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REDUCTIO WITHOUT ASSUMPTIONS?

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Quine has neatly summarized the received view of the *reductio* method: "The strategy of reductio ad absurdum ... consists in assuming the contradictory of what is to be proved and then looking for trouble" (1982: 254). Logic texts invariably treat *reductio ad absurdum* (hereafter, "*reductio*",[1] or indirect proof) as a method of deduction which relies essentially upon the use of an assumption. In particular, it is widely accepted that to accomplish a proof of an argument's validity by the *reductio* method, one must *assume* the negation of the argument's conclusion and use this assumption in conjunction with the argument's premisses in order to deduce a contradiction. The *reductio* method also finds employment in the validation of 0-premiss arguments. Here one simply *assumes* the contradictory of the proposition to be tested and deduces a contradiction from it alone. An important difference between these two uses of *reductio* is that in 0-premiss arguments the intention is to prove that some proposition is *logically true* whereas in one-or-more-premiss arguments, normally, the aim of a *reductio* is to prove that an argument is valid, not necessarily that the conclusion is true either logically or contingently. Hence, when logical truth is not the object of a *reductio* argument, the epistemic standards of proof must be invoked over and above the proof of validity in order to prove the truth of a conclusion.[2] The discussion that

[1] The term, "*reductio ad absurdum*" is the usual one used; however, strictly speaking, the term "*reductio ad impossibile*" is a better characterization of this method as it is practiced in formal systems since only impossible propositions are logically false, whereas the intension of "absurd proposition" is somewhat wider. Perhaps what some logicians have meant is "*logically* absurd", a term that may be taken as co-extensive with "logically impossible". Formal systems require that an explicit contradiction must be deduced. This is somewhat restrictive since a synthetically necessary falsehoods (if such there are), would do as well. Nevertheless, in this paper I dwell only on the problem as it arises in the most common formal systems.

[2] Copi (1979: 52-54) titles his discussion, "The rule of indirect proof," and means by that "proof of validity." Tapscott also treats of indirect proof and means by it a "rule for discharging assumptions" (1976: 129), and rationalizes its correctness with reference to the definition of validity (1976: 131). Gustason and Ulrich (1973: 112) also take indirect proof to be a method of establishing validity.

follows applies equally to both kinds of *reductio* arguments, but the exposition focuses on those that have at least one premiss.

In formal methods of deduction, what enables the introduction of the assumption required for a *reductio* is an explicitly stated rule. Either the rule permits (i) the introduction of *any* assumption and, hence, the particular assumption needed for a *reductio* proof,[3] or (ii) it permits the introduction of a particular assumption, i.e., the very proposition that is the contradictory of the argument's conclusion.[4] In the first instance, *reductio* is merely a special case of conditional proof; in the second case it is a unique rule especially designed for *reductio* proofs. A system that subsumes all *reductio* proofs under the more general rule of conditional proof has a sense of economy not shared by systems that state both rules (for the simple reason that it will have one less rule). From the point of view of systemic power nothing is gained by adding the *reductio* rule as an extra rule, although such a rule may be said to add "proof-efficiency" to a deductive system (for the reason that it allows one to immediately infer the falsity of the assumption, and hence the validity of the argument once a contradiction is reached, rather than having to 'argue through' the contradiction to the required conclusion[5]).

These two rival ways of effecting a *reductio* proof occasioned some interesting discussion in the literature of the 1970's. Scherer insisted on the importance of the difference between seeing the *reductio* method as just an instance of conditional reasoning and seeing it as a unique kind of reasoning. Treating *reductio* arguments as a kind of conditional proof, he wrote, "fails to manifest the basis upon which a *reductio* is informally conceived to rest" (Scherer 1971: 247). He wanted the rule for the *reductio* method to capture the pre-formal intuition that in a *reductio* proof one does not reason 'through' the contradiction--as one would if subsuming it under conditional proof--but one reasons 'back from' the contradiction to the negation of the assumption.

Whether or not one agrees with Scherer that there is an important difference in these two approaches to *reductio* proofs, it is clear that both methods depend on three things: (a) the use of an assumption, (b) the exploitation of a contradiction, and (c) a rule that allows one to discharge the assumption. We might think that these three elements are essential to any formal proof worthy of the name "*reductio*". The present paper, however, argues that it is only the second condition--the exploitation of a contradic-

[3] Examples are Copi's Strengthened Rule of Conditional Proof (1979: 558-60) and Gustason and Ulrich (1973: 111-13).

[4] E.g., Copi (1979: 53) and Tapscott (1976: 129-30).

[5] See, e.g., Scherer (1971: 248).

tion--that is, strictly speaking, essential to all methods of affecting *reductio* proofs. As the title of Quine's book intimates, there are a number of different methods available to the logician. The use of assumptions, and the inference rules used to discharge them, are essentially tied to only some of the *reductio* methods.

1. *Antilogisms*

What *is* essential to any *reductio* proof is that the conjunction of an argument's premisses and the negation of its conclusion is shown to be logically inconsistent, for, by definition, this guarantees the validity of the argument. Since only arguments that are not obviously valid need to be proven valid, what all methods of *reductio* proof have in common is that they are attempts to bring such a logical inconsistency to light. Any procedure that does not show an argument valid by demonstrating that the conjunction of the premisses with the negation of the conclusion is logically inconsistent, is not a *reductio* proof.

In effect, what this comes to, is that a *reductio* proof shows one argument valid by showing another, distinct argument to be valid also. For example, to show

$$(1) \quad <\{P_1, \ldots, P_n\}, Q>$$

valid by a *reductio* proof, one deduces a logical falsehood from

$$(2) \quad \{P_1, \ldots, P_n, -Q\}$$

But since the premiss set in (1) is distinct from the set of propositions in (2), argument (1) is distinct from any argument that takes the propositions of (2) as its premisses.

The relation of the argument in (1) to the set of propositions in (2), is the relation of an argument to its antilogism.[6] For our purposes, an "antilogism" is the set of propositions formed by conjoining the premisses of an argument with the negation of its conclusion. Hence, that an argument is valid if, and only if, its antilogism is logically inconsistent follows

[6] Here I use "antilogism" in a wider sense than it was originally given when introduced as a concept in syllogistic logic: "A triad of propositions two of which are the premises of a valid syllogism while the third is the contradictory of its conclusion, is called an *antilogism*" (Cohen and Nagel 1934: 92). In the present paper the term is used to denote the set formed by taking the premisses of any argument together with the negation of its attendant conclusion (whether or not the argument is valid).

directly from the definition of "validity". Given the concept of *antilogism* we may now restate the character of a *reductio* proof as one that shows an argument valid by proving that its antilogism is logically inconsistent.

There are a number of effective methods for showing that a set of propositions is inconsistent; for example, truth tables, semantic tableaux, and normal forms. However, closely associated with *reductio* reasoning is the idea of showing a set of propositions inconsistent by demonstrating that it *leads to* (i.e., entails) a logical falsehood. Since only inconsistent sets of propositions entail contradictions this is a sound approach (although, like natural deduction in general, it is not an effective method). Formal systems normally require that the logical falsehood must be an explicit contradiction; that is, a syntactically recognizable contradiction of the form p & $-p$. Thus, we get closer to the traditional conception of a *reductio* proof if we now describe them as consisting in a derivation of an explicit contradiction from an argument's antilogism.

Suppose F is any (explicit) contradiction; then the argument in (1) is shown valid by showing that (1)'s antilogism--the set of propositions in (2)--logically implies F; that is, that

$$(3)\ < \{P_1, \ldots, P_n, -Q\}, F >$$

is valid. In other words, showing (3) valid shows the antilogism in (2) to be inconsistent and, in turn, this shows (1) valid. These are all distinct steps, but considered together they have the essential feature of a *reductio* proof, namely, showing that the premisses and the negation of the conclusion, taken together, is logically impossible; however, none of the steps involve the deployment of an assumption.

2. *Argument augments and converses*

The claim that one can effect a *reductio* proof without using assumptions depends on demonstrating that there is a logically satisfactory way of relating distinct arguments to each other. We must ask how (3) is to be related to (1) without a rule that licenses the introduction of an assumption. To this end we will need to explain two ways of constructing new arguments from given arguments.

(i) Let Z be any argument. If we add a proposition to Z's set of premisses then we form an "augment of Z". If the proposition added is a tautology, this forms a "T-augment of Z". Z is valid if, and only if, any of its T-augments is valid.[7] In other words, if

$$(4) \quad <\{P\}, Q>$$

is any argument, and T is a tautology, then

$$(5) \quad <\{T, P\}, Q>$$

is a T-augment of (4), and it is valid just in case (4) is valid. This is because the addition or subtraction of tautologies to premiss sets does not affect validity (or invalidity). That is, whatever the validity-status of the original argument, it is not altered in the construction of its T-augment.

It should be noted that the proposition 'T' in (5) is not an assumption. It is not hyothetically advanced, and it is not to be withdrawn later.

(ii) From any argument one can generate *argument converses*. Take any of the premisses in a given argument, negate it, and make it the conclusion. Take the displaced conclusion, negate it, and substitute it for the premiss just removed. This generates a distinct argument from the initial one given. For example,

$$\text{(a)} \quad <\{P, Q\}, R>$$
$$\text{(b)} \quad <\{P, -R\}, -Q>$$
$$\text{(c)} \quad <\{-R, Q\}, -P>$$

are all argument converses of each other. An argument has as many converses as it has premisses, and it is valid if, and only if, at least one of its converses is valid.[8] Let us call this the Principle of Argument Conversion (PAC).

By PAC the argument form

$$(6) \quad <\{-Q, P\}, -T>$$

is valid if, and only if, (5) is valid, because (6) is an argument converse of (5). Thus a proof of (6)'s validity--by whatever method it is accomplished--is *eo ipso* a proof of (5)'s validity. But a proof of (5)'s validity is also a

[7] In fact, an argument is valid if, and only if, *all* its T-augments are valid. But one T-augment of an argument is valid only if all are. Hence, for purposes of the method being outlined here, it is sufficient to say that an argument is valid just in case *any* of its T-augments are valid.

[8] Valid arguments have all and only valid converses. Invalid arguments have all and only invalid converses.

proof of (4)'s validity. So, a proof of (6)'s validity is a proof that (4) is valid.

'T' was introduced as any tautology; hence, in (6), '-*T*' is a contradiction. We notice that the premiss set of (6) is (4)'s antilogism; therefore, the deduction of '-*T*' from the premisses in (6) shows that (4)'s antilogism is inconsistent. Thus, in showing that (6) is valid one satisfies the requirement of *reductio* reasoning that it must show an argument valid by showing that a contradiction is implied by its antilogism.[9]

Argument conversion no more involves making an assumption than does argument augmentation. It simply is an exploitation of the logical fact that an argument is valid if, and only if, another argument (a converse) is also valid. But does this method of combining argument augments and converses, together with the concept of an 'antilogism', capture enough of the traditional idea to qualify as a *reductio* method?

3. *Argumental deduction*

The method outlined in the preceding paragraphs relies not on assumptions, but on logical facts. It is a logical fact that deductive arguments are valid only if their *T*-augments are valid; it is a further logical fact that PAC is true. Are there, then, *systems* that do, or could, incorporate these logical facts? The history of logic contains a number of inviting suggestions, of which I shall mention three.

Corcoran (1983: 906) points out that Aristotle's *Prior Analytics* contains both a sentence-sequence deduction system and an argument-sequence reduction system. The former is a method of showing that a set of premisses imply a conclusion by "chaining simple inferences" together, the final sentence being the conclusion. The latter involves showing an initial argument valid by writing new arguments derived from it. Aristotle's important insight was that invalid syllogisms reduce only to other invalid syllogisms, whereas valid syllogisms will reduce both to some invalid and to some valid syllogisms. Hence, the rules governing reduction must insure that no invalid argument will reduce to one that is valid. Says Corcoran, "reduction is not a method of inference but rather a process for studying relationships among syllogisms ... an *almost* separate enterprise." (1983: 906).

[9] In the case of 0-premiss arguments the sequence of argument forms will be:

(4') $< \{ \}, L >$

where 'L' is thought to be a logical truth. The augment, with 'T' as a tautology, will be

(5') $< \{T, \{ \}\}, L >$

and the argument converse of the latter is

(6') $< \{-L, \{ \}\}, -T >$.

The Principle of Argument Conversion (PAC) is stated first in Aristotle's *Topics* and then finds employment as one of the methods of argument-sequence reduction in the *Prior Analytics*.[10] It is also one of the four Stoic *themata*. "[I]t seems clear that the *themata* ... were not conceived as supplementary premisses to be worked into arguments ... but rather as second-order rules governing the procedures followed by Chrysippus in his derivation of complicated moods" (Kneale and Kneale 1962: 169). The remark that *themata* are not supplementary premisses is consistent with our observation that PAC is not to be confused with the introduction of an assumption into an argument; that they are used for the derivation of complicated moods indicates that PAC is about relating argument forms to other argument forms.

Argument-sequence deduction, however, need not be restricted to the classical project of validating imperfect syllogisms (reduction). Corcoran in his paper on Stoic deduction gives a broader characterization of what he calls *argumental deduction*.

> Opposed to the sentential deductions (which are lists of sentences) there are those which are lists of arguments. Systems which consist entirely of lists of arguments are called *argumental deductive systems*. ... In creating an argumental deduction one does not start with premises and proceed to a conclusion but rather one takes *ab initio* certain simple arguments and constructs from them, line-by-line, increasingly complex arguments until the argument with desired premises and conclusion is reached. In argumental systems the rules produce arguments from arguments (not sentences from sentences) (Corcoran 1974: 176).

Argument-sequence reduction, then, is *a* system of argumental deduction. Other systems may be proposed.

Finally, Gentzen's sequent calculus is regarded as a meta-language in which "we make statements about deducibility relations in the object language" (Hacking 1993: 231). In particular, this calculus establishes that if a certain formula is a consequence of a given set of formulae (i.e., if it is a valid argument), then another formula is deducible from a related set of formulae (i.e., another, related argument is also valid).[11] Gentzen's method, being concerned to generate further deducibility relations from given deducibility relations, is not an argument-sequence system in Aristotle's limited sense since Aristotle was primarily concerned only with the

[10] See *Topics* VIII xiv 163a32 ff., and *Prior Analytics A*, V and VI.

[11] See Hacking (1993: 232-39).

asymmetrical relation of reduction, and he restricted propositions to the four categorical forms.[12] However, Gentzen's method is an argument-sequence method that relates valid arguments to other valid arguments.

4. *Conclusion*

It is not the purpose of the present paper to develop a system of argumental deduction; nor is it to claim that a system of argumental deduction can do everything that systems of sentenctial deduction can do. The intention is only to show that the essential aspect of *reductio ad absurdum* argumentation can be recast as an argumental deduction.

Still, it seems that a combined sentential and argumental deductive system is possible; that is, a logical system that permits both sentential and argumental deductions. For example, in Copi's 1979 system of natural deduction enriched by the rules of conditional and indirect proof we could drop the rule of indirect proof and replace it with two kinds of argumental inferences: argument augmentation and PAC. This would still allow us to show arguments valid by the *reductio method* as outlined above, but it might also involve the use of sentential deduction to show valid at least one of the arguments in the argumental sequence. Granted, such a system would be complicated, perhaps inelegant; yet satisfactory from a strictly logical point of view.

On practical grounds, perhaps, such a mixed system is not worthwhile developing. However, that it would be possible to do so allows us to see why the use of assumptions, and the formulation of rules whereby they arer discharged, is widely believed to be essential to *all* methods of *reductio* proofs. It is because, historically, sentence-sequence logics rather than argument-sequence logics, have come to represent the paradigm of deductive systems. Given that any *reductio* proof depends on the exploitation of an argument's antilogism, sentence-sequence logics, like the propositional and predicate calculi, can only obtain the requisite antilogism by *assuming* the contradictory of the conclusion to be demonstrated and treating it like a premiss in the ensuing deduction. Hence, as long as we think of sentential

[12] See Corcoran (1983: 906).

deduction as the only kind of deduction, we will be led to admit that *reductios* do essentially depend on the use of assumptions.[13]

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