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CANONICAL MODELS FOR TEMPORAL DEONTIC LOGIC

Patrice BAILHACHE

1. *Introduction*

In previous works I presented a temporal deontic system, called R5-D5[1], semantically built on a branching structure, mixing continuous lines which represent the possible non perfect evolutions of the world, and broken lines the possible perfect ones[2]. These evolutions corresponded to alternative *histories of the world* we can now call *paths*. Among them, one represented the real history of our world. When two paths, α, β, are branched at a certain instant, *t*, that is, when both paths have the same history up to *t*, we say that one is accessible to the other with respect to *t* ; i.e. we have then the ternary relation $Rt\alpha\beta$ ($R \subseteq T \times W \times W$, $T =$ *a* set of instants or times [3], W = a set of paths). When a path, β, is branched at a certain instant, *t*, to another, α, and β is perfect from the time *t*, we say that β is *permissible* to α with respect to *t*, that is, is a perfect alternative of α from *t*. This circumstance is formalized by the relation $St\alpha\beta$ ($S \subseteq T \times W \times W$) [4].

Besides classical propositional connectors, there are three modal operators in R5-D5, temporal, alethic and deontic:

R_tA = it is realized at the instant *t* that A.
$\Box A$ = it is necessary that A.
OA = it is obligatory that A.

The evaluation of a proposition should be made at a certain instant, the meta-symbol $\models_{\alpha,t}$ saying that "it is true in the world α at the instant *t*

[1] The original name was RS5-DS5. I abbreviate it here for the sake of simplicity.

[2] See the figure in [Bailhache 1993] p. 161.

[3] I will not specify here structural properties of this set, except that there exists a linear binary relation, <, on it, the intuitive content of which is that $t'<t$ just in case t' is earlier than t ($t' \leq t$ standing for not $t<t'$).

[4] R5-D5 is largely similar to one that was conceived by [Åqvist & Hoepelman 1981]. In this paper *my* relation $Rt\alpha\beta$ is denoted $\alpha \approx_t \beta$, while the deontic relation $St\alpha\beta$ is not introduced as primitve. See [Bailhache 1993].

that..." Obviously an evaluation without a time would be meaningless since a proposition can be true at a certain instant and false at another. The modal operators are conform to the three next rules:

$\models_{\alpha,t}$ $R_{t'}A$ iff $\models_{\alpha,t'}A$ [5]
$\models_{\alpha,t}$ $\Box A$ iff for every β such that $Tt\alpha\beta$, $\models_{\beta,t}A$
$\models_{\alpha,t}$ OA iff for every β such that $St\alpha\beta$, $\models_{\beta,t}A$

As proved in [Bailhache 1983], the branching feature of the structure is semantically assumed by the following properties[6]:

R is a ramified equivalence relation:
— reflexivity: $\forall\alpha\, Rt\alpha\alpha$[7]
— symmetry: $\forall\alpha\,\forall\beta(R\,t\alpha\beta \Rightarrow R\,t\beta\alpha)$
— transitivity: $\forall\alpha\,\forall\beta\,\forall\gamma\big[(R\,t\alpha\beta\,\&\,R\,t\beta\gamma) \Rightarrow R\,t\alpha\gamma\big]$
— ramification: $\forall t\,\forall t'\,\forall\alpha\,\forall\beta\big[(t' < t\,\&\,R\,t\alpha\beta) \Rightarrow R\,t'\,\alpha\beta\big]$
S is serial, secondary-reflexive and secondary-ramified:
— seriality: $\forall t\,\forall\alpha\,\exists\beta\,S\,t\alpha\beta$
— secondary-reflexivity: $\forall t\,\forall\alpha\,\forall\beta(S\,t\alpha\beta \Rightarrow S\,t\beta\beta)$[8]
— secondary-ramification:
$\forall t\,\forall t'\,\forall\alpha\,\forall\beta\,\forall\gamma\big[(t' < t\,\&\,S\,t'\,\alpha\beta\,\&\,S\,t\beta\gamma) \Rightarrow S\,t'\,\beta\gamma\big]$[9]

In addition, there are three properties common to R and S:
— SR-implication: $\forall t\,\forall\alpha\,\forall\beta\,(S\,t\alpha\beta \Rightarrow R\,t\alpha\beta)$

5 Since there is no ambiguity, we will never mention the reference to a model, writting only $\models_{\alpha,t}$ instead of $\models^{\mathfrak{M}}_{\alpha,t}$).

6 For an intuitive justification of these properties see [Bailhache 1991], p. 72, 80-81. Most of them are obvious; for example, reflexivity of R only says that, given a certain instant t, a path has the same history of himself, symmetry that if a path has the same history as another, then the latter has the same history as the former, too; and so on. Note that RS-transitivity has been forgotten in page 80.

7 Quantifiers are metalinguistic.

8 Cf. [Chellas 1980], p. 92.

9 The property trivially holds for $t' = t$, so $t' < t$ suffices.

— *RS*-transitivity: $\forall t\,\forall\alpha\,\forall\beta\,\forall\gamma\big[(R\,t\alpha\beta\;\&\;S\,t\beta\gamma)\Rightarrow S\,t\alpha\gamma\big]$

— *RS*-post-implication:

$\forall t\,\forall t'\,\forall\alpha\,\forall\beta\,\forall\gamma\big[(t'<t\;\&\;S\,t'\,\alpha\beta\;\&\;R\,t\beta\gamma)\Rightarrow S\,t'\,\beta\gamma\big]$[10]

To these semantic properties corresponds a system that can be axiomatized in the following manner. Each of the three operators, R_t, $\Box$, O, taken separately, complies with each of the systems, $\mathbb{R}$, S5 and KDU5, respectively ($\mathbb{R}$ is a system for temporal logic close to Rescher & Urquhart's)[11]. Concerning O, however, there is no need for the proper "5O"-axiom of KDU5, $\neg OA \rightarrow O\neg OA$, to be laid down, since it can be proved on the basis of S5 and KD[12]. Thus, we have first the three subsets of axioms and rules:

Kt	$R_t(A\rightarrow B)\rightarrow(R_tA\rightarrow R_tB)$
D!t	$R_tA\leftrightarrow\neg R_t\neg A$ [13]
Rtt'	$R_{t'}R_tA\leftrightarrow R_tA$
RNt	$A\,/\,R_tA$ [14]
K$\Box$	$\Box(A\rightarrow B)\rightarrow(\Box A\rightarrow\Box B)$
T$\Box$	$\Box A\rightarrow A$
5$\Box$	$\neg\Box A\rightarrow\Box\neg\Box A$
RN$\Box$	$A\,/\,\Box A$
KO	$O(A\rightarrow B)\rightarrow(OA\rightarrow OB)$
DO	$OA\rightarrow\neg O\neg A$
UO	$O(OA\rightarrow A)$
RNO	$A\,/\,OA$

10 This time the property does not hold for $t' = t$.

11 KDU5 is the system KD (= K + the D-axiom) plus the U-axiom and the 5-axiom.

12 The proof rests on the axioms $\Box O$ and $\Box O4$ below. See [Bailhache 1983], p. 766. The stronger theorem $\neg OA\rightarrow\Box\neg OA$ can even be proved.

13 See [Chellas 1980], p. 93. From a general point of view, the present notation of axioms and rules follows Chellas' one.

14 The notation $A\,/\,B$ means that if A is a thesis then B is a thesis, too. As known, other writings are $\vdash A\rightarrow\vdash B$, $\frac{A}{B}$. In my previous works appeared also the rule RN_ct $R_tA\,/\,A$ (t not free in A) which plays a role in [Rescher & Urquhart 1971]. I am not sure now that it is necessary, at least if no specification is made about the set of instants.

Some axioms should be added to characterize features common to two or three operators together.

R_t and $\Box$:

$\Box t1$	$R_t\Box A \leftrightarrow R_t\Box R_t A$
$\Box t2$	$t' \leq t \Rightarrow R_{t'}\Box R_t A \rightarrow R_t\Box R_t A$

R_t and O:

$Ot1$	$R_t OA \leftrightarrow R_t OR_t A$
$Ot2$	$t' < t \Rightarrow R_{t'} O\left(R_{t'} OR_t A \rightarrow R_t OR_t A\right)$ [15]

$\Box$ and O:

$\Box O$	$\Box A \rightarrow OA$
$\Box O4$	$OA \rightarrow \Box OA$

R_t, $\Box$ and O:

$\Box Ot$	$t' < t \Rightarrow R_{t'} OR_t\left(OA \rightarrow \Box A\right)$ [16]

Soundness and completeness have been proved earlier, the latter property by using a style of proofs like Hughes & Cresswell's 1968 textbook. Unfortunately, this method led to very complicated calculations compelling us to consider numerous systems increasingly rich (T, S4, S5, RT, R4, R5, T-DT, S4-D4, S5-D5, RT-DT, R4-D4, R5-D5)[17]. There is another way, however, to manage the problem, namely, by looking for *canonical models* adequate to the given axiomatics. The method, which exploits well-known Henkin's technique of maximal consistent sets of formulae, is implemented in Chellas' *Modal Logic* for different systems, only with one modality. Thus our task now is double. First, considering each modality separately, we have to produce canonical models for $\mathbb{R}$, S5, KDU (in fact the problem is already solved for S5 and KDU). Then, mixing the three modalities, R_t, $\Box$ and O, general canonical models should be constructed, which correspond to the remaining mixed axioms, $\Box t1$,...$\Box Ot$. As in Chellas' book, not only a canonical model is required, but also the *proper* canonical model, to prove the existence of canonical models[18].

[15] The formula is a trivial thesis for $t' = t$ (cf. the footnote about the secondary-ramification).

[16] The formula is false for $t' = t$ (cf. the footnote about the *RS*-post-implication).

[17] [Bailhache 1983], Annexe II, pp. 711-772. In this book, R4, R5, S4-D4, S5-D5, R4-D4 were respectively named RS4, RS5, S4-DS4, S5-DS5, RS4-DS4.

[18] [Chellas 1980], p. 173.

2. *Chellas' models*

2.1. *Standard models*

Standard models are semantic structures for normal systems. By definition, a system of modal logic is *normal* if and only if it contains Df$\Diamond$ and is closed under the rule RK[19]:

Df $\Diamond$ $\quad \Diamond A \leftrightarrow \neg\Box\neg A$

RK $\quad (A_1 \wedge \ldots \wedge A_n) \rightarrow A \,/\, (\Box A_1 \wedge \ldots \wedge \Box A_n) \rightarrow \Box A$

Another way to define normal systems is to use the axiom K and the rule RN [20]:

K $\quad \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

RN $\quad A \,/\, \Box A$

Chellas also considers systems which do not contain RK (or K and RN) and are called *classical*. It will not be necessary to consider them since, in the present paper, K and RN hold for the three modalities at issue (cf. Kt, RNt, K$\Box$, RN$\Box$, KO, RNO).

For reasons of simplicity we shall not introduce the related modalities of possibility, permission and *not realization at* t *that not*[21]. This does not mean that such modalities are not definable, but on the contrary that they raise no particular difficulty at all. So, in the definition of a standard model, we can neglect Df $\Diamond$.

$\mathfrak{M} =$ $\langle W, R, P\rangle$ is a *standard model* if and only if[22]:

(1) W is a set $\{\alpha, \beta, \ldots\}$.

(2) R is a binary relation on W (i.e. $R \subseteq W \times W$).

(3) P is a mapping from natural numbers to subsets of W (i.e. $P_n \subseteq W$, for each natural number n, formally: $P{:}\mathbb{N} \rightarrow P(W)$).

19 [Chellas 1980], p. 114. In this definition, of course, the modality $\Box$ should be understood as a general one which may stand for alethic, temporal, deontic,... concepts.

20 *Ibid.*, p. 115.

21 The axiom DD_ct shows that this modality does not differ from R_t.

22 *Ibid.*, p. 68.

Conditions (1) and (2) are well-known. Condition (3) can be understood as a description of the worlds with respect to the truth of atomic propositions, i.e. *the atomic proposition* $\mathbb{P}_n$ *is true at a world* α *in a model* $\mathfrak{M}$ *if* α *belongs to* P_n [23], in symbols:

$$\underset{\alpha}{\models} \mathbb{P}_n \text{ iff } \alpha \in P_n$$ [24]

The model is supplemented by a definition that gives us the way of evaluating a modal sentence:

$$\underset{\alpha}{\models} \Box A \text{ iff for every } \beta \text{ such that } \alpha R \beta, \underset{\beta}{\models} A.$$

2.2. Soundness

A system is sound with respect to a semantic interpretation if every theorem is valid. Since validity is truth is every world, the property can be written:

if $\vdash A$ then $\models A$

It is readily proved for normal systems. It suffices to show that each axiom is valid and rules of deduction preserve validity.

2.3. Completeness

Conversely, a system is complete with respect to a semantic interpretation if every valid formula is a theorem, formally:

if $\models A$ then $\vdash A$

Obviously the simple proof of soundness is not applicable for completeness. The method of canonical modes, although not very simple, does the job with a maximal elegance.

Definition of a *canonical standard model*, $\mathfrak{M} = <W, R, P>$, for a system Σ [25]:

[23] *Ibid.*, p. 5 and 35.

[24] In the sequel, I simply write $\models$ instead of $\overset{\mathfrak{M}}{\models}$ when no confusion may arise (cf. footnote 5).

[25] *Ibid.*, p. 171.

(1) $W = \{\Gamma : Max_\Sigma \Gamma\}$

(2) For every α $\Box A \in \alpha$ iff for every β such that $\alpha R \beta, A \in \beta$

(3) $P_n = |\mathbb{P}_n|_\Sigma, n \in \mathbb{N}$.

where

$Max_\Sigma \Gamma$ means that Γ is a set of sentences that is Σ-maximal, i.e. is Σ-consistent and has only Σ-inconsistent proper extensions, that is, Γ is consistent and contains as many sentences as it can without becoming inconsistent[26].

The symbols $|\ |_\Sigma$ denote a set such that

$|A|_\Sigma = \{Max_\Sigma \Gamma : A \in \Gamma\}$ i.e. $\Gamma \in |A|_\Sigma$ iff $A \in \Gamma$

$|A|_\Sigma$ is the *proof set* of A. This denomination is justified by the fact that, Γ being Σ-maximal, $A \in \Gamma$ is equivalent to $\Gamma \vdash_\Sigma A$.

The main property of a canonical model is that every formula is true at a world if and only if the formula *belongs to* this world (remember that a world of a canonical model is a set of formulae), in symbols:

$\models_\alpha$ A iff $A \in \alpha$ (MPCN)

This property, of course, has to be proved for each particular class of canonical models. Then since α is Σ-maximal $A \in \alpha$ is equivalent to $\alpha \vdash_\Sigma A$, as just explained. Thus, when all the worlds, α, are considered,

$\models A$ then $\vdash A$

so that completeness (and soundness) is established.

The proof of MPCN is by induction on the complexity of A. For example, assuming the proof is done for the classical propositional calculus, if a modality, $\Box$, is added, only the case of a formula $A = \Box B$ should be considered, where B is a formula complying with MPCN[27].

Notice that the existence of a canonical model is not yet proved. To this end, a particular canonical model must be exhibited, which is realized by

[26] *Ibid.*, p. 53.

[27] For propositional calculus see [Chellas 1980], p. 61, exercise 2.53; for the addition of $\Box$ *ibid.*, p. 172-173.

constructing the *proper* canonical model, defined by conditions (1)-(3) of a canonical standard model and:

$$\alpha R\beta \text{ iff } \{A:\Box A\in\alpha\}\subseteq\beta$$

But it remains to prove that this quasi definition of R is compatible with the condition (2) of a canonical standard model, i.e. that

For every α, $\Box A\in\alpha$ iff for every β such that $\{A:\Box A\in\alpha\}\subseteq\beta$, $A\in\beta$.

The equivalence is trivial only in the left-to-right direction. The reverse implication requires using of the rule RK (or the axiom K and the rule RN), which explains why the so defined proper canonical model is available only for normal systems (i.e. those which necessarily contain RK)[28].
Finally, an important feature of a proof of completeness with canonical models is how systems stronger than K are shown to be complete. When an axiom is added to the smallest normal system, K, *one has to prove that this addition corresponds to some semantic property*. For example,

$$D \quad \Box A\rightarrow\neg\Box\neg A$$

corresponds to the seriality of R:

For every α there is at least one β such that $\alpha R\beta$.

More precisely, to prove that seriality implies D is to show that system KD (= the system K + the axiom D) is *sound* with respect to the class of models where R is serial. Conversely, to prove that D implies the seriality of R is to show that KD is *complete* with respect to the same class of models[29]. Thus, since R5-D5 can be conceived as an extension[30] of the union of the three subsystems, $\mathbb{R}$, S5, KDU, when one has proved that these subsystems and their corresponding submodels are adequate, it will remain to establish that

[28] Chellas does not insist enough on this fact. The proof (p. 174) rests on theorem 4.30 (1), p. 158, which is shown thanks to *RK*.

[29] See [Chellas 1980], p. 175ff. For the *U*-axiom, see p. 92, 140, 193ff.

[30] An extension because there are supplementary axioms characteristic of two or three mixed modalities among R_t, $\Box$ and O.

each extending axiom implies (and is implied by) a corresponding semantic property.

3. *Searching for canonical models for* R5-D5

I shall now progressively define the structure of a canonical model for R5-D5 and, thus, give its formal definition only at the end of the paper.

3.1. *Worlds*

The previous semantics, in Hughes & Cresswell's style, was constructed out of worlds that were *paths* on which two relations existed with respect to time, $Rt\alpha\beta$ and $St\alpha\beta$. If we want to transfer this conception to canonical models, a difficulty arises. As known, in such models, worlds are maximal consistent sets of formulae, i.e. sets without any pair of formulae A and $\neg A$. But this is quite possible in a world α, since A may be true at some instant t and false at some other t'. Thus the sole solution of this dilemma is to define worlds of a canonical model not as paths, but as states of paths at a certain instant, that is, to replace α by α_t. Evaluation in α at t will be trivially replaced by evaluation in α_t in symbols $\models_{\alpha,t}$ by $\models_{\alpha_t}$.

Moreover, each of the three modalities, R_t, $\Box$, O, should be interpreted by means of *a specific binary relation between states of paths*. Let us choose T_t for R_t, R for $\Box$, S for O [31]. It is noteworthy that the relation T should be indexed by t as the operator R_t himself. In fact, there are as so many operators R_t as instants t [32].

3.2. *Canonical submodels for temporal logic* $\mathbb{R}$

Let us call u, v, w,... the members of W where formulae should be evaluated. The condition of evaluation of R_t should be similar to that of $\Box$ in standard models:

$$\models_u R_t A \text{ iff } \forall v \;\; st \;\; u T_t v, \models_v A \qquad (V\text{standard})$$

and, as known, it should amount to:

[31] R and S are now *binary* relations between states of paths, no longer the ternary relations $Rt\alpha\beta$ and $St\alpha\beta$ at the begining of Introduction.

[32] See footnote 1.

$$\vDash_{\alpha_{t'}} R_t A \text{ iff } \vDash_{\alpha_t} A \qquad (VR_t)$$

which means that it is true in α at t' that it is realized at t that A iff it is true in α at t that A. The problem is to prove, *only using the axioms and rules for* R_t, that condition (Vstandard) can be reduced to (VR_t).

As mentioned above, system $\mathbb{R}$ contains three axioms, *Kt, D!t, Rtt'* and one rule, *RNt*. Since *Kt* and *RNt* are the elements required for $\mathbb{R}$ to be a normal system, they have no effect on the semantic structure.

D!t is adequate with the *functionality* of T_t that is, with its seriality and semi-functionality[33]. For *D!t* can be viewed as the conjunction of

$$Dt \qquad R_t A \to \neg R_t \neg A$$

and

$$D_c t \qquad \neg R_t \neg A \to R_t A$$

The former is adequate with seriality[34], which says that for every element of W there is *at least* one accessible element, i.e.

$$\forall u \exists v \; u \, T_t v$$

The latter is adequate with semi-functionality, which means that for every element of W there is *at most* one accessible element, i.e.

$$\forall u \, v \, w \left(u \, T_t v \;\&\; u \, T_t w\right) \Rightarrow v = w$$

Thus *D!t* imposes that there is *exactly* one accessible element, which is functionality. Thus, instead of $u \, T_t v$, we are entitled to write

$$v = \phi(u, t)$$

where ϕ is a function of u and t.
Let us examine now what is the consequence of *Rtt'*. This axiom is an equivalence:

$$Rtt' \qquad R_{t'} R_t A \leftrightarrow R_t A$$

So we have to evaluate each equivalent proposition and see under what condition both evaluations yield the same result. For the left one:

33 [Chellas 1980], p. 167.

34 *Ibid.*, p. 175.

$\vDash_u R_{t'} R_t A$ iff $\forall v$ *st* $u\, T_{t'} v$, $\vDash_v R_t A$

where the universal quantifier can be dropped since functionality implies that there is only one v:

$$\vDash_u R_{t'} R_t A \text{ iff } u\, T_{t'} v \,\&\, \vDash_v R_t A \quad (1)$$

Again, the last term can be analyzed in the same way:

$$\vDash_v R_t A \text{ iff } v\, T_t w \,\&\, \vDash_w A \quad (2)$$

Similarly, for the proposition on the right :

$$\vDash_u R_t A \text{ iff } u\, T_t x \,\&\, \vDash_x A \quad (3)$$

These two ways of evaluating will be equivalent if and only if $x = w$.

Now let us employ the symbol α_t to name the function $\phi(u,t)$, attaching the letter α to the element u, so that $\phi(u,t')$ will be $\alpha_{t'}$, but $\phi(v,t)$ will be β_t. In virtue of $u\,T_{t'} v$ in (1) and $u\, T_t x$ in (3)

$$v = \alpha_{t'} \text{ and } x = \alpha_t$$

Taking into account $w = x$, the relation $vT_t w$ in (2) can be written

$$\alpha_t = \phi(\alpha_{t'}, t)$$

which shows us that the function ϕ preserves the part α and only changes the index. Thus one is allowed to interpret α as a path and the relation T_t as determining the instant t on this path. Therefore, in (2), v is $\alpha_{t'}$ and w is α_t under this interpretation. Finally (2) becomes

$$\vDash_{\alpha_{t'}} R_t A \text{ iff } \vDash_{\alpha_t} A \quad (VR_t)$$

which is the expected result.

This part of the proof is the most original one and cannot be simplified. Only when this stage has been reached, one is untitled to speak about *states of paths* and make evaluation in them.

3.3. *Introducing* $\Box$

Elements of W being states of path, the condition of evaluation of $\Box$ must be formulated

$$\vDash_{\alpha_t} \Box A \text{ iff } \forall \beta_{t'} \; st \; \alpha_t R \beta_{t'}, \vDash_{\beta_{t'}} A$$

Intuitively, the alethic modality changes path but not time (what is necessary at some time is what is true on every accessible world at this time). Thus, according to the already known semantic structure of R5-D5 (in Hughes & Cresswell's style), we have now to prove that the evaluation of $\Box$ amounts to:

$$\models_{\alpha_t} \Box A \text{ iff } \forall \beta_t \; st \; \alpha_t R \beta_t, \models_{\beta_t} A$$

where R is a ramified equivalence relation (see *Introduction*). The employed axioms and rules are $K\Box$, $T\Box$, $5\Box$, $RN\Box$ (for $\Box$ only) and $\Box t1$, $\Box t2$ (for R_t and $\Box$ together). The first four ones are characteristic of system S5. Chellas' proof can be taken up again to demonstrate that R must be an equivalence relation. So it remains to consider $\Box t1$ and $\Box t2$.

The former is

$$R_t \Box A \leftrightarrow R_t \Box R_t A$$

and we have to check that the evaluation of each equivalent member of it leads us to the same result. For the left-hand side[35]:

$$\models_{\alpha_1} R_t \Box A \text{ iff } \models_{\alpha_t} \Box A \text{ iff } \forall \beta_{t'} \; st \; \alpha_t R \beta_{t'}, \models_{\beta_{t'}} A$$

For the right-hand one:

$$\models_{\alpha_1} R_t \Box R_t A \text{ iff } \models_{\alpha_t} \Box R_t A \text{ iff } \forall \beta_{t'} \; st \; \alpha_t R \beta_{t'}, \models_{\beta_{t'}} R_t A \text{ iff} \models_{\beta_t} A$$

These two ways of evaluating will be equivalent if and only if $t' = t$, which implies that the evaluation of $\Box$ has the expected form

$$\models_{\alpha_t} \Box A \text{ iff } \forall \beta_t \; st \; \alpha_t R \beta_t, \models_{\beta_t} A$$

This result has a heavy import. It permits us to replace the former style relation $Rt\alpha\beta (R \subseteq T \times W \times W)$ by the new one $\alpha_t R \beta_t (R \subseteq W \times W)$, the set W being now constituted with states of paths instead of paths.

Let us examine finally the import of $\Box t2$:

$$t' \leq t \Rightarrow R_{t'} \Box R_t A \rightarrow R_t \Box R_t A$$

35 For convenience, we note certain instants by numbers, 1, 2,...

We have to search for on what semantic condition the antecedent implies the consequent, assuming $t' \leq t$. Evaluating the former gives us

$$\vDash_{\alpha_1} R_{t'}\Box R_t A \text{ iff } \vDash_{\alpha_{t'}} \Box R_t A \text{ iff } \forall \beta_{t'} \; st \; \alpha_{t'} R \beta_{t'}, \vDash_{\beta_{t'}} R_t A \text{ iff} \vDash_{\beta_t} A$$

and the latter

$$\vDash_{\alpha_1} R_t\Box R_t A \text{ iff } \vDash_{\alpha_t} \Box R_t A \text{ iff } \forall \beta_t \; st \; \alpha_t R \beta_t, \vDash_{\beta_t} R_t A \text{ iff} \vDash_{\beta_t} A$$

Now we have seen, in the Introduction, how the property of ramification was formulated in [Bailhache 1983]:

$$\forall t \forall t' \; \forall \alpha \; \forall \beta \left[(t' \leq t \;\&\; R\, t \alpha \beta) \Rightarrow R\, t' \, \alpha \beta \right]$$

In the present context $R\, t\alpha\beta$ must be replaced by $\alpha_t R \beta_t$, so that ramification (RAM) will be false if there is a path γ such that

$$\alpha_t \; R \; \gamma_t \text{ and not } \alpha_{t'} R \gamma_{t'}$$

Assuming not-RAM, if we suppose $\nvDash_{\gamma_t} A$, the former evaluation may be true since $\gamma \notin \{\beta \colon \alpha_{t'} R \beta_{t'}\}$ while the latter is false since $\gamma \in \{\beta \colon \alpha_t R \beta_t\}$. Thus $\Box t2$ implies RAM, Q.E.D.[36]

3.4. *Introducing O*

The deontic operator will be evaluated according to the following condition which involves a relation, *S*, of *permissibility*

$$\vDash_{\alpha_t} OA \text{ iff } \forall \beta_{t'} \; st \; \alpha_t S \beta_{t'}, \vDash_{\beta_{t'}} A$$

The case is similar to that of alethic modality, so that this condition must be reduced to one in which the time does not change:

$$\vDash_{\alpha_t} O\text{A iff } \forall \beta_t \; st \; \alpha_t S \beta_t, \vDash_{\beta_t} A$$

Moreover, our previous study in Hughes & Cresswell's style permits us to know that *S* possesses three properties, seriality, secondary-reflexivity, secondary-ramification, and that there are three additional properties common to R and S, *SR*-implication, *RS*-transitivity, *RS*-post-implication (see *Introduction*). The corresponding axiomatic elements are K*O*, D*O*, UO,

[36] In the present case as in the sequel, converse implications are readily proved, which establishes systems to be not only complete but also sound, i.e. adequate.

RN*O* and, for mixed modalities, *O*t1, *O*t2, □*O*, □*O*4, □*O*t. As already mentioned, the first four ones are adequate with the system *KDU*; proof can be found in [Chellas 1980]. Thus we have to consider the remaining five axioms.

The first is

$$R_t OA \leftrightarrow R_t OR_t A$$

quite similar to □*t*1, but with *O* instead of □. The consequence will be similar, too, namely, $t' = t$ in the condition of evaluation. Thus we reach the expected form

$$\vDash_{\alpha_t} OA \text{ iff } \forall \beta_t \; st \; \alpha_t S\beta_t, \vDash_{\beta_t} A$$

and, as in the alethic case, the relation $St\alpha\beta$ can be replaced by $\alpha_t S\beta_t$.

The second

$$t' < t \Rightarrow R_{t'} O(R_{t'} OR_t A \rightarrow R_t OR_t A)$$

offers some similarity with □*t*2 ($t' \leq t \Rightarrow R_{t'} \Box R_t A \rightarrow R_t \Box R_t A$). It is too different, however, not to be studied separately. Assuming that $t' \leq t$, we can start its evaluation

$$\vDash_{\alpha_1} R_{t'} O(R_{t'} OR_t A \rightarrow R_t OR_t A)$$

i.e.

$$\vDash_{\alpha_{t'}} O(R_{t'} OR_t A \rightarrow R_t OR_t A)$$

This evaluation is equivalent to

$$\forall \beta_{t'} \; st \; \alpha_{t'} \; S \; \beta_{t'}, \; \vDash_{\beta_{t'}} R_{t'} OR_t A \rightarrow R_t OR_t A$$

Thus, under the condition $\forall \beta_{t'} \; st \; \alpha_{t'} \; S \; \beta_{t'}$

$$\vDash_{\beta_{t'}} R_{t'} OR_t A \text{ must imply } \vDash_{\beta_{t'}} R_t OR_t A$$

The former amounts to

$$\vDash_{\beta_{t'}} OR_t A \text{ i.e. } \forall \gamma_{t'} \; st \; \beta_{t'} \; S \; \gamma_{t'}, \; \vDash_{\gamma_{t'}} R_t A \text{ or } \vDash_{\gamma_t} A$$

and the latter

$$\vDash_{\beta_t} OR_t A \text{ i.e. } \forall \delta_t \; st \; \beta_t \; S \; \delta_t, \; \vDash_{\delta_t} R_t A \text{ or } \vDash_{\delta_t} A$$

As in the case of ramification, the secondary-ramification (S-RAM) of S, formulated in [Bailhache 1983] by

$$\forall t \forall t' \forall \alpha \forall \beta \forall \gamma [(t' < t \;\&\; S\,t'\,\alpha\beta \;\&\; S\,t\beta\gamma) \Rightarrow S\,t'\,\beta\gamma]$$

should be re-written with $\alpha_t S\beta_t$ instead of $St\alpha\beta$. This property will be false if there is a path ε such that

$$\alpha_{t'} \; S \; \beta_{t'}, \; \beta_t \; S \; \varepsilon_t \text{ and not } \beta_{t'} \; S \; \varepsilon_{t'}$$

Assuming not-S-RAM, if we suppose $\not\models_{\varepsilon_t} A$, the former evaluation may be true since $\varepsilon \notin \{y: \beta_{t'}\ S\ \gamma_{t'}\}$, while the latter is false since $\varepsilon \in \{\delta: \beta_t\ S\ \delta_t\}$. Thus S-RAM is a consequence of *Ot*2, Q.E.D[37].

The third axiom is

$\Box O \qquad \Box A \rightarrow OA$

Again we have to search for on what semantic condition the antecedent implies the consequent, i.e. when

$\models_{\alpha_1} \Box A$, that is $\forall \beta_1\ st\ \alpha_1\ R\ \beta_1,\ \models_{\beta_1} A$

implies

$\models_{\alpha_1} OA$, that is $\forall \beta_1\ st\ \alpha_1\ S\ \beta_1,\ \models_{\beta_1} A$

Obviously, this inference is bound to the property of *SR*-implication presented above:

$\forall t \forall \alpha \forall \beta (\alpha_t\ S\ \beta_t \Rightarrow \alpha_t\ R\ \beta_t)$

It is falsified if there is a path γ such that

$\alpha_1\ S\ \gamma_1$ and not $\alpha_1\ R\ \gamma_1$

Assuming not-*SR*-implication, if we suppose $\not\models_{\gamma_1} A$, the former evaluation may be true since $\gamma \notin \{\beta: \alpha_1\ R\ \beta_1\}$, while the latter is false since $\gamma \in \{\beta: \alpha_1\ S\ \beta_1\}$. Thus *SR*-implication is a consequence of $\Box O$. Q.E.D. To abbreviate the proof, it will be sufficient to say that, for the inference between evaluations to be saved, it is necessary that every β such that $\alpha_1\ S\ \beta_1$ is among the γ's such that $\alpha_1\ R\ \gamma_1$, which is just *SR*-implication.

The fourth axiom,

$\Box O4 \qquad OA \rightarrow \Box OA,$

leads to a very similar proof. The evaluation of antecedent

$\models_{\alpha_1} OA$, that is $\forall \beta_1\ st\ \alpha_1\ S\ \beta_1,\ \models_{\beta_1} A$

must imply that of consequent

$\models_{\alpha_1} \Box OA$, that is $\forall \beta_1\ st\ \alpha_1\ R\ \beta_1,\ \models_{\beta_1} OA$

or $\forall \beta_1\ st\ \alpha_1\ R\ \beta_1,\ \forall \gamma_1\ st\ \beta_1\ S\ \gamma_1,\ \models_{\gamma_1} A$

Adopting the abbreviated style of proof just used for the previous axiom, the inference requires that every γ_1 such that $\alpha_1\ R\ \beta_1\ \&\ \beta_1\ S\ \gamma_1$ is among the δ_1's such that $\alpha_1\ S\ \delta_1$, therefore:

$\forall t \forall \alpha \forall \beta \forall \gamma [(\alpha_t\ R\ \beta_t\ \&\ \beta_t\ S\ \gamma_t) \Rightarrow \alpha_t\ S\ \gamma_t]$

37 As for ramification of *R*, the converse inference is readily proved, which permits to establish soundness relatively to *Ot*2. The same remark holds for the remaining axioms.

This property precisely is the *RS*-transitivity. Q.E.D.

Finally it remains to examine the case of

$\Box Ot \qquad t' < t \Rightarrow R_{t'} OR_t (OA \rightarrow \Box A)$

Again assuming $t' < t$, we have to evaluate $R_{t'} OR_t (OA \rightarrow \Box A)$:

$\models_{\alpha_1} R_{t'} OR_t (OA \rightarrow \Box A)$

i.e.

$\models_{\alpha_{t'}} OR_t (OA \rightarrow \Box A)$

which is equivalent to

$\forall \beta_{t'}$ *st* $\alpha_{t'} \, S \, \beta_{t'}$, $\models_{\beta_{t'}} R_t (OA \rightarrow \Box A)$

i.e. $\models_{\beta_t} OA \rightarrow \Box A$

Thus under the prescribed condition $(\forall \beta_{t'}$ *st* $\alpha_{t'} \, S \, \beta_{t'})$

$\models_{\beta_t} OA$, i.e. $\forall \gamma_t$ *st* $\beta_t \, S \, \gamma_t$, $\models_{\gamma_t} A$

must imply

$\models_{\beta_t} \Box A$, i.e. $\forall \delta_t$ *st* $\beta_t \, R \, \delta_t$, $\models_{\delta_t} A$

For this inference to be true it is necessary that every δ_t such that $\beta_t \, R \, \delta_t$ are among the γ_t's such that $\beta_t \, S \, \gamma_t$, therefore:

$\forall t \forall t' \forall \alpha \forall \gamma [(t' < t \; \& \; \alpha_{t'} \, S \, \beta_{t'} \; \& \; \beta_t \, R \, \gamma_t) \Rightarrow \beta_t \, S \, \gamma_t]$

This property is the *RS*-post-implication. Q.E.D.

4. *Conclusion*

This last result ends the proof of completeness of R5-D5. Since soundness is easily proved, this establishes the biunivocity between semantics and axiomatics, in other words the *adequation*. Axioms and rules, namely, those of {ℝ}, S5, KDU, and the mixed ones, □*t*1, □*t*2, *Ot*1, *Ot*2, □*O*, □*O*4, □*Ot*, are in the following correspondence with the above described semantic properties:

Kt of $\{\mathbb{R}\}$	standardness of the model
D!t of $\{\mathbb{R}\}$	functionality of T_t
Rtt' of $\{\mathbb{R}\}$	uniqueness of path for evaluation of R_t
RNt of $\{\mathbb{R}\}$	standardness of the model
K□ of S5	standardness of the model
T□ of S5	reflexivity of R
5□ of S5	transitivity & symmetry of R
RN□ of S5	standardness of the model
KO of KDU	standardness of the model
DO of KDU	seriality of S
UO of KDU	secondary-reflexivity of S
RNO of KDU	standardness of the model
□t1	$t' = t$ in $\forall \beta_{t'}$ st $\alpha_t R \beta_{t'}$
□t2	ramification of R
Ot1	$t' = t$ in $\forall \beta_{t'}$ st $\alpha_t S \beta_{t'}$
Ot2	secondary-ramification of S
□O	SR-implication
□O4	RS-transitivity
□Ot	RS-post-implication

Finally, we are now in a position to give a formal definition of a canonical standard model for R5-D5 [38] in Chellas' style. Such a model is a 6-tuple

$<W, T, <, R, S, P>$

with

(1) $W = \{\Gamma\text{: } Max_{\text{R5-D5}}\Gamma\}$

(2) $T = \{t,\ t',\ t'',\ ...\}$

(3) $< \subseteq T \times T$.

(4) For every α_t, $\Box A \in \alpha_t$ iff for every β_t such that $\alpha_t R \beta_t$, $A \in \beta_t$ ($\alpha_t R \beta_t$ is an alternative formulation for $Rt\alpha\beta$).

[38] To remain strictly in Chellas' style, I will not use the notion of a *frame*, which is, as known, that of a model minus the assignment of truth values (or valuation) function.

(5) For every $\alpha_t, OA \in \alpha_t$ iff for every β_t such that $\alpha_t S \beta_t$, $A \in \beta_t$ ($\alpha_t S \beta_t$ is an alternative formulation for $St\alpha\beta$).

(6) $P_n = |\mathbb{P}_n|_{R5-D5}$, $n \in \mathbb{N}$.

where

— (1), (4), (5) and (6) are as in the general definition of a canonical standard model[39].

— T is a set of instants (times).

— $<$ is a binary linear relation on T, $t < t'$ meaning that t is anterior to t'.

In addition, formulae where the R_t-operator appears are evaluated according to the condition:

For every $\alpha_{t'}$, $R_t A \in \alpha_{t'}$ iff $A \in \alpha_t$.

Département de Philosophie, Rue de la Censive du Tertre
B.P. 81227, F-44312 Nantes Cedex 3, France

REFERENCES

ÅQVIST L. & HOEPELMAN J.

1981 "Some theorems about a 'tree' system of deontic tense logic", in R. Hilpinen, *New Studies in Deontic Logic*, Reidel, pp. 187-221.

BAILHACHE P.

1983 *Normes et modalités, essai d'analyse formelle du devoir-être*, thèse d'Etat, defended may 83, 10 at the Université de Paris IV (Sorbonne), 785 p., reprinted in *Les normes dans le temps et sur l'action*, Université de Nantes, 1986.

1990 "Perspectives ontiques et temporelles pour la logique déontique (axiomatique, sémantique et complétude)", *Acta Universitatis Lodziensis, Folia philosophica* 7, p. 5-18.

1991 Essai de logique déontique, Vrin, coll. "*Mathesis*", 232 p.

1993 "The Deontic Branching Time: Two Related Conceptions", *Logique et analyse*, 141-142, p. 159-175 (published in 1995).

[39] Now, members of W, which are sets of sentences, can be interpreted as states of paths, α_t, β_t,, etc., instead of paths. T is the set of time indices. As expressed above, the relations $Rt\alpha\beta$, $St\alpha\beta$ ($\subseteq T \times W \times W$) of earlier formulations have to be replaced by $\alpha_t R \beta_t$, $\alpha_t S \beta_t$ ($\subseteq W \times W$)..

CHELLAS B.F.
1980 Modal Logic: an Introduction, *Cambridge University Press,* p.xii-295.

HUGHES G.E. & CRESSWELL M.J.
1968 An Introduction to Modal Logic, *Methuen*, xii-388 p.
1984 A Companion to Modal Logic, *Methuen*, xv-203 p.
1996 A New Introduction to Modal Logic, Routledge, x-421 p. (I did not know this book when writing the article).

RESCHER N. & URQUHART A.
1971 Temporal Logic, *Springer-Verlag*, xviii-273 p.