Logique & Analyse **146** (1994), 367-377

GENERALISING TAUTOLOGICAL ENTAILMENT[1]

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The first degree fragment of relevant logics is captured by the system E_{fde}, which corresponds to an intuitive notion of variable sharing as described by Anderson and Belnap[2]. I show how this notion of variable sharing can be generalised in an intuitively appealing way to entailments of arbitrary degree. However where the underlying logic of implication is weaker than that of the system *R*, the resulting full degree "variable sharing" logics are strictly weaker than the corresponding "standard" relevant systems. Thus there is a tension between satisfying this extended notion of variable sharing (which seems desirable on relevantist grounds) and obtaining some of the nice features of the standard systems (such as algebraic properties centred around fusion).

I will begin with a description of tautological entailments and the attendant notion of variable sharing, and then show how these ideas can be generalised.

The system of *tautological entailments* comprises entailments between zero degree sentences[3], and corresponds to the first degree fragment of the standard relevant logics. It is based on the following:

- A conjunction of atomic sentences entails a disjunction of atomic sentences iff at least one of the former is identical with (in meaning) one of the latter.
- A sentence entails a conjunction of sentences iff it entails each conjunct.
- A disjunction of sentences entails a sentence iff each disjunct entails the sentence.
- Negation has De Morgan properties.

[1] This paper is based on part of my M.A. thesis and I wish to thank Chris Mortensen for his help while I was conducting that research.

[2] A.R. Anderson and N.D. Belnap, Jr., *Entailment: The Logic of Relevance and Necessity*, Vol.1, Princeton University Press, Princeton, 1975, Chapter 3.

[3] A *zero degree* or *atomic* sentence contains conjunction, disjunction or negation connectives only. A *first degree* entailment is an entailment between atomic sentences.

These ideas are put on a formal footing as follows.[4]

An *atom* is a sentential variable or the negation of such. A *primitive conjunction* is a conjunction of atoms. A *primitive disjunction* is a disjunction of atoms. $A \rightarrow B$ is a *primitive entailment* iff A is a primitive conjunction and B is a primitive disjunction.

A first degree entailment is in *normal form* iff the antecedent is a disjunction of primitive conjunctions and the consequent is a conjunction of primitive disjunctions.

A primitive entailment is *explicitly tautological* iff at least one atom of the antecedent is identical with an atom of the consequent. A first degree entailment is *explicitly tautological* iff it is in normal form, and each of the primitive entailments between each disjunct of the antecedent and every conjunct of the consequent is explicitly tautological.

This notion is extended to the class of all first degree entailments using disjunctive and conjunctive normal forms for zero degree wff:

Where A is a zero degree wff, a *disjunctive normal form* of A is a wff of the form

$$\left(A_1^1 \wedge A_2^1 \wedge \ldots \wedge A_{m_1}^1\right) \vee \left(A_1^2 \wedge A_2^2 \wedge \ldots \wedge A_{m_2}^2\right) \vee \ldots \left(A_1^n \wedge A_2^n \wedge \ldots \wedge A_{m_n}^n\right)$$

(where each $A_{j_k}^i$ is an atom) obtained from A by substitution of equivalents using commutation, association, distribution, double negation and DeMorgan's laws. A *conjunctive normal form* is the $\wedge / \vee$ dual form:

$$\left(A_1^1 \vee A_2^1 \vee \ldots \vee A_{m_1}^1\right) \wedge \left(A_1^2 \vee A_2^2 \vee \ldots \vee A_{m_2}^2\right) \wedge \ldots \left(A_1^n \vee A_2^n \vee \ldots \vee A_{m_n}^n\right).$$

Observe that the disjunctive normal form of a wff A is unique up to commutation, association, and repetition of conjuncts and disjuncts.

One way to see this is to construct a model comprised of sets of sets of strings 'p' and '$\neg p$' for the sentential variables p of the language. Treating these sets as corresponding to disjunctive normal forms define operations, $\wedge$, $\vee$ and $\neg$ in the obvious way[5]. Now define an assignment function I from the class of zero degree wffs into this model in the obvious composi-

[4] c.f. *Entailment* chapter 3.

[5] That is $\{x, y, \ldots\} \wedge \{u, v, \ldots\} \equiv \{x \cup u, x \cup v, \ldots, y \cup u, y \cup v, \ldots\}$ and $X \vee Y \equiv X \cup Y$, and $\neg X$ is obtained from the set of all choices of one member from each member of X by replacing each string of form 'p' by '$\neg p$' and vice-versa.

tional manner, beginning with $I(p)=\{\{p\}\}$ for sentential variables p. The fact that it is a function establishes the above observation.

Clearly the same holds of conjunctive normal forms.

I will speak of *the* disjunctive/conjunctive normal form of a wff with a degree of infelicity indicated by the above observation. The notation *dnf(A)* and *cnf(A)* will be used, to denote disjunctive and conjunctive normal forms (respectively) of a zero degree wff A.

A first degree entailment $A \rightarrow B$ is a *tautological entailment* iff *dnf(A)* $\rightarrow$ *cnf(B)* is explicitly tautological[6]

Anderson and Belnap encapsulate the path used to arrive at the above definition by the notion of variable sharing:

Zero degree wff A and B satisfy *variable sharing* iff every disjunct of *dnf(A)* and each conjunct of *cnf(B)* have an atom in common.

Clearly $A \rightarrow B$ is a tautological entailment iff A and B satisfy variable sharing.

My aim is to show how these ideas can be nontrivially extended beyond first degree entailments.

Note that we can regard the notion of tautological entailment as involving three ingredients. The first is an intensional basis of meaning inclusion, which is taken to be identity of atoms. The second is a set of principles incorporating the meaning of extensional disjunction and conjunction, while the third is De Morgan negation.

Different candidate notions of meaning inclusion could be used as the first ingredient, while still retaining the others, and so in a sense incorporating the "same" conjunction, disjunction and negation connectives[7].

The following generalisation of tautological entailment is based on this idea. Rather than identity of atoms for the first ingredient, simply require the corresponding entailment between "atoms"[8] to be included in a set of given entailments which, it is supposed, captures some already established notion of meaning inclusion.

Let W be a set of wff. Then an entailment which is a substitution instance of a first degree entailment in normal form, that is of form

[6] This use of notation is OK — observe that if the requirements for explicit tautologyhood are met by a disjunctive normal form of A and conjunctive normal form of B, they are also met by every disjunctive normal form of A with every conjunctive normal form of B.

[7] In fact weaker negations can be accommodated by modifying the transformations permitted to obtain normal forms of wff, in such case *atom* would be defined to include prefixing by arbitrary length strings of "$\neg$" such as "$\neg\neg\neg p$".

[8] Scare-quotes because wffs of arbitrary degree will in fact play the role of atoms in what follows.

$$\left(A_1^1 \wedge A_2^1 \wedge \ldots \wedge A_{m_1}^1\right) \vee \left(A_1^2 \wedge A_2^2 \wedge \ldots \wedge A_{m_2}^2\right) \vee \ldots \left(A_1^n \wedge A_2^n \wedge \ldots \wedge A_{m_n}^n\right)$$

$$\rightarrow \left(B_1^1 \vee B_2^1 \vee \ldots \vee B_{u_1}^1\right) \wedge \left(B_1^2 \vee B_2^2 \vee \ldots \vee B_{u_2}^2\right) \wedge \ldots \left(B_1^r \vee B_2^r \vee \ldots \vee B_{u_r}^r\right).$$

is *W-augmented explicitly tautological* iff for each $i=1,\ldots n$ and $j=1,\ldots r$ there is a pair A_s^i and B_t^j such that $A_s^i \rightarrow B_t^j \in W$.

Let A be a wff in the full logical vocabulary. A *generalised disjunctive normal form* of A is a wff of form $\left(A_1^1 \wedge A_2^1 \wedge \ldots \wedge A_{m_1}^1\right) \vee \left(A_1^2 \wedge A_2^2 \wedge \ldots \wedge A_{m_2}^2\right) \vee \ldots \left(A_1^n \wedge A_2^n \wedge \ldots \wedge A_{m_n}^n\right)$ from which A can be obtained by substitution of equivalents using commutation, association, distribution, double negation and De Morgan's laws, with the proviso that no occurrence in A_j^i of a subwff of A_j^i can be substituted for.

Thus the A_j^i's are fixed and treated as if they were atoms, as in the definition of tautological entailment. We abbreviate 'generalised disjunctive normal form' by 'gdnf'. *Generalised conjunctive normal form* is defined in the obvious analogous way. Note that A' is a gdnf of A iff there is a zero degree wff B from which A can be obtained by uniform substitution, where the corresponding substitution applied to the corresponding *dnf(B)* delivers A'.

An entailment $A \rightarrow B$ is a *W-augmented tautological entailment*, or *W-ate*, given a set of wff W, iff A has a gdnf A' and B has a gcnf B' such that $A' \rightarrow B'$ is W-augmented explicitly tautological.

W-ates are those entailments corresponding to a generalised notion of variable sharing, where:

- any wff can play the role of the atoms in the definition of variable sharing
- the requirement of identity of atoms is weakened to a requirement that appropriate entailments be in W.

Suppose we begin with a class of wff W, and form W' the class of entailments which are W-ates, and then W'' the class of $(W' \cup W)$-ates, and so on to obtain entailments of arbitrary degree generated from W by augmented variable sharing. This is equivalent to adopting the procedure in the definition of augmented variable sharing as a rule. It is also necessary to close under substitution of commutation, association, distribution, double negation and De Morgan equivalents, at each step. This is to allow "mixing" of wffs introduced at different steps in the process. This is in accord with our assumption that logical equivalence is preserved by these rules, inherent in the normal form manoeuvres sanctioned in the definition of disjunctive and conjunctive normal forms.

Call this extended closure of a set of wff W its vs-closure (for variable sharing).

As an example, begin by letting W be the set of all substitution instances of the negation/implication axioms of a fixed logic, E say. Then take the vs-closure to obtain the full logic generated by those negation/implication principles but with $\wedge/\vee$ interplay determined by augmented variable sharing. The question of interest is whether the resulting system is E, and the answer is no; in fact (as we will see) it is strictly weaker.

To facilitate axiomatic presentation the following rules which have the same effect as forming the vs-closure, are used.

The *augmented tautological entailment* rules are:

(1) $A \rightarrow C \Rightarrow A \wedge B \rightarrow C$
(2) $A \rightarrow B \Rightarrow A \rightarrow B \vee C$
(3) $A \rightarrow B$ and $A \rightarrow C \Rightarrow A \rightarrow B \wedge C$
(4) $A \rightarrow C$ and $B \rightarrow C \Rightarrow A \vee B \rightarrow C$
(5) *Substitution of commutation, association, distribution, double negation and De Morgan equivalents.*

Lemma 1 Let W be a set of wff closed under (5) above, then the vs-closure of W is equal to the set of wffs obtained by closure of W under the above rules

Proof: Clearly the vs-closure of W is closed under the rules (1)-(4), as these are simple applications of the procedure for obtaining W-ates treating A, B and C as atoms.

Furthermore by stipulation vs-closure includes closure under (5). Thus the above rules don't deliver any *more* than the vs-closure.

Conversely, suppose AB is contained in the vs-closure of W. Then we have

$$\left(A_1^1 \wedge A_2^1 \wedge \ldots \wedge A_{m_1}^1\right) \vee \left(A_1^2 \wedge A_2^2 \wedge \ldots \wedge A_{m_2}^2\right) \vee \ldots \left(A_1^n \wedge A_2^n \wedge \ldots \wedge A_{m_n}^n\right)$$

$$\rightarrow \left(B_1^1 \vee B_2^1 \vee \ldots \vee B_{u_1}^1\right) \wedge \left(B_1^2 \vee B_2^2 \vee \ldots \vee B_{u_2}^2\right) \wedge \ldots \left(B_1^r \vee B_2^r \vee \ldots \vee B_{u_r}^r\right).$$

from which $A \rightarrow B$ can be obtained by applications of (5). Furthermore if we can establish that for each antecedent disjunct A^i and consequent conjunct B^j we have $A^i \rightarrow B^j$, then the above entailment can be obtained from these by application(s) of (3) and (4). But for each such antecedent disjunct and consequent conjunct there is a pair A_s^i and B_t^j with $A_s^i \rightarrow B_t^j$

holding as a premiss for application of the augmented tautological entailment rule. (1) and (2) deliver the required $A^i \to B^j$. So we have (upon reversal) a proof of $A \to B$ using the above rules.

Logic systems which espouse just this form of interaction between $\wedge/\vee$ and the intensional part of the logic are grounded directly on the intuitions underlying variable sharing. We aim to see how these compare to more standard systems.

Lemma 2 Let L be a logic system which contains all *identities* $A \to A$ and is closed under the rules

- *modus ponens:* $\vdash A$ and $\vdash A \to B \Rightarrow \vdash B$
- *rule prefixing* $\vdash A \to B \Rightarrow \vdash B \to C \to . A \to C$
- *rule suffixing:* $\vdash A \to B \Rightarrow \vdash C \to A \to . C \to B$
- *rule contraposition:* $\vdash A \to \neg B \Rightarrow \vdash B \to \neg A$

as well as under augmented tautological entailment.

Then L can be presented with the following axioms in place of the rules (1), (2) and (5).

- (6) $\vdash A \to A$
- (7) $\vdash A \wedge B \to A \quad \vdash A \wedge B \to B$
- (8) $\vdash A \to A \vee B \quad \vdash B \to A \vee B$
- (9) $\vdash A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
- (10) $\vdash \neg\neg A \to A$

Proof: All of the above axioms are theorems of L as they are L-ates, given that L contains all identities. Conversely (1) and (2) follow from the corresponding axioms (7) and (8) above using closure under transitivity (which is ensured by modus ponens and the prefixing and suffixing rules). For (5) it is a straightforward exercise to establish the corresponding entailments using the axioms, (3) and (4) and rule contraposition; and substitution of these equivalents is facilitated by modus ponens, rule prefixing and rule suffixing.

Thus a basic minimum system *BB* with extensional/intensional interplay determined by augmented variable sharing (given that identity, rule prefixing, rule suffixing and rule contraposition are also espoused) can be formulated as follows:

Definition 1
The logic *BB* has as axioms (6) - (10) (above), and rules (3), (4), modus ponens, rule suffixing, rule prefixing, rule contraposition, and also adjunction[9].

This differs from the standard minimal relevant logic *B* in that the rules (3) and (4) replace the following axioms of *B*

- (A3) $A \rightarrow B \wedge A \rightarrow C \rightarrow . A \rightarrow (B \wedge C)$
- (A4) $A \rightarrow C \wedge B \rightarrow C \rightarrow . (A \vee B) \rightarrow C$

Similarly augmented variable sharing logics can be obtained from the standard formulations of corresponding relevant logics by weakening the above axioms to rules exactly as in the case of *BB*. I label these with a prefix '*B*' also. So corresponding to the systems[10] *B*, *TW*, *T*, *E*, and *R* (in increasing strength) we have *BB*, *BTW*, *BT*, *BE*, and *BR*, etc..

The obvious question is whether the "*B*-ed" systems are really distinct from their standard counterparts. In fact those logics *L* weaker than (and including) *E* have *BL* strictly weaker than *L*, while *BR* = *R*. We first demonstrate the latter fact[11] and then prove the former using a suitable algebraic model.

Lemma 3 If the logic *BL* contains *BB* and includes the following rule and axiom then in fact *BL* includes as theorems (A3) and (A4):

- *permutation* $\vdash A \rightarrow . B \rightarrow C \Rightarrow \vdash B \rightarrow . A \rightarrow C$
- *contraposition* $\vdash A \rightarrow \neg B \rightarrow . B \rightarrow \neg A$

Proof: For (A4):

$\vdash A \rightarrow C \wedge B \rightarrow C \rightarrow . A \rightarrow C$ using (7), and then permutation gives
$\vdash A \rightarrow . (A \rightarrow C \wedge B \rightarrow C) \rightarrow C$ and
$\vdash B \rightarrow . (A \rightarrow C \wedge B \rightarrow C) \rightarrow C$ is obtained similarly.
hence $\vdash A \vee B \rightarrow . (A \rightarrow C \wedge B \rightarrow C) \rightarrow C$ using (4),
and so we have $\vdash A \rightarrow C \wedge B \rightarrow C \rightarrow . A \vee B \rightarrow C$ by permutation.

9 $\vdash A$ and $\vdash B \Rightarrow \vdash A \wedge B$

10 For their formulations see *Entailment* page 341ff, or R.Routley et al, *Relevant Logics and Their Rivals*, vol.1, Ridgeview Publishing Co., Atascadero, 1982, 287ff.

11 Shown by Bob Meyer.

Now (A3) can be obtained using contraposition from the following instances of (A4):

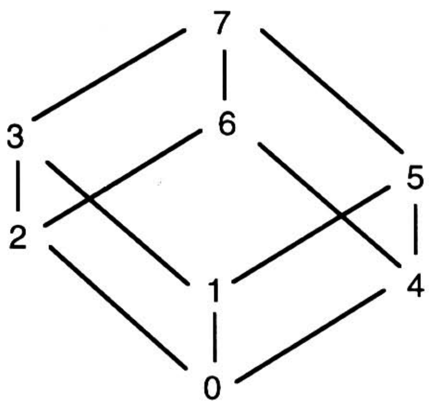
$$\vdash \neg A \rightarrow \neg C \wedge \neg B \rightarrow \neg C \rightarrow . \neg A \vee \neg B \rightarrow \neg C$$

Theorem 1 $BR = R$: Immediate corollary of the above lemma.

This result shows that conceptually R can be regarded as E_{fde} (the first degree entailments) plus its intensional component $R_{\neg\rightarrow}$. Thus the intensional/extensional interplay within R can be characterised precisely in terms of (augmented) variable sharing. We shall next see that such is not the case for E, by use of a suitable "algebraic" model.

Definition 2 BM_0 is the eight element structure defined as follows.

It is a De Morgan lattice with meet and join determined from the following Hasse diagram in the standard way.



The arrow operator is defined:

→	0	1	2	3	4	5	6	7
0	3	3	3	3	3	3	3	3
1	0	3	0	3	0	3	0	3
2	0	1	3	3	1	1	3	3
*3	0	1	0	3	0	1	0	3
4	0	1	0	1	3	3	3	3
5	0	0	0	0	0	3	0	3
6	0	1	0	1	1	1	3	3
*7	0	0	0	0	0	0	0	3

A negation operator is defined on these elements as follows:

x	0	1	2	3	4	5	6	7
$\neg x$	7	6	5	4	3	2	1	0

The designated elements or "truth filter" are {3,7}, as indicated by the stars above.

For definiteness we note that E can be formulated by adding the following axioms and rule to B, while BE is obtained by adding them to BB:

Prefixing
$A \to B \to . B \to C \to . A \to C$
Suffixing
$A \to B \to . C \to A \to . C \to B$
Contraction
$A \to (A \to B) \to . A \to B$

Negation introduction
$A \to \neg A \to . \neg A$
Contraposition
$A \to \neg B \to . B \to \neg A$
Necessitation
$\vdash A \Rightarrow \vdash (A \to B) \to B$

Theorem 2 BM_0 is sound for *BE* in the sense that for every assignment of values $\{0,1,2,3,4,5,6,7\}$ to the propositional variables of a wff, the value assigned (in the usual inductive manner) to the wff is a designated value (that is, 3 or 7), if that wff is a theorem of *BE*.

The proof is a tedious but routine checking of cases, to ensure that the axioms come out designated and the rules preserve designated-hood. I have confirmed my manual check[12] using the program Finder.[13]

Theorem 3 BM_0 falsifies (A3) $A \to B \wedge A \to C \to . A \to (B \wedge C)$ and (A4) $A \to C \wedge B \to C \to . (A \vee B) \to C$.

Proof: Assign $A = 4$, $B = 3$, $C = 6$ to give (A3) value 0, which is not designated. Assign $A = 4$, $B = 1$, $C = 3$ to give (A4) value 0.

Theorem 4 *BE* is distinct from *E*.

This follows directly from the previous two theorems.

Thus for logics weaker than (and including) *E* the distinction between the axioms (A3) and (A4) and their weaker rule forms (3) and (4) can be non-trivially maintained.[14]

Anderson and Belnap claim that *E* is composed of E_{fde} and $E_{\neg\to}$.[15] However $BE = BB + E_{\neg\to}$ and E_{fde} is a sublogic of *BB*, so it is rather more accurate to say that it is *BE* which is composed of E_{fde} and $E_{\neg\to}$. This is conceptually more appealing since the interplay between and the inten-

12 Thanks again to Chris Mortensen for his help in this.

13 Finite Domain Enumerator, developed by John Slaney. This program is described in John Slaney, *Finder: Notes and Guides*, Technical Report TR-ARP-1/92, Automated Reasoning Project, Australian National University, April 1992.

14 I have developed algebraic and relational semantics for these systems. Underpinning both is a particular notion of theory-hood which I believe provides further conceptual stability to these logics. These are described in P.Lavers, *Generating Intensional Logics*, M.A. thesis, University of Adelaide, 1986.

15 *Entailment* p. 231

sional connectives is determined by augmented variable sharing in BE, which is just the natural generalisation of E_{fde} variable sharing.

Let us take stock. I have proposed a generalisation of tautological entailment and variable sharing which has two key features. Tautological entailment applies to just first-degree entailments whereas this generalisation applies to entailments of any degree. Secondly the underlying warrant for tautological entailment, the identity of atoms (and so *Identity*) in variable sharing, has been relativised to membership in some given class of entailments, for augmented variable sharing.

Axioms and rules equivalent to augmented variable sharing have been established and I have then used these to formulate augmented variable sharing logics corresponding to the standard relevant logics.

The procedure in *Entailment* is to separately examine conjunction and disjunction (giving E_{fde}), and relevant logics of implication and negation (the intensional part), and graft these two components together to give the full logics. This "grafting" comprises adding to the pure intensional theses axioms which are intended to lift E_{fde} to the full degree context. I have similarly added to the intensional theses axioms and rules which capture augmented variable sharing. But augmented variable sharing itself only warrants the addition of the rules (3) and (4) rather than the stronger axiom forms (A3) and (A4). So the axiomatic presentation of extensions of the implication/negation logics to include conjunction and disjunction, while also satisfying augmented variable sharing, is strictly weaker than the full logics presented by Anderson and Belnap.

Viewed from the W-ate perspective, the starting class of entailments W for our augmented variable sharing procedure is all instances (in the full language) of just a standard formulation for the implication/negation fragment of the given relevant logic. So the "mixed" axioms (such as (A3) and (A4)) are not included in the initial class of entailments for application of the augmented variable sharing procedure, and we should not expect these mixed axioms to necessarily get validated by the procedure.

We have seen that for $R_{\neg\rightarrow}$ (A3) and (A4) follow from (3) and (4) anyway, so there is no real logical difference, while for $E_{\neg\rightarrow}$ and weaker the systems *are* distinct; the logic E is too strong from the point of view of augmenting $E_{\neg\rightarrow}$ to obtain a full logic which still satisfies our generalised notion of variable sharing.

Anderson and Belnap produce the slightly stronger systems in part because they also set out to provide Fitch-style formulations for the systems. The natural Fitch-style introduction and elimination rules for conjunction immediately deliver (A3) (and are actually posited to do so[16]). Moreover

[16] *Entailment* p. 271.

the rules for conjunction and disjunction can be replaced by a Fitch-style Tautological Entailment rule[17]. In this context the medium for expressing an analogue of Tautological Entailment takes one further than our generalisation of variable sharing permits. If a logic is formulated with extensional part just augmented variable sharing, then the intensional import is out in the open, so to speak.

I will conclude by noting that augmented variable sharing is also applicable (and E_{fde} variable sharing is not) where identity is not espoused[18]. This highlights the fact that augmented variable sharing is independent of any particular semantic assumptions one may make about entailment.

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17 *Entailment* p. 277.

18 For motivation of this see E.P.Martin and R.K.Meyer, Solution to the {P-W} problem, *The Journal of Symbolic Logic*, 47(4), 1982