Int} \gamma) \\
& \text{iff } (\nexists w' \in W)\,(wRw' \wedge w' \in \mathrm{dom}\,(m'(T_I(\beta))) \wedge w' \notin \mathrm{dom}\,(m'(T_I(\gamma)))) \\
& \text{iff } (\nexists w' \in A)\,(wR'\,w' \wedge w' \in \mathrm{dom}\,(m'(T_I(\beta))) \wedge w' \notin \mathrm{dom}\,(m'(T_I(\gamma)))) \\
& \text{iff } w \in \mathrm{dom}\,(\overline{R';(m'(T_I(\alpha)) \cdot \overline{m'(T_I(\beta))}}))) \\
& \text{iff } w \in \mathrm{dom}\,(m'(\overline{R';(T_I(\alpha) \cdot \overline{T_I(\beta)}}))) \\
& \text{iff } w \in \mathrm{dom}\,(m'(T_I(\alpha \rightarrow \beta))).
\end{aligned}$$

■

Theorem 6.9 *Let* $\mathfrak{F} = \langle\langle\mathfrak{A}, R\rangle, m\rangle$ *be a fork model satisfying conditions* (C1)-(C6). *Then there exists an intuitionistic model* $\mathfrak{I} = \langle W, R', m'\rangle$ *constructed from* $\mathfrak{F}$ *such that for all* $w \in W$ *and for all* $\varphi \in IntFor$

$$w \in \mathrm{dom}\,(m(T_I(\varphi))) \Leftrightarrow \mathfrak{I}, w \models_{Int} \varphi.$$

Proof. Let us define $W = Urel_{\mathfrak{A}}$, $R' = R$, and for all $p_i \in PropVar$ define $m'(p_i) = \mathrm{dom}\,(m(R_i))$. Notice that by conditions (C1)-(C6) R' is a reflexive and transitive relation on W, and the heredity condition is satisfied by m'. The remaining part of the proof proceeds by induction on the structure of the formula φ.

$\varphi = p_i$:

$w \in \mathrm{dom}\,(m(T_I(p_i)))$
iff $w \in \mathrm{dom}\,(m(R_i))$
iff $w \in m'(p_i)$
iff $\mathfrak{I}, w \models_{Int} p_i$.

$\varphi = \neg\alpha$:

$w \in \mathrm{dom}\,(m(T_I(\neg\alpha)))$
iff $w \in \mathrm{dom}\,(m(\overline{R' ; T_I(\alpha)}))$
iff $w \in \mathrm{dom}\,(\overline{R; m(T_I(\alpha))})$
iff $(\nexists w' \in A)\,(wRw' \wedge w' \in \mathrm{dom}\,(m(T_I(\alpha))))$
iff $(\nexists w' \in W)\,(wRw' \wedge w' \in \mathrm{dom}\,(m(T_I(\alpha))))$
iff $(\forall w' \in W)\,(wRw' \Rightarrow w' \notin \mathrm{dom}\,(m(T_I(\alpha))))$
iff $(\forall w' \in W)\,(wRw' \Rightarrow \mathfrak{I}, w' \not\models_{Int} \alpha$
iff $\mathfrak{I}, w \models_{Int} \neg\alpha$

$\varphi = \alpha \vee \beta$:

$w \in \mathrm{dom}\,(m(T_I(\alpha \vee \beta)))$
iff $w \in \mathrm{dom}\,(m(T_I(\alpha) + T_I(\beta)))$
iff $w \in \mathrm{dom}\,(m(T_I(\alpha)) + m(T_I(\beta)))$
iff $w \in \mathrm{dom}\,(m(T_I(\alpha)))$ or $w \in \mathrm{dom}\,(m(T_I(\beta)))$
iff $\mathfrak{I}, w \models_{Int} \alpha$ or $\mathfrak{I}, w \models_{Int} \beta$
iff $\mathfrak{I}, w \models_{Int} \alpha \vee \beta$.

$\varphi = \alpha \wedge \beta$:

$w \in \mathrm{dom}\,(m(T_I(\alpha \wedge \beta)))$

iff $w \in \mathrm{dom}\,(m(T_I(\alpha) \cdot T_I(\beta)))$

iff $w \in \mathrm{dom}\,(m(T_I(\alpha)) \cdot m(T_I(\beta)))$

iff $w \in \mathrm{dom}\,(m(T_I(\alpha)))$ and $w \in \mathrm{dom}\,(m(T_I(\beta)))$

iff $\mathfrak{I}, w \models_{Int} \alpha$ and $\mathfrak{I}, w \models_{Int} \beta$

iff $\mathfrak{I}, w \models_{Int} \alpha \wedge \beta$.

$\varphi = \alpha \rightarrow \beta$:

$w \in \mathrm{dom}\,(m(T_I(\alpha \rightarrow \beta)))$

iff $w \in \mathrm{dom}\,\overline{(m(R;(T_I(\alpha) \cdot \overline{T_I(\beta)})))}$

iff $w \in \mathrm{dom}\,\overline{(R;(m(T_I(\alpha)) \cdot \overline{m(T_I(\beta))}))}$

iff $(\nexists w' \in W)\,(wRw' \wedge w' \in \mathrm{dom}\,(m(T_I(\alpha))) \wedge w' \notin \mathrm{dom}\,(m(T_I(\beta))))$

iff $(\forall w' \in W)\,(wRw' \wedge w' \in \mathrm{dom}\,(m(T_I(\alpha))) \Rightarrow w' \in \mathrm{dom}\,(m(T_I(\beta))))$

iff $(\forall w' \in W)(wR'\, w' \wedge \mathfrak{I}, w' \models_{Int} \alpha \Rightarrow \mathfrak{I}, w' \models_{Int} \beta)$

iff $\mathfrak{I}, w \models_{Int} \alpha \rightarrow \beta$.

■

Let us denote by $\mathfrak{K}$ the class of those fork models $\langle\langle \mathfrak{A}, R\rangle, m\rangle$ where $\mathfrak{A}$ and m satisfy conditions (C1)-(C6). In the remaining parts of the paper we will denote by FL' the fork logic induced by the class of fork models $\mathfrak{K}$ in the following way

$$\models_{FL'} xTy \Leftrightarrow \forall \mathfrak{M} \in \mathfrak{K}(\mathfrak{M} \models_{FL} xTy).$$

Theorem 6.10 Let $\psi \in Intfor$. Then, given $x \in UreVar$ and $y \in CompVar$,

$$\models_{Int} \psi \text{ iff } \models_{FL'} xT_I(\psi)y.$$

Proof. Let us prove the contrapositive. If $\not\models_{Int} \psi$ then there exists an intuitionistic model $\mathfrak{I} = \langle W, R, m\rangle$ and $w \in W$ such that $\mathfrak{I}$, $w \not\models_{Int} \psi$. Then, by Thm. 6.8 there exists a fork model $\mathfrak{F} = \langle\langle \mathfrak{A}, R'\rangle, m'\rangle$ such that $w \notin \mathrm{dom}(m'(T_I(\psi)))$. Let v be a valuation satisfying $v(x) = w$, then $\mathfrak{F}$, $v \not\models_{FL} xT_I(\psi)y$, and thus $\not\models_{FL} xT_I(\psi)y$.

If $\not\models_{FL} xT_I(\psi)y$, then there exists a fork model $\mathfrak{F} = \langle\langle \mathfrak{A}, R\rangle, m\rangle$ satisfying (C1)-(C6) and a valuation v such that $\langle v(x), v(y)\rangle \notin m(T_I(\psi))$. Thus, $v(x) \notin \mathrm{dom}(m(T_I(\psi)))$. By Thm. 6.9 there exists an intuitionistic model $\mathfrak{I} = \langle A, R', m'\rangle$ such that $\mathfrak{I}$, $v(x) \not\models_{Int} \psi$, and thus $\not\models_{Int} \psi$. ■

6.3 *A Fork Logic Based Calculus for Intuitionistic Logic*

In this subsection we will present a calculus for intuitionistic logic based on the calculus *FLC*. The calculus will be obtained by adding specific rules and modifying the notion of fundamental sequence in the calculus *FLC*.

Throughout this subsection we assume we are working with a fork language $\mathfrak{L}(R)$ with only one constant symbol as in Def. 6.7.

Definition 6.11 A sequence of fork formulas Γ is *Int-fundamental* if any of the following conditions is true:

1. Γ is fundamental according to Def. 5.3, or
2. the fork formula $xRx \in \Gamma$ for some $x \in UreVar$.

Condition 2 reflects the property that the intuitionistic accessibility relation R is reflexive on the set of urelements.

We define the intuitionistic calculus *Int-FLC* by adding the following specific rules to those of *FLC*.

$$\frac{\Gamma, xRy, \Delta}{\Gamma, xRz, xRy, \Delta \quad \Gamma, zRy, xRy, \Delta}\ (TranR)$$

$$\frac{\Gamma, xR_iy, \Delta}{\Gamma, zRx, xR_iy, \Delta \quad \Gamma, zR_iy, xR_iy, \Delta}\ (H)$$

$$\frac{\Gamma, xR_iy, \Delta}{\Gamma, xR_iy, xR_iz, \Delta}\ (RI)$$

$$\frac{\Gamma}{\Gamma, xRy}\ (RUr) \qquad \frac{\Gamma}{\Gamma, xR_iy}\ (VarUr)$$

In rules (*TranR*), (*H*) and (*RI*), $z \in UreVar$ is arbitrary. In rule (*RUr*) either x or y belong to $IndTerm \setminus UreVar$, and in rule (*VarUr*), $x \in IndTerm \setminus UreVar$. The admissibility of rule (*TranR*) is equivalent to the transitivity of the relation R. The admissibility of rule (*H*) is equivalent to the heredity condition. The admissibility of rule (*RI*) is equivalent to relational variables being interpreted as right-ideal relations. The admissibility of rule (*RUr*) is equivalent to R being defined only on urelements. Finally, the admissibility of rule (*VarUr*) is equivalent to variables having urelements in their domain.

Notice that the last comments imply the soundness of the calculus *Int-FLC*.

Theorem 6.12 *The calculus Int-FLC is sound w.r.t. the logic FL', i.e., given* $tQt' \in ForkFor$

$$\vdash_{Int-FLC} tQt' \Rightarrow \models_{FL'} tQt'.$$

Definition 6.13 A proof tree T of a sequence of formulas Γ is *Int*-saturated if in all open branches B, the following conditions are satisfied.

1. Conditions (1) through (19) from Def. 5.7,
2. If $xRy \in B$ then for each $z \in IndTerm$ either $xRz \in B$ or $zRy \in B$ by an application of rule (*TranR*).
3. If $xR_iy \in B$ ($R_i \in RelVar$), then for each $z \in IndTerm$ either $zRx \in B$ or $zR_iy \in B$ by an application of rule (*H*).
4. If If $xR_iy \in B$ ($R_i \in RelVar$), then for all $z \in IndTerm$ $xR_iz \in B$ by an application of rule (*RI*).
5. For all x, $y \in IndTerm$ such that $x \in IndTerm \setminus UreVar$ or $y \in IndTerm \setminus UreVar$, $xRy \in B$ by an application of rule (*RUr*).
6. For all $x \in IndTerm \setminus UreVar$ and $y \in IndTerm$, $xR_iy \in B$ by an application of rule (*VarUr*).

Theorem 6.14 *The calculus Int-FLC is complete w.r.t. the logic FL', i.e., given* $tQt' \in ForkFor$

$$\models_{FL'} tQt' \Rightarrow \vdash_{Int-FLC} tQt'.$$

Proof. The proof will follow the lines of the proof of Thm. 5.9, and therefore the reader will be directed there for some parts of the proof.

Assume tQt' is not provable in *Int-FLC*. Then no proof tree exists that provides a proof for tQt'. In particular, no *Int*-saturated tree with root tQt' provides a proof. Therefore, if T is an *Int*-saturated tree, there must exist an infinite branch B in T.

Let $\equiv$ be the binary relation on *IndTerm* defined by

$$x \equiv y \Leftrightarrow x1'y \notin B.$$

The proof that $\equiv$ is an equivalence relation is the same as in Thm. 5.9.

Let $\mathfrak{A}$ be the Full*PFAU* with set of urelements $\{|x| : x \in UreVar\}$ and pairing function $*$ defined by $|x| * |y| = |x * y|$. Proving that $*$ is well defined and injective is done as in Thm. 5.9.

Let us define, for $R' \in RelVar \cup \{R\}$,

$$\langle |t_1|, |t_2| \rangle \in m(R') \Leftrightarrow t_1 R' t_2 \notin B.$$

That m is well-defined is proved as in Thm. 5.9.

The relation R is reflexive, for if there exists $x \in UreVar$ such that $\langle |x|, |x| \rangle \notin m(R)$, then $xRx \in B$. Then B would be a closed branch which is a contradiction.

The relation R is transitive, for if there are x_1, x_2, $x_3 \in UreVar$ such that $\langle |x_1|, |x_2| \rangle \in m(R)$, $\langle |x_2|, |x_3| \rangle \in m(R)$ and $\langle |x_1|, |x_3| \rangle \notin m(R)$, then $x_1Rx_2 \notin B$, $x_2Rx_3 \notin B$ and $x_1Rx_3 \in B$. Then, applying rule (*TranR*) either $x_1Rx_2 \in B$ or $x_2Rx_3 \in B$ which is a contradiction.

The heredity condition holds, for if there are x_1, $x_2 \in UreVar$ and $t \in IndTerm$ such that $\langle |x_1|, |t| \rangle \in m(R_i)$ $(R_i \in RelVar)$, $\langle |x_1|, |x_2| \rangle \in m(R)$ and $\langle |x_2|, |t| \rangle \notin m(R_i)$, then $x_1R_it \notin B$, $x_1Rx_2 \notin B$ and $x_2R_it \in B$. Applying rule (*H*) either $x_1R_it \in B$ or $x_1Rx_2 \in B$, which is a contradiction.

In a similar way we show that relational variables are interpreted as right-ideal relations.

$R \subseteq UreVar \times UreVar$, for if there is $t \in IndTerm \setminus UreVar$ such that $|t| \in \text{dom}\,(R)$ or $|t| \in \text{ran}\,(R)$, then applying rule (*RUr*) we arrive to a contradiction.

For all $R_i \in RelVar$, dom $(R_i) \subseteq UreVar$, for if there are $t \in IndTerm \setminus UreVar$ and $t' \in IndTerm$ such that $\langle |t|, |t'| \rangle \in m(R_i)$ then $tR_it' \notin B$. Applying rule (*VarUr*) we arrive to a contradiction.

Therefore the structure $\langle \langle \mathfrak{A}, m(R) \rangle, m \rangle$ is a fork model that satisfies (C1)-(C6). The rest of the proof is analogous to the respective part of the proof of Thm. 5.9. ■

From Thm. 6.14 the corollary below immediately follows.

Corollary 6.15 *Given a formula* $\varphi \in IntFor$ $x \in UreVar$ *and* $y \in CompVar$, *we have*

$$\models_{Int}\varphi \Leftrightarrow \vdash_{Int-FLC} xT_I(\varphi)y.$$

Proof From Thm. 6.10, for any formula $\varphi \in IntFor$ $x \in UreVar$ and $y \in CompVar$,

$$\models_{Int}\varphi \Leftrightarrow \models_{FL'} xT_I(\varphi)y. \tag{2}$$

From Thms. 6.12 and 6.14, we then obtain

$$\models_{FL'} xT_I(\varphi)y \Leftrightarrow \vdash_{Int-FLC} xT_I(\varphi)y. \tag{3}$$

Joining (2) and (3), we then obtain

$$\models_{Int}\varphi \Leftrightarrow \vdash_{Int-FLC} xT_I(\varphi)y.$$ ■

Example In order to see how the calculus works let us consider a proof of the intuitionistic tautology $\neg\neg\neg\alpha \to \neg\alpha$. According to Coro. 6.15, it suffices to prove that $\vdash_{Int-FLC} xT_I(\neg\neg\neg\alpha \to \neg\alpha)y$. In order to keep an economic notation we will not apply the mapping T_I entirely from the beginning, but by parts according to our needs. To do so we will use a rule called (T_I).

$$\dfrac{\dfrac{xT_I(\neg\neg\neg\alpha \to \neg\alpha)y \quad (T_I)}{x\overline{R;(T_I(\neg\neg\neg\alpha)\cdot\overline{T_I(\neg\alpha)})}y} \quad (N;)}{\underbrace{x\overline{R}z_1, z_1\overline{T_I(\neg\neg\neg\alpha)\cdot\overline{T_I(\neg\alpha)}}y}_{\Lambda_1} \quad \underbrace{x\overline{R}z_2, z_2\overline{T_I(\neg\neg\neg\alpha)\cdot\overline{T_I(\neg\alpha)}}y}_{\Lambda_2}}$$

In sequences Λ_1 and Λ_2, $z_1 \in UreVar$ and $z_2 \in CompVar$.

Regarding sequence Λ_2, we have

$$\dfrac{x\overline{R}z_2, z_2\overline{T_I(\neg\neg\neg\alpha)\cdot\overline{T_I(\neg\alpha)}}y \quad (RUr)}{x\overline{R}z_2, z_2\overline{T_I(\neg\neg\neg\alpha)\cdot\overline{T_I(\neg\alpha)}}y, xRz_2}$$

The last sequence is fundamental, and thus the branch is closed.

Regarding sequence Λ_1, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x\overline{R}z_1, z_1\overline{T_I(\neg\neg\neg\alpha)\cdot\overline{T_I(\neg\alpha)}}y \quad (N\cdot)}{x\overline{R}z_1, z_1\overline{T_I(\neg\neg\neg\alpha)}y, z_1\overline{\overline{T_I(\neg\alpha)}}y \quad (N^-)}}{x\overline{R}z_1, z_1\overline{T_I(\neg\neg\neg\alpha)}y, z_1T_I(\neg\alpha)y \quad (T_I)}}{x\overline{R}z_1, z_1\overline{\overline{R;T_I(\neg\neg\alpha)}}y, z_1T_I(\neg\alpha)y \quad (N^-)}}{x\overline{R}z_1, z_1R;T_I(\neg\neg\alpha)y, z_1T_I(\neg\alpha)y \quad (T_I)}}{x\overline{R}z_1, z_1R;T_I(\neg\neg\alpha)y, z_1\overline{R;T_I(\alpha)}y \quad (N;)}}{\underbrace{x\overline{R}z_1, z_1R;T_I(\neg\neg\alpha)y, z_1\overline{R}t_1, t_1\overline{T_I(\alpha)}y}_{\Lambda_3} \quad \underbrace{x\overline{R}z_1, z_1R;T_I(\neg\neg\alpha)y, z_1\overline{R}t_2, t_2\overline{T_I(\alpha)}y}_{\Lambda_4}}$$

In sequences Λ_3 and Λ_4, $t_1 \in UreVar$ and $t_2 \in CompVar$.

Regarding sequence Λ_3 we have:

$$\dfrac{x\overline{R}z_1, z_1R;T_I(\neg\neg\alpha)y, z_1\overline{R}t_1, t_1\overline{T_I(\alpha)}y \quad (P;)}{\underbrace{x\overline{R}z_1, z_1Rt_1, z_1\overline{R}t_1, t_1\overline{T_I(\alpha)}y}_{\Lambda_5} \quad \underbrace{x\overline{R}z_1, t_1T_I(\neg\neg\alpha)y, z_1\overline{R}t_1, t_1\overline{T_I(\alpha)}y}_{\Lambda_6}}$$

Since the fork formulas z_1Rt_1 and $z_1\overline{R}t_1$ occur in Λ_5, this branch is closed.
Regarding sequence Λ_6 we have

$$\frac{\dfrac{x\overline{R}z_1,\, t_1T_I(\neg\neg\alpha)y,\, z_1\overline{R}t_1,\, t_1\overline{T_I(\alpha)}y \quad (T_I)}{x\overline{R}z_1,\, t_1\overline{R;\overline{T_I(\neg\alpha)}}y,\, z_1\overline{R}t_1,\, t_1\overline{T_I(\alpha)}y \quad (N;)}}{\underbrace{x\overline{R}z_1,\, z_1\overline{R}v_1,\, v_1\overline{T_I(\neg\alpha)}y,\, z_1\overline{R}t_1,\, t_1\overline{T_I(\alpha)}y}_{\Lambda_7} \quad \underbrace{x\overline{R}z_1,\, t_1\overline{R}v_2,\, v_2\overline{T_I(\neg\alpha)}y,\, z_1\overline{R}t_1,\, t_1\overline{T_I(\alpha)}y}_{\Lambda_8}}$$

In sequences Λ_7 and Λ_8, $v_1 \in UreVar$ and $v_2 \in CompVar$.
Regarding sequence Λ_7 we have:

$$\frac{\dfrac{\dfrac{x\overline{R}z_1,\, t_1\overline{R}v_1,\, v_1\overline{T_I(\neg\alpha)}y,\, z_1\overline{R}t_1,\, t_1\overline{T_I(\alpha)}y \quad (T_I)}{x\overline{R}z_1,\, t_1\overline{R}v_1,\, v_1\overline{\overline{R;T_I(\alpha)}}y,\, z_1\overline{R}t_1,\, t_1\overline{T_I(\alpha)}y \quad (N^-)}}{x\overline{R}z_1,\, t_1\overline{R}v_1,\, v_1R;T_I(\alpha)y,\, z_1\overline{R}t_1,\, t_1\overline{T_I(\alpha)}y \quad (P;)}}{\underbrace{x\overline{R}z_1,\, t_1\overline{R}v_1,\, v_1Rv_1,\, z_1\overline{R}t_1,\, t_1\overline{T_I(\alpha)}y}_{\Lambda_9} \quad \underbrace{x\overline{R}z_1,\, t_1\overline{R}v_1,\, v_1T_I(\alpha)y,\, z_1\overline{R}t_1,\, t_1\overline{T_I(\alpha)}y}_{\Lambda_{10}}}$$

Since the fork formula v_1Rv_1 accurs in Λ_9, this branch is closed.
Regarding sequence Λ_{10} we have:

$$\frac{x\overline{R}z_1,\, t_1\overline{R}v_1,\, v_1T_I(\alpha)y,\, z_1\overline{R}t_1,\, t_1\overline{T_I(\alpha)}y \quad (H)}{\underbrace{x\overline{R}z_1,\, t_1\overline{R}v_1,\, t_1Rv_1,\, v_1T_I(\alpha)y,\, z_1\overline{R}t_1,\, t_1\overline{T_I(\alpha)}y}_{\Lambda_{11}} \quad \underbrace{x\overline{R}z_1,\, t_1\overline{R}v_1,\, t_1T_I(\alpha)y,\, v_1T_I(\alpha)y,\, z_1\overline{R}t_1,\, t_1\overline{T_I(\alpha)}y}_{\Lambda_{12}}}$$

Since the fork formulas $t_1\overline{R}v_1$ and t_1Rv_1 occur in Λ_{11}, the branch is closed. In a similar way, since the fork formulas $t_1T_I(\alpha)y$ and $t_1\overline{T_I(\alpha)}y$ occur in Λ_{12}, also this branch is closed.
Regarding sequence Λ_8, we have:

$$\frac{x\overline{R}z_1,\, t_1\overline{R}v_2,\, v_2\overline{T_I(\neg\alpha)}y,\, z_1\overline{R}t_1,\, t_1\overline{T_I(\alpha)}y \quad (RUr)}{\underbrace{x\overline{R}z_1,\, t_1\overline{R}v_2,\, v_2\overline{T_I(\neg\alpha)}y,\, z_1\overline{R}t_1,\, t_1\overline{T_I(\alpha)}y,\, t_1Rv_2}_{\Lambda_{13}}}$$

Since the fork formulas $t_1\overline{R}v_2$ and t_1Rv_2 occur in Λ_{13}, the branch is closed.
Finally, regarding sequence Λ_4 we have:

$$\frac{x\overline{R}z_1,\, z_1R;T_I(\neg\neg\alpha)y,\, z_1\overline{R}t_2,\, t_2\overline{T_I(\alpha)}y \quad (RUr)}{\underbrace{x\overline{R}z_1,\, z_1R;T_I(\neg\neg\alpha)y,\, z_1\overline{R}t_2,\, t_2\overline{T_I(\alpha)}y,\, z_1Rt_2}_{\Lambda_{14}}}$$

Since $z_1\overline{R}t_2$ and z_1Rt_2 occur in Λ_{14}, the branch is closed.

7. *A Relational Proof System for Minimal Intuitionistic Logic*

Minimal intuitionistic logic J has been introduced by Johansson in 1936 [18]. It differs from the intuitionistic logic in that the axiom $\neg\alpha \rightarrow (\alpha \rightarrow \beta)$ is deleted. In [4], Fitting introduced a Kripke-style semantics for the logic J. A Kripke model for J is a system $M = \langle W, R, Q, m\rangle$ where W is a nonempty set, R is a reflexive and transitive relation on W, $Q \subseteq W$ is a R-closed subset of W, that is, if $w \in Q$ and $\langle w, w'\rangle \in R$, then $w' \in Q$, and m is a meaning function which is defined as for the Kripke semantics of the intuitionistic logic *Int* with the exception of the evaluation of negations:

$$M, w \models_J \neg\alpha \text{ iff for all } w', \textit{if } \langle w, w'\rangle \in R, \text{ then } M, w' \not\models_J \alpha \text{ or } w' \in Q.$$

Q is to be thought of as the set of those states of information which are inconsistent. The notion of truth of a formula in a model and validity are the same as for *Int*. It is known that a formula α is valid in J iff α is true in every finite model of J with antisymmetric relation R.

Interpretability of J in fork logic FL is established by a translation T_J of formulas of J into relational terms. It coincides with translation T_I except for the translation of negated formulas:

$$T_J(\neg\alpha) = \overline{R;(T_J(\alpha)\cdot\overline{Q})}$$

where R and Q are relational constants interpreted as the accessibility relation from models of J and the right ideal relation that is a counterpart of the set Q from these models.

The relational proof system for J (that we will denote by *J-FLC*) consists of all the rules of the proof system of the fork logic, the specific rules for *Int* and the following specific rules:

$$\frac{\Gamma, xQy, \Delta}{\Gamma, zQy, \Delta, xQy \quad \Gamma, zRx, \Delta, xQy} \; (Q1)$$

$$\frac{\Gamma, xQy, \Delta}{\Gamma, xQz, \Delta, xQy} \; (Q2)$$

$$\frac{\Gamma}{\Gamma, xQy} \; (Q3)$$

In rule $(Q1)$ $z \in UreVar$, in rule $(Q2)$ $z \in IndTerm$, and in rule $(Q3)$, $x \in IndTerm \setminus UreVar$ and $y \in IndTerm$.

Rule $(Q1)$ is admissible iff Q is R-closed, rule $(Q2)$ is admissible iff Q is a right-ideal relation, and rule $(Q3)$ is admissible iff Q has only urelements in its domain.

Notice that the abstract fork-algebraic equations

(C7) : $Q = Q ; 1,$
(C8): $\overline{(R \cdot Q); \overline{Q}} = 1,$
(C9)]: $Q \leq {}_U 1,$

state that Q is an R-closed, right-ideal relation whose domain is made of urelements.

Let $\mathfrak{L}(R, Q)$ be a fork language with two constant symbols. Let $\mathfrak{K}$ be the class of those fork models $\langle\langle \mathfrak{A}, R, Q\rangle, m\rangle$ for the language $\mathfrak{L}(R, Q)$ satisfying conditions (C1) - (C9). Then $\mathfrak{K}$ induces a fork logic FL'' as follows:

$$\models_{FL''} xTy \Leftrightarrow \forall \mathfrak{M} \in \mathfrak{K} (\mathfrak{M} \models_{FL} xTy).$$

From the admissibility of the specific rules (Q1) - (Q3) we obtain the following theorem on the soundness of the calculus *J-FLC*.

Theorem 7.1 *The calculus J-FLC is sound w.r.t. the logic FL", i.e.,*

$$\vdash_{J-FLC} xTy \Rightarrow \models_{FL''} xTy.$$

Definition 7.2 A proof tree T of a sequence of formulas Γ is J-saturated if in all open branches B, the following conditions are satisfied.

1. T is *Int*-saturated,
2. If $xQy \in B$ then for all $z \in UreVar$ either $zQy \in B$ or $zRx \in B$ applying rule $(Q1)$,

3. If $xQy \in B$ then for all $z \in IndTerm$ $xQz \in B$ applying rule ($Q2$),
4. For all $x \in IndTerm \setminus UreVar$ and $y \in IndTerm$, $xQy \in B$ applying rule ($Q3$).

Theorem 7.3 *The calculus J-Int is complete w.r.t. the logic FL'', i.e.,*

$$\models_{FL''} xSy \Rightarrow \vdash_{J-FLC} xSy.$$

Proof. The proof will follow the lines of the proof of Thm. 5.9, and therefore the reader will be directed there for some parts of the proof.

Assume $tSt' \in ForkFor$ is valid in FL'' but is not provable in *J-FLC*. Then no proof tree exists that provides a proof for tSt'. In particular, no J-saturated tree with root tSt' provides a proof. Therefore, if T is a J-saturated tree, there must exist an infinite branch B in T.

Let $\equiv$ be the binary relation on *IndTerm* defined by

$$x \equiv y \Leftrightarrow x1'y \notin B.$$

The proof that $\equiv$ is an equivalence relation is the same as in Thm. 5.9.

Let $\mathfrak{A}$ be the Full$PFAU$ with set of urelements $\{|x| : x \in UreVar\}$ and pairing function $*$ defined by $|x|*|y| = |x*y|$. Proving that $*$ is well defined and injective is done as in Thm. 5.9.

Let us define, for $R' \in RelVar \cup \{R, Q\}$,

$$\langle |t_1|, |t_2| \rangle \in m(R') \Leftrightarrow t_1 R' t_2 \notin B.$$

That m is well-defined is proved as in Thm. 5.9.

That R is reflexive, transitive and that the heredity condition holds are all proved as in Thm. 5.9.

Assume that $\langle |w|, |w'| \rangle \in m(R)$, $\langle |w|, |x| \rangle \in m(Q)$ and $\langle |w'|, |x| \rangle \notin m(Q)$. Then $wRw' \notin B$, $wQx \notin B$, and $w'Qx \in B$. Applying rule ($Q1$) we immediately arrive to a contradiction, and thus Q is R-closed.

Assume $\langle |x|, |y| \rangle \in m(Q)$, but $\langle |x|, |t| \rangle \notin m(Q)$ for some $T \in IndTerm$. Then, $xQt \in B$. Since the tree T is J saturated, applying rule ($Q2$) we arrive to a contradiction, and thus Q is right-ideal.

In a similar way we show that relational variables are interpreted as right-ideal relations.

If there are $x \in IndTerm \setminus UreVar$ and $y \in IndTerm$ such that $\langle |x|, |y| \rangle \in m(R)$, then $xRy \notin B$. Applying rule ($Q3$), $xRy \in B$, which is a contradiction.

Therefore the structure $\langle \langle \mathfrak{A}, m(R), m(Q) \rangle, m \rangle$ is a fork model that satisfies ($C1$)-($C9$).

Let v be the valuation defined by $v(x)=|x|$, for $x \in IndVar$. In Thm. 5.9 it is shown by induction that $v(t)=|t|$ for all $t \in IndTerm$.

The remaining part of the proof is as in Thm. 5.9 ■

In order to be able to reason in the calculus *J-FLC* for proving minimal intuitionistic properties we still need to show the interpretability of the logic *J* in the logic *FL*".

Theorem 7.4 *Let* $\mathfrak{I} = \langle W, R, Q, m\rangle$ *be a minimal intuitionistic model. Then there exists a fork model* $\mathfrak{F} = \langle\langle \mathfrak{A}, R', Q'\rangle, m'\rangle$ *constructed from* $\mathfrak{I}$ *satisfying conditions* (C1)-(C9) *such that for all* $w \in W$ *and for all* $\varphi \in IntFor$

$$\mathfrak{I}, w \models_J \varphi \Leftrightarrow w \in \mathrm{dom}(m'(T_J(\varphi))).$$

Proof Define $\mathfrak{A}$ as the Full*PFAU* with set of urlements W, let $R' = R$, let $Q' = \{\langle x, y\rangle : x \in Q\}$, and for each $R_i \in RelVar$ define $m'(R_i) = \{\langle x, y\rangle : x \in m(p_i)\}$. Conditions $(C1)$ - $(C9)$ hold because of the way R', Q' and m' are defined.

The remaining part of the proof proceeds by induction on the structure of the formula φ and equals the proof of Thm. 6.8 except for the case of the negation.

$$\begin{aligned}
&\mathfrak{I}, w \models_J \neg\alpha \\
&\text{iff } (\forall w' \in W)(wRw' \Rightarrow \mathfrak{I}, w' \not\models_J \alpha \vee w' \in Q) \\
&\text{iff } (\forall w' \in W)(wRw' \Rightarrow w' \notin \mathrm{dom}(m'(T_J(\alpha))) \vee w' \in \mathrm{dom}(Q')) \\
&\text{iff } (\nexists w' \in W)(wRw' \wedge w' \in \mathrm{dom}(m'(T_J(\alpha))) \wedge w' \notin \mathrm{dom}(Q')) \\
&\text{iff } (\nexists w' \in A)(wR'w' \wedge w' \in \mathrm{dom}(m'(T_J(\alpha))) \wedge w' \notin \mathrm{dom}(Q')) \\
&\text{iff } w \in \mathrm{dom}\,\overline{(R' ;(m'(T_J(\alpha)) \cdot \overline{Q'}))} \\
&\text{iff } w \in \mathrm{dom}\,(m'\,\overline{(R' ;(T_J(\alpha) \cdot \overline{Q'}))}) \\
&\text{iff } w \in \mathrm{dom}\,(m'(T_J(\neg\alpha))).
\end{aligned}$$

■

Theorem 7.5 *Let* $\mathfrak{F} = \langle\langle \mathfrak{A}, R, Q\rangle, m\rangle$ be *a fork model satisfying conditions* $(C1)$ - $(C9)$. *Then there exists a minimal intuitionistic model* $\mathfrak{I} = \langle$W, $R', Q', m'\rangle$ *constructed from* $\mathfrak{F}$ *such that for all* $w \in Urel_{\mathfrak{A}}$ *and for all* $\varphi \in IntFor$

$$w \in \mathrm{dom}(m(T_J(\varphi))) \Leftrightarrow \mathfrak{I}, w \models_J \varphi.$$

Proof. Let us define $W = Urel_{\mathfrak{A}}$, $R' = R$, $Q' = \text{dom}\,(Q)$ and for all $p_i \in PropVar$ define $m'(p_i) = \text{dom}\,(m(R_i))$. Notice that by conditions $(C1)$ - $(C9)$ R' is a reflexive and transitive relation on W, the heredity condition is satisfied by m' and Q' is an R-closed right-ideal relation. The remaining part of the proof proceeds by induction on the structure of the formula φ and equals the proof of Thm. 6.9 except for the case of the negation.

$$
\begin{aligned}
& w \in \text{dom}\,(m(T_J(\neg\alpha))) \\
& \text{iff } w \in \text{dom}\,(m(\overline{R;(T_J(\alpha)\cdot\overline{Q})})) \\
& \text{iff } w \in \text{dom}\,(\overline{R;(m(T_J(\alpha))\cdot\overline{Q})}) \\
& \text{iff } (\nexists w' \in A)\,(wRw' \wedge w' \in \text{dom}\,(m(T_J(\alpha))) \wedge w' \notin \text{dom}\,(Q)) \\
& \text{iff } (\nexists w' \in W)\,(wRw' \wedge w' \in \text{dom}\,(m(T_J(\alpha))) \wedge w' \notin \text{dom}\,(Q)) \\
& \text{iff } (\forall w' \in W)\,(wRw' \Rightarrow w' \notin \text{dom}\,(m(T_J(\alpha))) \vee w' \in Q) \\
& \text{iff } (\forall w' \in W)\,(wRw' \Rightarrow \mathfrak{I}, w' \not\models_J \alpha \vee w' \in Q) \\
& \text{iff } \mathfrak{I}, w \models_J \neg\alpha.
\end{aligned}
$$

■

Theorem 7.6 *Let $\psi \in IntFor$. Then, given $x \in UreVar$ and $y \in IndVar$,*

$$\models_J \psi \Leftrightarrow \models_{FL''} xT_J(\psi)y.$$

Proof. Let us prove the contrapositive. If $\not\models_J \psi$, then there exists a minimal intuitionistic model $\mathfrak{I} = \langle W, R, Q, m\rangle$ and $w \in W$ such that $\mathfrak{I},\ w \not\models \psi$. Then, by Thm. 7.4 there exists a fork model $\mathfrak{F} = \langle\langle \mathfrak{A}, R', Q'\rangle, m'\rangle$ such that $w \notin \text{dom}(m'(T_J(\psi)))$. Let v be a valuation satisfying $v(x) = w$, then $\mathfrak{F}$, $v \not\models_{FL''} xT_J(\psi)y$ and thus $\not\models_{FL''} xT_J(\psi)y$.
If $\not\models_{FL''} xT_J(\psi)y$, then there exists a fork model $\mathfrak{F} = \langle\langle \mathfrak{A}, R, Q\rangle, m\rangle$ satisfying $(C1)$ - $(C9)$ and a valuation v such that $\langle v(x), v(y)\rangle \notin m(T_J(\psi))$. Thus, $v(x) \notin \text{dom}(m(T_J(\psi)))$. By Thm. 7.5 there exists a minimal intuitionistic model $\mathfrak{I} = \langle A\ R', Q', m'\rangle$ such that $\mathfrak{I},\ v(x) \not\models_J \psi$ and thus $\not\models_J \psi$.

■

From Thm. 7.3 the corollary below immediately follows.

Corollary 7.7 *Given a formula $\varphi \in IntFor$, $x \in UreVar$* and *$y \in CompVar$, we have*

$$\models_J \varphi \Leftrightarrow \vdash_{J\text{-}FLC} xT_J(\varphi)y.$$

Proof. From Thm. 7.6, for any formula $\varphi \in IntFor$,

$$\models_J \varphi \Leftrightarrow \models_{FL''} xT_J(\varphi)y. \qquad 4$$

From Thms. 7.1 and 7.3, we then obtain

$$\models_{FL''} xT_J(\varphi)y \Leftrightarrow \vdash_{J-FLC} xT_J(\varphi)y. \qquad 5$$

Joining (4) and (5), we then obtain

$$\models_J \varphi \Leftrightarrow \vdash_{J-FLC} xT_J(\varphi)y.$$

■

8. *Relational Reasoning in Intermediate Logics*

Intermediate logics are the logics whose valid formulas include all the formulas that are valid in intuitionistic logic but not necessarily all the tautologies of classical logic. In that sense these logics are between intutionistic and classical logic. For many intermediate logics a Kripke semantics is known. Below we give examples of conditions that the accessibility relation is supposed to satisfy in Kripke models of some intermediate logics.

(I1) $\exists x \forall y (xRy)$
(I2) $\forall x \exists y (xRy \wedge \forall z (yRz \rightarrow y = z))$
(I3) $\forall x \forall y \exists z (zRx \wedge zRy \wedge \forall t (tRx \wedge tRy \rightarrow tRz))$
(I4) $\forall x \forall y \forall z (xRy \wedge xRz \rightarrow yRz \vee zRy \vee \forall t (yRt \rightarrow zRt))$
(I5) $\forall x \forall y \forall z (xRy \wedge xRz \rightarrow \exists t (yRt \wedge zRt))$
(I6) $\exists x \forall y (y \neq x \rightarrow xRy) \wedge \forall x \forall z \forall t (xRz \wedge xRt \rightarrow \neg zRt)$

The translation from formulas of intermediate logics into relational terms is the same as for formulas of intuitionistic logic. There are three methods of developing relational means of reasoning for intermediate logics within the framework of fork logic.

Method 1 We define a specific rule or a fundamental sequence for every condition on the accessibility relation in the underlying Kripke models of a given logic. The relational proof system for the logic consists in all the rules and fundamental sequences from the prof system of fork logic together with those new specific rules and/or fundamental sequences. For example, the rule corresponding to condition (I4) is the following:

$$\frac{\Gamma, \overbrace{yRz, zRy, zRt}^{\Phi}, \Delta \quad (R4)}{\Gamma, xRy, \Phi, \Delta \quad \Gamma, xRz, \Phi, \Delta \quad \Gamma, yRt, \Phi, \Delta}$$

where x is a variable.

Proposition 8.1 *Rule* ($R4$) *is admissible in fork logic iff in every fork model the relation R satisfies condition* ($I4$).

Proof $\Rightarrow$) Observe that condition ($I4$) is equivalent to the following:

$$\forall x \forall y \forall z \forall t (zRy \wedge xRz \wedge yRt \rightarrow yRz \vee zRy \vee zRt).$$

Assume that rule ($R4$) is admissible and suppose that in some fork model condition ($I4$) is not satisfied. Hence, for some valuation v in this model we have $\langle v(x), v(y)\rangle \in R, \langle v(x), v(z)\rangle \in R, \langle v(y), v(t)\rangle \in R$, $\langle v(y), v(z)\rangle \notin R, \langle v(z), v(y)\rangle \notin R$, and $\langle v(z), v(t)\rangle \notin R$. Consider an instance of rule ($R4$) with $\Gamma = x\overline{R}y, x\overline{R}z, y\overline{R}t$ and with Δ empty. Then all the lower sequences of the rule are valid, so the upper sequence must be valid as well. But in the above model none of the formulas of the upper sequence is true under valuation v, a contradiction.

$\Leftarrow$) It is clear that this implication also holds. ■

Method 2 We use the following deduction theorem for fork algebras with urelements.

Theorem 8.2 *Let* $\gamma_1, \ldots, \gamma_n$ *and* γ *be fork algebra terms. Then*

$$\{\gamma_1 = 1, \ldots, \gamma_n = 1\} \models_{PFAU} \gamma = 1 \Leftrightarrow \models_{PFAU} 1; \overline{\gamma_1 \cdot \ldots \cdot \gamma_n}; 1 + \gamma = 1.$$

Proof. $\Rightarrow$) If

$$\{\gamma_1 = 1, \ldots, \gamma_n = 1\} \models_{PFAU} \gamma = 1$$

then, since $SPFAU \subseteq PFAU$,

$$\{\gamma_1 = 1, \ldots, \gamma_n = 1\} \models_{SPFAU} \gamma = 1.$$

Let $\mathfrak{A} \in SPFAU$ and m:*RelConst*$\rightarrow$*A* be arbitrary. If $\mathfrak{A} \models \gamma_1 = 1, \ldots, \gamma_n = 1$, then by hypothesis $\mathfrak{A} \models \gamma = 1$. Then

$$\mathfrak{A} = 1; \overline{\gamma_1 \cdot \ldots \cdot \gamma_n}; 1 + \gamma = 1.$$

If $\mathfrak{A} \not\models \gamma_1 = 1, \ldots, \gamma_n = 1$, then $\overline{\gamma_1 \cdot \ldots \cdot \gamma_n} \neq 0$.. Then, since $\mathfrak{A}$ is simple (and thus satisfies (1) from Def. 3.9, $\mathfrak{A} \models 1; \overline{\gamma_1 \cdot \ldots \cdot \gamma_n}; 1 = 1$. Thus, $\mathfrak{A} \models 1; \overline{\gamma_1 \cdot \ldots \cdot \gamma_n}; 1 + \gamma = 1$. Then, $\models_{SPFAU} 1; \overline{\gamma_1 \cdot \ldots \cdot \gamma_n}; 1 + \gamma = 1$. By Thm. 3.10, we then have

$$\models_{PFAU} 1; \overline{\gamma_1 \cdot \ldots \cdot \gamma_n}; 1 + \gamma = 1.$$

$\Leftarrow$) Notice that if $\gamma_1 = 1, \ldots, \gamma_n = 1$ then $\overline{\gamma_1 \cdot \ldots \cdot \gamma_n} = 0$. Thus,

$$1; \overline{\gamma_1 \cdot \ldots \cdot \gamma_n}; 1 = 0.$$

Thus, if $\models_{PFAU} 1; \overline{\gamma_1 \cdot \ldots \cdot \gamma_n}; 1 + \gamma = 1$, must be $\models_{PFAU} \gamma = 1$. ■

Let $L(\Gamma)$ be an intuitionistic logic, where $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ is a finite set of first-order sentences imposing conditions on the accessibility relation.
Let $\mathfrak{K}_\Gamma$ be the class of those fork models satisfying the set of equations $\{T_{\langle\rangle}(\gamma) = 1 : \gamma \in \Gamma\}$ and let FL_Γ be the fork logic induced by the class of fork models $\mathfrak{K}_\Gamma$.

Theorem 8.3 *Let* $\mathfrak{I} = \langle W, R, m \rangle$ *be a model for the intuitionistic logic* $L(\Gamma)$. *Then there exists a fork model* $\mathfrak{F} = \langle\langle \mathfrak{A}, R' \rangle, m' \rangle$ *for the fork logic* FL_Γ *constructed from* $\mathfrak{I}$ *such that for all* $w \in W$ *and for all* $\varphi \in IntFor$

$$\mathfrak{I}, w \models_{L(\Gamma)} \varphi \Leftrightarrow w \in \mathrm{dom}(m'(T_I(\varphi))).$$

Proof. Define $\mathfrak{A}$ as the Full$PFAU$ with set of urelements W, let $R' = R$, and for each $R_i \in RelVar$ define $m'(R_i) = \{\langle x, y \rangle : x \in m(p_i)\}$. The equations in the set $\{T_{\langle\rangle}(\gamma) = 1 : \gamma \in \Gamma\}$ hold because of the way R' and R_i are defined.
The remaining part of the proof proceeds by induction on the structure of the formula φ, and is as in Thm. 6.8. ■

Theorem 8.4 *Let* $\mathfrak{F} = \langle\langle \mathfrak{A}, R \rangle, m \rangle$ *be a model for the fork logic* FL_Γ. *Then there exists an intuitionistic model* $\mathfrak{I} = \langle W, R', m' \rangle$ *for the logic* $L(\Gamma)$ *constructed from* $\mathfrak{F}$ *such that for all* $w \in W$ *and for all* $\varphi \in IntFor$

$$w \in \mathrm{dom}(m(T_I(\varphi))) \Leftrightarrow \mathfrak{I}, w \models_{L(\Gamma)} \varphi.$$

Proof. Let us define $W = Urel_{\mathfrak{A}}$, $R' = R$, and for all $p_i \in PropVar$ define $m'(p_i) = \mathrm{dom}(m(R_i))$. Notice that the validity of the set of equations $\{T_{\langle\rangle}(\gamma) = 1 : \gamma \in \Gamma\}$ in $\mathfrak{F}$ implies the validity of the sentences Γ in $\mathfrak{I}$. The remaining part of the proof proceeds by induction on the structure of the formula φ as in Thm. 6.9. ■

Theorem 8.5 *Let* $\psi \in IntFor$. *Then, given* $x \in UreVar$ *and* $y \in CompVar$,

$$\models_{L(\Gamma)} \psi \Leftrightarrow \models_{FL(\Gamma)} xT_I(\psi)y.$$

Proof. Let us prove the contrapositive. If $\not\models_{Int} \psi$, then there exists an intuitionistic model $\mathfrak{I} = \langle W, R, m \rangle$ for $L(\Gamma)$ and $w \in W$ such that $\mathfrak{I}, w \not\models_{Int} \psi$. Then, by Thm. 8.3 there exists a fork model $\mathfrak{F} = \langle \langle \mathfrak{A}, R' \rangle, m' \rangle$ such that $w \notin \mathrm{dom}(m'(T_I(\psi)))$ Let v be a valuation satisfying $v(x) = w$, then $\mathfrak{F}$, $v \not\models_{FL_\Gamma} xT_I(\psi)y$, and thus $\not\models_{FL_\Gamma} xT_I(\psi)y$.

If $\not\models_{FL_\Gamma} xT_I(\psi)y$, then there exists a fork model $\mathfrak{F} = \langle \langle \mathfrak{A}, R \rangle, m \rangle$ satisfying the set of equations $\{T_{\langle\rangle}(\gamma) = 1 : \gamma \in \Gamma\}$ and a valuation v such that $\langle v(x), v(y) \rangle \notin m(T_I(\psi))$. Thus, $v(x) \notin \mathrm{dom}(m(T_I(\psi)))$. By Thm. 8.4 there exists an intuitionistic model $\mathfrak{I} = \langle A, R', m' \rangle$ such that $\mathfrak{I}$, $v(x) \not\models_{L(\Gamma)} \psi$, and thus $\not\models_{L(\Gamma)} \psi$. ■

Definition 8.6 We define the calculus $Int - FLC_\Gamma$ by the condition

$$\vdash_{Int-FLC_\Gamma} xQy \Leftrightarrow \vdash_{Int-FLC} x1; \overline{T_{\langle\rangle}(\gamma_1) \cdot \ldots \cdot T_{\langle\rangle}(\gamma_k)}; 1 + Qy$$

Theorem 8.7 *The calculus* $Int - FLC_\Gamma$ *is sound and complete w.r.t. the fork logic* FL_Γ.

Proof.

$$\begin{aligned}
\vdash_{Int-FLC_\Gamma} xQy &\Leftrightarrow \vdash_{Int-FLC} x1; \overline{T_{\langle\rangle}(\gamma_1) \cdot \ldots \cdot T_{\langle\rangle}(\gamma_k)}; 1 + Qy && \text{(by Def. 8.6)} \\
&\Leftrightarrow \models_{FL'} x1; \overline{T_{\langle\rangle}(\gamma_1) \cdot \ldots \cdot T_{\langle\rangle}(\gamma_k)}; 1 + Qy && \text{(by Thm. 6.14)} \\
&\Leftrightarrow \models_{\mathfrak{R}} x1; \overline{T_{\langle\rangle}(\gamma_1) \cdot \ldots \cdot T_{\langle\rangle}(\gamma_k)}; 1 + Qy && \text{(by Def. } FL'\text{)} \\
&\Leftrightarrow \{T_{\langle\rangle}(\gamma) = 1 : \gamma \in \Gamma\} \models_{\mathfrak{R}} Q = 1 && \text{(by Thm. 8.2)} \\
&\Leftrightarrow \models_{\mathfrak{R}_\Gamma} Q = 1 && \text{(by Def.} \mathfrak{R}_\Gamma\text{)} \\
&\Leftrightarrow \models_{FL_\Gamma} xQy. && \text{(by Def. } FL_\Gamma\text{)}
\end{aligned}$$

■

To prove validity of a formula of an intermediate logic that admits a Kripke semantics with a finite set Γ of constraints on the accessibility relation, we should show that for every Kripke model, if all the constraints from Γ are true in the model, then the formula in question is true in this model. We translate the constraints from Γ and the given formula into relational terms and, applying the deduction theorem, we verify validity of the respective term in fork logic. In order to do this we use the proof system $Int - FLC_\Gamma$.

Method 3 We translate constraints from Γ and the formula in question into relational terms and we verify whether from the terms obtained from the members of Γ the term obtained from the formula is derivable. In order to do this we apply the equational means of reasoning within the theory of fork algebras as in [13].

9. *An Intuitionistic Logic for Hardware Verification*

An intuitionistic logic referred to as a propositional lax logic (PLL) has been recently proposed as a tool for a formal verification of computer hardware (Fairtlough and Mendler [3], Mendler [24, 25]). We show how such a logic can be handled in a relational framework. The logic is obtained from the propositional intuitionistic logic by augmenting its language with a unary propositional connective $\bigcirc$ that models delay propagation of signals. Signals are conceived as Boolean valued functions, the Boolean values being denoted by 1 and 0. With input and output signals of combinatorial gates we associate propositional variables. Then if α is such a variable, then truth of α in a model is interpreted as "α is stable at 1", truth of $\neg\alpha$ means "α is stable at 0", truth of $\bigcirc\alpha$ means "α is going to stabilize to 1", and truth of $\bigcirc\neg\alpha$ means "α is going to stabilize to 0". Formally, by a model for PLL we mean a system of the form $M = \langle W, R, S, m\rangle$ where $\langle W, R, m\rangle$ is an intuitionistic model and S is a binary relation in W that is reflexive and transitive, and $S \subseteq R$. The satisfiability of formulas of the form $\bigcirc\alpha$ is defined as follows:

$$M, w \models \bigcirc\alpha \Leftrightarrow$$
$$\text{for all } y \in W, \text{ if } xRy \text{ then there is } z \in W \text{ such that } ySz \text{ and } M, z \models \alpha.$$

Operator $\bigcirc$ is a modal-like operator that shares some features of both necessity and possibility.

The specific axioms that characterize $\bigcirc$ are as follows:

1. $\alpha \to \bigcirc\alpha$,
2. $\bigcirc\bigcirc\alpha \to \bigcirc\alpha$,
3. $(\alpha \to \beta) \to (\bigcirc\alpha \to \bigcirc\beta)$.

Intuitionistic modal logics have been extensively investigated by Vakarelov [36, 37]. Interpretability of PLL in fork logic is established by a translation T_{PLL} that coincides with the translation T_I on the formulas without operator $\bigcirc$ and is extended to formulas of the form $\bigcirc\alpha$ in the following way:

$$T_{\mathrm{PLL}}(\bigcirc\alpha) = \overline{R;\overline{S;T_{\mathrm{PLL}}(\alpha)}},$$

where R and S are relational constants interpreted as the accessibility relations from models of PLL.

The relational proof system for PLL consists of the rules and fundamental sequences for *Int-FLC*, reflexivity and transitivity rules for the relation S and the following rule that reflects the condition $S \subseteq R$:

$$\frac{\Gamma, xRy, \Delta}{\Gamma, xSy, \Delta, xRy} \quad (SR)$$

10. *Conclusions*

In this paper we presented a Rasiowa-Sikorski style proof system *FLC* for the logic *FL* (fork logic) of relations that is based on fork algebras. We proved soundness and completeness of *FLC*. In the spirit of our work on relational formalisation of nonclassical logics ([13, 28, 30]) we developed a methodology of constructing relational logics and relational proof systems for intuitionistic logic, minimal intuitionistic logic, and some superintuitionistic logics. For each logic L from this class of logics we defined the respective relational logic *L-FL* and we proved interpretability of L in *L-FL*. Each of the logics *L-FL* was accompanied by a proof system *L-FLC* obtained as an extension of the proof system *FLC*.

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REFERENCES

[1] Brink, Ch., Kahl, W., and Schmidt, G. (eds) *Relational Methods in Computer Science*. Springer, to appear.

[2] Chin, L.H.C. and Tarski, A., *Distributive and Modular Laws in the Arithmetic of Relation Algebras,* University of California Publications in Mathematics. Ed. University of California, 1951, pp. 341-384.

[3] Fairtlough, M. and Mendler, M. *An intutionistic modal logic with applications to the formal verification of hardware*. Technical Report ID-TR: 1994-13. Department of Computer Science, Technical University of Denmark, 1994.

[4] Fitting, M. *Intuitionistic logic, model theory and forcing*. North Holland, Amsterdam, 1969.

[5] Frias, M.F. and Baum, G.A., *On the Exact Expressiveness and Probability of Fork Algebras*, in Abstracts of the Tenth Latinamerican Symposium on Mathematical Logic, Colombia, 1995.

[6] Frias, M.F., Baum, G.A. and Haeberer A.M. *Adding Design Strategies to Fork Algebras,* in Perspectives of System Informatics, LNCS, 1181, pp. 214-226, 1996.

[7] Frias, M.F., Baum, G.A. and Haeberer A.M. *Representability and Program Construction in Fork Algebras,* Journal of the Interest Group in Pure and Applied Logic, to appear.

[8] Frias, M.F., Baum, G.A., Haeberer, A.M. and Veloso, P.A.S., *Fork Algebras are Representable*, Bull. Section of Logic, Polish Academy of Sciences (1995) vol. 24/2, pp. 64-75.

[9] Frias, M.F. and Gordillo, S.E., *Semantic Optimization of Queries in Deductive Object-Oriented Database*, Proceedings of ADBIS'95, Moscow, Russia, June 1995, Springer-Verlag, pp. 55-72.

[10] Frias, M.F., Haeberer, A.M. and Veloso, P.A.S., *Metalogical Properties of Fork Algebras*, Abstracts of the Winter Meeting of the ASL, San Francisco, California, January 1995. The Bulletin of Symbolic Logic, vol. 1, No. 3, 1995, pp.364-365.

[11] Frias, M.F., Haeberer, A.M. and Veloso, P.A.S., *A Finite Axiomatization for Fork Algebras,* Journal of the IGPL vol. 5, N. 3, pp. 311-319, 1997.

[12] Frias, M.F., Haeberer, A.M., Veloso, P.A.S. and Baum, G.A., *Representability of Fork Algebras*, in Proceedings of the Logic Colloquium '94, July, 1994, p. 51. Also in Bulletin of Symbolic Logic, vol. 1, N. 2, June 1995, pp. 234-235.

[13] Frias, M. and Orlowska, E., *Equational reasoning in non-classical logics*. Journal of Applied Non-Classical Logics, to appear.

[14] Haeberer, A.M., Baum, G.A. and Schmidt G., *On the Smooth Calculation of Relational Recursive Expressions out of First-Order Non-Constructive Specifications Involving Quantifiers*, in Proceedings of the International Conference on Formal Methods in Programming and Their Applications, LNCS 735, Springer-Verlag, 1993, pp. 281-298.

[15] Haeberer, A.M. and Veloso, P.A.S., *Partial Relations for Program Derivation: Adequacy, Inevitability and Expressiveness,* in Constructing Programs from Specifications - Proceedings of the IFIP TC2 Working Conference on Constructing Programs from Specifications. North Holland., IFIP WG. 2.1, Bernhard Möller (ed.), 1991, pp. 319-371.

[16] Henkin, L., Monk, D. and Tarski, A., *Cylindric Algebras* 1, Studies in Logic and the Foundations of Mathematics, vol.64, North-Holland, 1971.

[17] Jaoua, A. (ed) *Proceedings of the Third International Seminar on the Use of Relational Methods in Computer Science*, January 1997, Hammamet, Tunisia.

[18] Johansson, I. *Der minimalkalkuel, ein reduzierter intuitionistischer Formalismus*. Compositio Mathematica 4, 119-136, 1936.

[19] Jónsson, B. and Tarski, A. *Boolean Algebras with Operators PART II*. American Journal of Mathematics, vol. 74, 1952. 127-162.

[20] Lambek, J. *Binary relations in the social and mathematical sciences*. Manuscript, 1996.

[21] Lyndon, R.C., *The representation of relational algebras*, Annals of Mathematics (series 2) 51 (1950), pp. 707-729.

[22] MacCaull, W. *Relational proof system for linear and other substructural logics*. Journal of the Interest Group in Pure and Applied Logic, to appear.

[23] Maddux, R., *Topics in Relation Algebras*, Ph.D. Thesis, University of California at Berkeley, 1978.

[24] Mendler, M., *Constrained proofs: A logic for dealing with behavioural constraints in formal hardware verification*. In: Jones, G. and Sheeran, M. (eds) Designing Correct Circuits. Springer, 1-28, 1990.

[25] Mendler, M., *A modal logic for handling behavioural constraints in formal hardware verification*. PhD thesis, Edinburgh University, Department of Computer Science, 1993.

[26] Németi, I. *Algebraisation of quantifier logics, an introductory overview*. Studia Logica 50, 485-570, 1991.

[27] Orlowska, E., *Relational interpretation of modal logics*. In: Andreka, H., Monk,D. and Nemeti,I. (eds) Algebraic Logic. Colloquia Mathematica Societatis Janos Bolyai vol 54, North Holland, 1988, 443-471.

[28] Orlowska, E., *Relational proof system for relevant logics*. Journal of Symbolic Logic 57, (1992), 1425-1440.

[29] Orlowska,E., *Relational semantics for nonclassical logics: Formulas are relations*. In: Wolenski, J. (ed) Philosophical Logic in Poland. Kluwer, 1994, 167-186.

[30] Orlowska, E., *Relational proof systems for modal logics*. In: Wansing,H. (ed) Proof Theory of Modal Logics. Kluwer, 1995, 55-77.

[31] Polos, L. and Masuch, M. (eds) *Arrow Logic and Multi-Modal Logic*. CSLI Publications, Stanford, California, 141-187, 1996.

[32] Rasiowa, H. and Sikorski, R. *The Mathematics of Metamathematics*. Polish Science Publishers, Warsaw, 1963.

[33] Schmidt, G. and Ströhlein, T., *Relations and Graphs,* EATCS Monographs on Theoretical Computer Science, Springer-Verlag, 1993.

[34] Tarski, A., *On the Calculus of Relations*. Journal of Symbolic Logic (1941), vol. 6, pp. 73-89.

[35] Tarski, A., and Givant, S., *A Formalization of Set Theory without Variables,* A.M.S. Coll. Pub. (1987), vol. 41.

[36] Vakarelov, D., *Intuitionistic modal logic incompatible with the law of excluded middle*. Studia Logica XL, 103-111, 1981.

[37] Vakarelov, D., *An application of Rieger-Nishimura formulas to the intuitionistic modal logic.* Studia Logics XLIV, 79-85, 1985.

[38] Vakarelov, D., *A modal theory of arrows.* Arrow logics I. Lecture Notes in Artificial Intelligence 633, 1-24, 1992.

[39] Veloso, P.A.S. and Haeberer, A.M., *A Finitary Relational Algebra for Classical First-Order Logic,* Bull. Section of Logic, Polish Acad. Sciences (1991), vol. 20/2, pp. 52-62.

[40] Veloso, P.A.S., Haeberer, A.M., and Frias, M.F., *Fork Algebras as Algebras of Logic,* in Abstracts of the Logic Colloquium '94, July, 1994, p. 127. Also in Bulletin of Symbolic Logic, June 1995, 265-266.

[41] Venema, Y. *Cylindric modal logic.* Journal of Symbolic Logic 58, 1003-1034, 1995.