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QUANTIFIED DEONTIC LOGIC WITH DEFINITE DESCRIPTIONS

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I

In a recent paper [1] I argued that deontic operators should be considered extensional with respect to all singular terms, including definite descriptions. That is, the rule of substitution for identically referring singular terms, as well as standard, unrestricted rules of existential generalization and universal instantiation, seem valid even when applied to statements containing deontic modalities. This is in sharp contrast to the situation familiar from the logic of necessity and belief contexts. In order that these rules be valid without restriction, it is necessary to modify the standard semantics for deontic statements slightly. In [1] I proposed a way to do this. Here I present that proposal more rigorously by defining a model theory for a first-order deontic language containing identity and definite descriptions. Further, to show that this way of interpreting deontic statements has the kind of stability one might reasonably demand, I will present an axiomatization of the set of logical truths and inferences valid under this interpretation, and prove strong consistency and completeness theorems for this system.

That deontic contexts are extensional with respect to singular terms, and that the standard rules for quantifiers and identity should hold for them without restriction, is somewhat surprising, especially to those trained in the ways of alethic and intentional modal logics. I will not try to rehearse the arguments of [1] here; they may be summarized very briefly in four main points:

(i) Our ordinary use of deontic statements, unprejudiced by interpretations drawn from other areas of modal logic, seems to treat them as extensional; the inferences by the classical rules seem easy and unproblematic. There are no natural examples of the rules' failure, no cases obviously analogous to the morning star/evening star or Scott/the author of Waverly. The variables bound within deontic contexts seem to range over ordinary persons and things; there is no temptation to invoke individual concepts or possibilia or other entities of that ilk in order to interpret quantifying into deontic modalities.

(ii) Deontic contexts must be regarded as extensional in order to explain certain conflicts of obligation. There seems to be an important sense in which a statement of the form '*s* ought to do *A* with respect to the one who is the-*F*' — $OAs\iota x.Fx$ — is incompatible with the corresponding statement '*s* ought not to do *A* with respect to the one who is the-*G*' — $O{\sim}As\ \iota x.Gx$ — when the-*F* is one and the same individual as the-*G*. After all, *s* cannot do what he ought to do in the first instance without violating what he ought not to do in the second. This appearance of incompatibility can only be explained if 'ought' is extensional, since in that case the first statement $OAs\ \iota x.Fx$ and the identity $\iota x.Fx = \iota x.Gx$ imply $OAs\ \iota x.Gx$ which is contrary to the second statement.

(iii) It is sometimes proposed that deontic contexts must be non-extensional in order to prevent the Good Samaritan paradox.[1] Such an appeal does not adequately address the paradox, however. It leaves the possibility open, indeed it invites it to be the case, that there are true statements in which definite descriptions will lie within the scope of deontic operators, whereupon the inference that leads to the paradox comes into play. Since the paradox is not resolved if deontic contexts are non-extensional, its spectre provides no reason to think they are not extensional.

(iv) Furthermore, to try to resolve the Good Samaritan paradox in this way, especially through appeals to scope ambiguities, requires quantifying into deontic contexts. If those contexts are non-extensional, then it is difficult to see how one should interpret that quantification. The kinds of proposals made to account for quantifying into necessity or belief contexts do not seem appropriate to the deontic setting.

If these considerations give reason to think deontic contexts are extensional, there are, nevertheless, also reasons to think they cannot be, if the principles of standard deontic logic are valid without modification. For one thing, if deontic contexts are extensional, then the scope solution for the Good Samaritan paradox truly becomes unavailable, for statements with wide scope for the definite description become equivalent to statements with narrow scope. For another, there are powerful arguments that if deontic contexts are extensional, then the deontic modalities reduce to triviality, that if there is any true statement '*s* ought to do *A* with respect to *t*', then any statement '*b* does *B*' would imply '*b* ought to do *B*', that is, $OAst \supset (Bb \supset OBb)$ would be logically true, and, further, any statement '*s* ought

[1] For example, by van Fraassen [6] p. 424, and Loewer and Belzer [3] p. 118.

to do A with respect to t' would imply 's does A with respect to t', i.e., $OAst \supset Ast$ would be logically true.[2]

Hence, if, in light of considerations (i)–(iv), deontic contexts are indeed to be taken to be extensional and the classical rules for identity and quantification are to hold without restriction, then some modification in the standard principles of deontic logic must be made. In particular, a slight restriction on the rule

P.1) If $\vdash A \supset B$, then $\vdash OA \supset OB$

and also on replacement for logical equivalents

P.2) If $\vdash A \equiv B$, then $\vdash C \equiv C'$, where C' is the result of replacing one or more occurrences of A in C by B

is required. These will not hold in just those cases where the entailment in the antecedent depends essentially on the occurrence of a definite description in A or B. That is exactly the situation that gives rise to the Good Samaritan paradox and that is required for the argument that would reduce the modalities to triviality. This restriction follows naturally from the very aspects of the interpretation presented below that enable the deontic operators to be fully extensional with respect to singular terms. In all other respects, the interpretation and the system of deontic logic that follows from it, will be entirely within the spirit of standard deontic logic.

II

Standard propositional deontic logic would interpret statements Op within the framework of a possible world semantics by saying, roughly, that such a statement is true at a world w just in case p is true at every world w_i that is a deontic alternative to w. The deontic alternatives to a world are those worlds that are best or ideal (from the point of view of the given world). To say that it ought to be that p, or that s ought to do A, in this world is thus to

[2] See [2] p. 409 for such a demonstration. Very briefly one argues thus: Suppose $OAst$ and Bb are both true; $t = \iota x(x = t \,\&\, Bb)$; hence, by substitutivity $OAs\iota x(x = t \,\&\, Bb)$ is true. $As\iota x(x = t \,\&\, Bb)$ entails Bb; hence $OAs\iota x(x = t \,\&\, Bb)$ entails OBb, by the principle of standard deontic logic that if A entails B, then OA entails OB. Hence, OBb by *modus ponens*. To show that $OAst$ implies Ast, suppose that $OAst$ is true but not Ast; hence $\sim Ast$ is true. So, by the preceding $O\sim Ast$, contrary to the standard postulate that $OAst$ entails $\sim O\sim Ast$. This kind of argument is familiar from the work of Quine, e.g. [4] esp. p. 159, though such arguments stem from Church, Gödel, and perhaps Frege.

say that that is what occurs, or what s does, in the best of all alternative worlds, in the worlds where all that ought to be is, or something like that.[3] When this sort of account is extended to apply to statements of a first-order deontic language, one might expect to interpret statements OFt, where t is a singular term and F is an elementary predicate, by saying, roughly, that such a statement is true at w just in case Ft is true at every world w_i that is a deontic alternative to w, and that will be so just in case, for each such w_i the denotation of t at w_i is a member of the extension of F at w_i.

To apply such an interpretation to first-order deontic statements in a language containing definite descriptions will lead, however, to deontic statements being non-extensional. Thus, suppose that $\iota x.Fx$ and $\iota x.Gx$ are two definite descriptions such that on interpretation $\iota x.Fx = \iota x.Gx$ is true at a world w (which we may suppose to be the actual world), and suppose that $OA\iota x.Fx$ is also true at w. Hence, by the preceding, $A\iota x.Fx$ would be true at every w_i that is a deontic alternative to w. But there is no guarantee that $\iota x.Fx = \iota x.Gx$ would be true at any such w_i and no guarantee that $A\iota x.Gx$ would be true at w_i; hence $OA\iota x.Gx$ might well be false at w despite the identity of $\iota x.Fx$ and $\iota x.Gx$.

What is needed to enable extensionality for deontic statements is a way to evaluate statements $A\iota x.Fx$ at a world w_i that is keyed to the referent of $\iota x.Fx$ not in w_i but in w. Then if $A\iota x.Fx$ is true at w_i under this mode of evaluation, so would $A\iota x.Gx$ be true, since the referent of $\iota x.Fx$ at w would be the same as the referent of $\iota x.Gx$. That 'cross-world' mode of evaluation is the key to the present proposal.

In this section I will define model structures of a familiar sort in terms of which truth conditions may be defined for deontic statements in a language of first-order quantification including identity and definite descriptions. In this I follow Thomason's treatment of definite descriptions in quantified alethic modal logic (cf. [5], esp. p. 70f.), although one could, I am sure, adapt the proposed interpretation of the deontic operator to other ways of treating definite descriptions in modal contexts if one wanted. Then, in the next section, I will present an axiomatic system which will be proved both strongly consistent and complete with respect to the proposed interpretation. The proof of those results follows familiar Henkin-style techniques.

[3] Many variations on this theme have been developed by different deontic logicians. It is often considered necessary to specify more exactly how the deontic alternatives are related to the given world, and to add other factors to the evaluation of deontic statements. I mean the present account to be a generic description that might apply more or less to most such variations; their differences in detail should not matter for present purposes.

Let $\mathfrak{DL}$ be a language for first-order predicate logic, with identity and definite descriptions, and also containing the monadic deontic operator O; that is, $\mathfrak{DL}$ contains the vocabulary:

i) An infinite set, V, of *individual variables*
ii) A set, C, of *individual constants*
iii) A set, P^n, of n-ary predicates, for $n \geq 1$
iv) Logical symbols: $\sim$, $\supset$, $\forall$, $=$, ι , and O (and parentheses for punctuation)

with the usual formation rules. Other truth-functional connectives and quantifiers may be introduced by definition in the usual ways. I use 'A & B', '$A \vee B$', '$A \equiv B$', '$\exists xA$', '$\exists!xA$' to abbreviate '$\sim(A \supset \sim B)$', '$\sim A \supset B$', '$(A \supset B)$ & $(B \supset A)$', '$\sim\forall x\sim A$', '$\exists y\forall x(A \equiv x = y)$', respectively (where, in the last, y is the alphabetically first individual variable not occurring in A). Small italic letters 'x', 'y', 'z', with or without subscripts, are variables ranging over V, and small italic letters, 'a', 'b', 'c', similarly modified, are variables ranging over C. Definite descriptions are terms of the form $\iota x.B$, where B may be any well-formed formula of $\mathfrak{DL}$. Small italic letters 't', often with subscripts, range over the entire class T of singular terms, i.e. individual variables, constants and definite descriptions. Italic capital letters, 'A', 'B', 'C', etc. are variables ranging over the class W of well-formed formulas of $\mathfrak{DL}$. Where $A \in W$, and $t_1, t_2 \in T$, At_1 / t_2 is the result of replacing all occurrences of t_1 in A by t_2 (with the usual constraints concerning bound and free variables), and $At_1 // t_2$ is the result of replacing one or more occurrences of t_1 in A by t_2 (with similar constraints).

To define truth-conditions for formulas $A \in W$, a *model-structure*, m, is the quintuple $< w_0, \mathcal{W}, \mathfrak{R}, \mathfrak{D}, \mathfrak{D}'>$ where

a) $w_0 \in \mathcal{W}$ ($\mathcal{W}$ is the set of possible worlds; w_0 is the actual world, according to m.)
b) $\mathfrak{R}$ is a function assigning a set of worlds to each world $w_i \in \mathcal{W}$; $\mathfrak{R} w_i \subseteq \mathcal{W}$. Further, for each $w_i \in \mathcal{W}$, $\mathfrak{R} w_i \neq \Lambda$. ($\mathfrak{R}$ selects the deontic alternatives to w_i).
c) $\mathfrak{D}$ is a function assigning a non-empty domain of individuals, $\mathfrak{D} w_i$ to each world $w_i \in \mathcal{W}$
d) $\mathfrak{D}'$ is a non-empty set disjoint from $\bigcup_{w_i \in \mathcal{W}} \mathfrak{D}w_i$.

Let $\mathfrak{D}^m = \mathfrak{D}' \cup \bigcup_{w_i \in \mathcal{W}} \mathfrak{D}w_i$ be the domain of individuals for m. (Note that for all $w_j \in \mathcal{W}$, $\mathfrak{D}^m - \mathfrak{D}w_j \neq \Lambda$).

Given $m = < w_0, \mathcal{W}, \mathfrak{R}, \mathfrak{D}, \mathfrak{D}'>$, an interpretation function, I^m, on m is, as usual, a function such that for each $w_i \in \mathcal{W}$

1) If x is an individual variable, then $I^m(x, w_i) \in \mathfrak{D}^m$, and for all $w_i, w_j \in \mathcal{W}$, $I^m(x, w_i) = I^m(x, w_j)$
2) If c is an individual constant, then $I^m(c, w_i) \in \mathfrak{D}^m$, and for all $w_i, w_j \in \mathcal{W}$, $I^m(c, w_i) = I^m(c, w_j)$
3) If F^n is an n-ary predicate, then $I^m(F^n, w_i)$ is a set of ordered n-tuples of members of $\mathfrak{D}^m$.

In the ordinary way of things, as discussed above, we would now define truth conditions for sentences in the language so that, e.g., a sentence *Ft* is true at a world $w_j \in \mathcal{W}$ if and only if the denotation of t at w_j belongs to the extension of F at w_j i.e., iff $I^m(t, w_j) \in I^m(F, w_j)$ In keeping with the goal that the deontic operator O be extensional with respect to singular terms, it is necessary first to have a method of evaluating sentences in worlds w_j that are deontic alternatives to a world w_i in light of the references of singular terms that are determined according to the other world w_i itself. Accordingly, let us have a function I_i^m, the interpretation *fixed by* w_i, that is derived from I^m as follows: If e is a primitive expression as in (1)–(3) above, i.e., if e is an individual variable, individual constant, or primitive n-ary predicate, then for all $w_i, w_j \in \mathcal{W}$

1')–(3') $\quad I_i^m(e, w_j) = I^m(e, w_j)$

Truth-value assignments for formulas $A \in W$ and denotations for other singular terms under such fixed interpretations I_i^m are defined by simultaneous induction in much the usual way, except that they allow for such cross-world reference, as for instance in clause (4) below. For every $w_i, w_j \in \mathcal{W}$:

4) $I_i^m(F^n t_1, \ldots, t_n, w_j) = T$ iff $< I_i^m(t_1, w_i), \ldots, I_i^m(t_n, w_i) > \in I_i^m(F^n, w_j)$;
$I_i^m(F^n t_1, \ldots, t_n, w_j) = F$ otherwise.
5) $I_i^m(t_1 = t_2, w_j) = T$ iff $I_i^m(t_1, w_i) = I_i^m(t_2, w_i)$;
$I_i^m(t_1 = t_2, w_j) = F$, otherwise.

Suppose that $I_i^m(A, w_j)$ and $I_i^m(B, w_j)$ are defined for all $w_i, w_j \in \mathcal{W}$, then, as usual, (where in all cases, $I_i^m(A, w_j) = F$ iff $I_i^m(A, w_j) \neq T$):

6) $I_i^m(\sim A, w_j) = T$ iff $I_i^m(A, w_j) \neq T$
7) $I_i^m(A \supset B, w_j) = T$ iff $I_i^m(A, w_j) \neq T$ or $I_i^m(B, w_j) = T$

8) $I_i^m(\forall xA, w_j) = T$ iff for every $d \in \mathfrak{D} w_j$ $I_i^m d / x(A, w_j) = T$

where $I_i^m d/x$ is exactly like I_i^m except perhaps in assigning d to x. Further, for deontic formulas

9) $I_i^m(\mathrm{O}A, w_j) = T$ iff for all $w_k \in \mathfrak{R} w_j$ $I_i^m(A, w_k) = T$

Likewise, interpretations for definite descriptions are defined for each w_i, $w_j \in \mathcal{W}$ thus:

10) $I_i^m(\iota x. B, w_j)$ = the unique $d \in \mathfrak{D} w_i$ such that $I_i^m d/x(B, w_i = T)$ if there is one. If not, then $I_i^m(\iota x. B, w_j)$ = an arbitrary member of $\mathfrak{D}^m - \mathfrak{D} w_i$ (subject to the constraint that in that case $I_i^m(\iota x. B, w_j)$ = $I_i^m(\iota x. B, w_k)$ for all w_j, $w_k \in \mathcal{W}$)

It is worth noting that, given that constraint, for all terms t, including definite descriptions, and for all w_i and w_j, $I_i^m(t, w_j) = I_i^m(t, w_i)$, and hence, for all w_j and w_k, $I_i^m(t, w_j) = I_i^m(t, w_k)$. Thus under this sort of interpretation, i.e., interpretations fixed by worlds w_i, definite descriptions behave like fixed, or rigid, designators, as familiar from the work of Kripke and others. So construed, their values are the same in all alternative worlds. Nevertheless, it is not always the case that $I_i^m(t, w_j) = I_j^m(t, w_j)$, nor, as it will turn out given (12) below, is it always the case that $I^m(t, w_i) = I^m(t, w_j)$ when t is a definite description. In that respect definite descriptions are definitely not rigid designators; they are evaluated at each world in quite the usual way.

The definition of the basic interpretation function I^m may now be completed. For each $w_i \in \mathcal{W}$, for each $A \in W$:

11) $I^m(A, w_i) = I_i^m(A, w_i)$

and for each term $t \in T$

12) $I^m(t, w_i) = I_i^m(t, w_i)$

So, for example, just as we described more informally above, for a sentence $\mathrm{O}Ft$, $I^m(\mathrm{O}Ft, w_i) = T$ iff $I_i^m(\mathrm{O}Ft, w_i) = T$ iff $I_i^m(Ft, w_j) = T$ for all $w_j \in \mathfrak{R} w_i$ iff for all $w_j \in \mathfrak{R} w_i$, $I_i^m(t, w_i) \in I_i^m(F, w_j)$ iff for all $w_j \in \mathfrak{R} w_i$,

$I^m(t, w_i) \in I^m(F, w_j)$, that is, iff the denotation of t at w_i is a member of the extension of F at every w_j that is a deontic alternative to w_i.

Given a model-structure $m = <w_0, \mathcal{W}, \mathfrak{R}, \mathfrak{D}, \mathfrak{D}'>$ and I^m defined for m, then as usual a formula $A \in W$ is *true on* I^m *for m* iff $I^m(A, w_0) = T$, and A is *valid for m* iff for every interpretation, I^m, defined for m, A is true on I^m for m. A is *valid* or *logically true*— $\Vdash A$ —iff A is valid for every model-structure m. A set of formulas, Γ, is *simultaneously satisfiable* or *consistent* iff there is a model-structure m and an interpretation, I^m, defined on m such that for every $C \in \Gamma$, C is true on I^m for m. Finally, A is a *logical consequence* of a set of formulas Γ— $\Gamma \Vdash A$ — iff $\Gamma \cup \{\sim A\}$ is not consistent iff for every model-structure m and every interpretation, I^m, defined for m, if every member C of Γ is true on I^m for m, then A is true on I^m for m.

Given this account, it is now easy to verify the following principles of standard deontic logic:

P.3) $\Vdash \mathrm{O}(A \& B) \equiv \mathrm{O}A \& \mathrm{O}B$

P.4) $\Vdash \mathrm{O}(A \supset B) \supset (\mathrm{O}A \supset \mathrm{O}B)$

P.5) $\Vdash \mathrm{O}A \supset \sim \mathrm{O} \sim A$

for all formulas $A, B \in W$.

As I discussed earlier (and also at greater length in [1]), we must not expect the usual principle of standard deontic logic

P.1) If $\Vdash A \supset B$, then $\Vdash \mathrm{O}A \supset \mathrm{O}B$

to hold without restriction, and indeed it does not hold on the present interpretation; similarly for (P.2), replacement of logical equivalents. Nevertheless, these principles are sound when their hypotheses contain no definite descriptions that are essential to its validity. (A definite description is essential to the validity of a statement when replacing it throughout the statement by any other singular term, including individual constants and variables, fails to preserve validity.) For any $A \in W$, let A^* be the result of substituting a distinct variable y_i new to A for every definite description t_i in A, then this slight modification of the standard principle does hold in general:

P.1* If $\Vdash (A \supset B)^*$, then $\Vdash \mathrm{O}A \supset \mathrm{O}B$

Also, generally, for any formulas $C, C' \in W$:

P.2* If $\Vdash (A \equiv B)^*$, then $\Vdash C \equiv C'$, where C' is the result of replacing one or more occurrences of the formula A in C by B.

It is the restriction of P.1 to P.1* and P.2 to P.2* that, as remarked earlier, prevents the collapse of the deontic modalities to triviality. It likewise prevents the appearance of the Good Samaritan paradox in its usual forms.

Regarding quantification and identity, on this form of interpretation the classical rules are valid without restriction. For any $A \in W$, including formulas containing the deontic operator O, and for any singular terms in T, including definite descriptions:

I) $\vdash t_1 = t_2 \supset (A \supset At_1 // t_2)$

EG) $\vdash \exists y(y = t) \supset (Ax/t \supset \exists xA)$

UI) $\vdash \exists y(y = t) \supset (\forall xA \supset Ax/t)$[4]

Thus the proposed interpretation succeeds, as advertised, in giving an account of deontic statements according to which they are fully extensional with respect to singular terms, while at the same time keeping very close to the original spirit of standard deontic logic.

Finally, since this is a deontic logic, we do require

$\nVdash \mathrm{O}A \supset A$ and $\nvdash A \supset \mathrm{O}A$

(with '$\nVdash$' for 'is not valid'). These are easily verified; they demonstrate further that the modalities do not collapse to triviality.

III

This way of interpreting deontic statements is well-behaved and stable, in the sense that the set of logical truths and inferences valid under this interpretation may be formally axiomatized. Let the deductive system *DLQID* (for Deontic Logic with Quantification, Identity and Definite Descriptions) be defined by the following axiom and rule schemata For all A, $B \in W$ and all $t \in T$:

A.0 If A is a truth-functional tautology, then $\vdash A$

R.0 If $\vdash A$ and $\vdash A \supset B$, then $\vdash B$

A.1 $\vdash \forall xA \supset (\exists y(y = t) \supset Ax/t)$

[4] The antecedent clauses on (EG) and (UI) ensure that the term t denotes; they are required since many singular terms are definite descriptions, where, obviously, denotation might fail. This is independent of considerations concerning the deontic character of the modalities.

A.2 $\vdash \forall x(\exists y(x = y) \supset A) \supset \forall xA$

A.3 $\vdash \exists x \exists y(x = y)$

A.4 $\vdash t = t$

A.5 $\vdash t_1 = t_2 \supset (A \supset At_1 // t_2)$

A.6 $\vdash \exists y(y = \iota x.\, A) \supset \exists! yA$

A.7 $\vdash \forall y[\forall x(A \equiv x = y)] \supset y = \iota x.\, A$

R.1 If $\vdash A \supset B$, then $\vdash A \supset \forall xB$, if x is not free in A

R.2 If $\vdash A \supset\sim t = x$ then $\vdash \sim A$, if x is not free in A

A.8 $\vdash \mathrm{O}A \supset\sim \mathrm{O} \sim A$

A.9 $\vdash \mathrm{O}(A \supset B) \supset (\mathrm{O}A \supset \mathrm{O}B)$

A.10 $\vdash t_1 = t_2 \supset \mathrm{O}t_1 = t_2$

A.11 $\vdash \sim t_1 = t_2 \supset \mathrm{O} \sim t_1 = t_2$

R.3 If $A^* = At_1 \ldots t_n / y_1 \ldots y_n$ for every definite description, t_i in A, where every y_i is foreign to A, then if $\vdash A^*$, then $\vdash \mathrm{O}A$

A.0 and R.0, obviously, yield classical propositional logic; A.1–R.2 yield first-order logic with identity and definite descriptions; and A.8–R.3 yield the deontic component of the logic.

Note that with A.9 and R.3, even with its restriction, the deductive analog of (P.3) is derivable:

T.1 $\vdash \mathrm{O}(A \& B) \equiv \mathrm{O}A \& \mathrm{O}B$

as are the counterparts of (P.1*):

DR.1 If $\vdash (A \supset B)^*$, then $\vdash \mathrm{O}A \supset \mathrm{O}B$, when $(A \supset B)^*$ is as A^* in R.3.

and of (P.2*):

DR.2) If $\vdash (A \equiv B)^*$, then $\vdash C \equiv C'$, where C' is the result of replacing one or more occurrences of A in C by B.

Thus *DLQID* contains, along with standard quantification theory, all the key deontic principles mentioned above, namely, P.1*, P.2*, P.3, P.4 and P.5, though with the necessary restriction on P.1 and P.2.

DLQID is both strongly consistent and strongly complete. Where Γ is a set of formulas of $\mathfrak{DL}$ and A is a formula of $\mathfrak{DL}$,

Theorem I (Consistency) — If $\Gamma \vdash A$, then $\Gamma \Vdash A$.

This is readily shown by induction on the structure of proofs in *DLQID*, since all axioms are valid and the rules preserve validity. The converse theorem

Theorem II (Completeness) — If $\Gamma \Vdash A$, then $\Gamma \vdash A$

is proved using standard Henkin-style techniques. In this I follow Thomason's proof of (strong) completeness for his system Q3 in [5], although the constructions are easier for *DLQID* because of the extensionality of deontic contexts.

As usual, we show that any deductively consistent (*d*-consistent) set of formulas, Γ, included in W is simultaneously satisfiable by letting Γ generate out of itself, as it were, a model-structure and interpretation upon which all its members are true.

Lemma 1 — Every *d*-consistent set $\Gamma \subseteq W$ has a saturated extension in any ω-extension of $\mathfrak{DL}$.

This lemma is proved as for classical logic, with the understanding that a set Δ is saturated iff it is *d*-consistent, maximal (for every C, $C \in \Delta$ or $\sim C \in \Delta$), ω-complete (for all $C \in W$ and all $x \in V$, if $\exists xC \in \Delta$, then there is a $y \in V$ such that $Cx/y \in \Delta$) and further, for all $t \in T$, there is an $x \in V$ such that $t = x \in \Delta$. An ω-extension of $\mathfrak{DL}$ is $\mathfrak{DL}$ plus denumerably many individual variables not in $\mathfrak{DL}$. (In what follows designations '*V*', and '*T*' will refer to the variables and singular terms of whatever extension of $\mathfrak{DL}$ includes the set of formulas being addressed.).

Where Δ and Δ' are saturated sets of formulas (in an ω-extension of $\mathfrak{DL}$), define a relation $\mathfrak{R}$ whereby $\Delta \, \mathfrak{R} \, \Delta'$ iff for all formulas *B* such that *B* contains no singular terms other than variables, if $OB \in \Delta$ then $B \in \Delta'$.

Lemma 2 — If Δ is a saturated set of formulas and *B* contains no singular terms other than variables and $\sim OB \in \Delta$, then there is a saturated set Δ' such that $\sim B \in \Delta'$ and $\Delta \, \mathfrak{R} \, \Delta'$.

Proof by construction: Let $X = \{C$: C contains no singular terms other than variables and $OC \in \Delta\}$. $X \cup \{\sim B\}$ is *d*-consistent, for suppose it were not. Then $X \vdash B$, and so there are formulas $C_1, \ldots, C_n \in X$, such that $C_1, \ldots, C_n \vdash B$. Hence $\vdash (C_1 \& \ldots \& C_n) \supset B$; so $\vdash O(C_1 \& \ldots \& C_n) \supset OB$ by DR.1, and $\vdash (OC_1 \& \ldots \& OC_n) \supset OB$ by T.1, etc. Since all $C_i \in X$, all $O\,C_i \in \Delta$; hence, $OB \in \Delta$, and $\sim OB \notin \Delta$, contrary to the hypothesis of the lemma.

Therefore, $X \cup \{\sim B\}$ is d-consistent; hence, by Lemma 1, $X \cup \{\sim B\}$ has a saturated extension. Call one such Δ'. It is trivial that $\Delta \ \mathfrak{R} \ \Delta'$.

Next, define a quintuple $m^* = \langle L_0, \mathscr{L}, \mathscr{R}, \mathscr{D}, \mathscr{D}' \rangle$ thus:

i) L_0 is a saturated set of formulas (in an ω-extension of $\mathfrak{D}\mathfrak{L}$)
ii) $\mathscr{L}$, is the closure of $\{ L_0 \}$ under $\mathfrak{R}$ (as defined for Lemma 2)
iii) $L_j \in \mathscr{R} L_i$ iff $L_i \mathfrak{R} L_j$

Partition the set of individual variables by an equivalence relation $\approx$ whereby for all $x, y \in V$, $x \approx y$ iff $x = y \in L_0$ Select one member from each partition π of V. Then, for all $x \in V$ and for all partitions π, then if $x \in \pi$, let $f(x)$ = the selected member of π.

iv) $\mathfrak{D}L_i = \{ f(x) : x \in V \,\&\, \exists y(y = x) \in L_i \}$
v) $\mathfrak{D}' = \{ f(x) : x \in V \,\&$ for all $L_i \in L, \sim \exists y(y = x) \in L_i \}$

Let $\mathfrak{D}^* = \mathfrak{D}' \cup \bigcup_{L_i \in \mathscr{L}} \mathfrak{D}L_i$

Lemma 3 — m^* is a model-structure.

The proof is merely a matter of rehearsing the definition of a model-structure. That $\mathscr{L}$, is non-empty and that $L_0 \in \mathscr{L}$, are trivial. That for all $L_i \in \mathscr{L}$, $\mathscr{R} L_i \neq \Lambda$ is given by Lemma 2. That for all $L_i \in \mathscr{L}$, $\mathfrak{D} L_i \neq \Lambda$ results from Axiom A.3. That $\mathfrak{D}'$ is disjoint from $\bigcup_{L_i \in \mathscr{L}} \mathfrak{D}L_i$ follows immediately from the definitions and the fact that for all $L_i \in \mathscr{L}$ and all variables $x, y \in V$, $x = y \in L_i$ iff $x = y \in L_0$, which is easily shown given A.10 and A.11 and $\mathscr{L}$ the closure of $\{ L_0 \}$ under $\mathfrak{R}$. $\mathfrak{D}'$ is non-empty since, by saturation, there is an $x \in V$ such that $x = \imath y.(Fy \,\&\! \sim Fy) \in L_0$ and $f(x) \notin \mathfrak{D}L_j$ for any $L_j \in \mathscr{L}$.

Given m^*, next define a function I^* on it. First, for all $t \in T$, let $g(t, L_i)$ = $f(x)$ for some $x \in V$ such that $t = x \in L_i$ By saturation there always will be such a variable, and it is easy to show that it doesn't matter which such variable is chosen to determine the value of $g(t, L_i)$. Then define I^* on m^* as follows:

a) $I^*(x, L_i) = g(x, L_i) = f(x)$ for all $x \in V$
b) $I^*(c, L_i) = g(c, L_i)$ for all $c \in C$
c) $I^*(F^n, L_i) = \{ \langle f(x_1) \ldots f(x_n) \rangle : F^n x_1 \ldots x_n \in L_i \}$ for all $F^n \in P^n$

Further, define the function I_i^* on the basis of I^*, as so far specified, following the pattern of (1')–(3') and (4)–(10) of the definition of interpretation functions fixed by worlds that was given in Section II, with the stipulation under (10) that if there is no unique $d \in \mathfrak{D}L_i$ such that $I_i^* d/x(A, L_i) = T$, then for all $L_j \in \mathcal{L}$, $I_i^*(\imath x.A, L_j) = g(\imath x.A, L_i)$

Lemma 4 — For all $L_j \in \mathcal{L}$, all $A \in W$, and all $t \in T$, $I_i^*(t, L_j) = g(t, L_i)$. and $A \in L_j$ if and only if $I_i^*(A, L_j) = T$

Proof is by induction, rehearsing the definition of I_i^*, and is straight-forward. Here is the case where the formula A is of the form OB. For this it is useful to have the following

Sublemma — For all $L_i, L_j \in \mathcal{L}$, if $I_i^*(t_1, L_j) = I_i^*(y, L_j)$, then $I_i^*(A, L_j) = I_i^*(Ay/t_1, L_j)$ and $I_i^*(t_2, L_j) = I_i^*(t_2 y/t_1, L_j)$ (with the usual restrictions on free and bound variables).

which is easily proved by induction on the structure of A or t_2 Given the sublemma, the case for the lemma follows easily. Suppose that $OB. \in L_j$ By saturation and A.5, there are individual variables $x_i \ldots x_n \in V$, such that $x_i = t_1 \in L_j$ and ... and $x_n = t_n \in L_j$ and $OB t_1 \ldots t_n / x_1 \ldots x_n \in L_j$ for all terms t_i in B. Hence, by definition of $\mathfrak{R} L_j$, $B t_1 \ldots t_n / x_1 \ldots x_n \in L_k$ for all $L_k \in \mathfrak{R} L_j$ By the inductive hypothesis, $I_i^*(Bt_1 \ldots t_n / x_1 \ldots x_n, L_k) = T$ for all $L_k \in \mathfrak{R} L_j$. Hence $I_i^*(\mathrm{O}\, Bt_1 \ldots t_n / x_1 \ldots x_n, L_j) = T$. Since by the inductive hypothesis $I_i^*(t_i, L_j) = I_i^*(x_i, L_j)$, $I_i^*(\mathrm{O}B, L_j) = T$ by the sublemma.

Similar steps establish the converse. Suppose $I_i^*(\mathrm{O}B, L_j) = T$ but that $OB \notin L_j$ Then $\sim OB \in L_j$ by saturation, and then there will be individual variables $x_1 \ldots x_n \in V$, such that $x_1 = t_1 \in L_j$ and ... and $x_n = t_n \in L_j$ and $\sim OB t_1 \ldots t_n / x_1 \ldots x_n \in L_j$ for all terms t_i in B. By Lemma 2 above, there is an $L_k \in \mathfrak{R} L_j$ such that $\sim Bt_1 \ldots t_n / x_1 \ldots x_n \in L_k$ Hence, $Bt_1 \ldots t_n / x_1 \ldots x_n \notin L_k$ and so, by the inductive hypothesis, $I_i^*(Bt_1 \ldots t_n / x_1 \ldots x_n, L_k) \neq T$ which yields $I_i^*(\mathrm{O}Bt_1 \ldots t_n / x_1 \ldots x_n, L_j) \neq T$. Then, by the sublemma, $I_i^*(\mathrm{O}B, L_j) \neq T$, contrary to the original assumption. The other cases in the induction for this lemma are even more routine, and may be left to the reader.

Given I_i^*, the definition of I* is completed as by (11) and (12) of Section II. That is, $I^*(A, L_i) = I_i^*(A, L_i)$ and $I^*(t, L_i) = I_i^*(t, L_i)$. Lemma 4 then yields immediately this

Corollary — For all $L_i \in \mathcal{L}$, all $A \in W$, and all $t \in T$, $I^*(t, L_i) = g(t, L_i)$, and $A \in L_i$ if and only if $I^*(A, L_i) = T$.

It remains to establish

Lemma 5 — I^* is an interpretation function.

This follows directly from the definitions of I^* and I_i^* provided that it can be shown that if there is no unique $d \in \mathfrak{D}L_i$ such that $I_i^* d/x(A, L_i) = T$, then for all $L_j \in \mathcal{L}$, $I_i^*(\iota x.A, L_j) =$ a member of $\mathfrak{D}^* - \mathfrak{D}L_i$ and for all $L_j, L_k \in \mathcal{L}$, $I_i^*(\iota x.A, L_j) = I_i^*(\iota x.A, L_k)$. The latter is immediate from the definition of I_i^*. Of necessity, $I_i^*(\iota x.A, L_i) = g(\iota x.A, L_i)$ is a member of $\mathfrak{D}^*$, so we must show that $g(\iota x.A, L_i) \notin \mathfrak{D}L_i$ If there is no unique $d \in \mathfrak{D}L_i$ such that $I_i^* d/x(A, L_i) = T$, then $I_i^*(\exists!xA, L_i) \neq T$, so, by Lemma 4, $\exists!xA \notin L_i$. Hence, by Axiom 6, $\exists y(y = \iota x.A) \notin L_i$. By saturation, there is a $z \in V$ such that $z = \iota x.A \in L_i$ and such that $g(\iota x.A, L_i) = f(z)$, by definition of g. Suppose, then, that $f(z) \in \mathfrak{D}L_i$; by definition, $\exists y(y = z) \in L_i$. Since $z = \iota x.A \in L_i$, $\exists y(y = \iota x.A) \in L_i$ by Axiom 5. But $\exists y(y = \iota x.A) \notin L_i$, so $f(z) = g(\iota x.A, L_i)$ is not a member of $\mathfrak{D}L_i$.

These results combine to yield

Lemma 6 — If a set of formulas Γ is d-consistent, then Γ is simultaneously satisfiable.

Suppose that Γ is d-consistent. Then Γ has a saturated extension; call it L_0 (Lemma 1). Let m^* be as for Lemma 3 and I^* as for Lemma 4. m^* is a model-structure (Lemma 3) and I^* is an interpretation function (Lemma 5). For all A, $A \in L_0$ iff $I^*(A, L_0) = T$ (Corollary to Lemma 4). Hence for all $A \in L_0$, there is a model-structure, m, and an interpretation, I, such that A is true on I for m. Hence L_0 is simultaneously satisfiable. Since $\Gamma \subseteq L_0$, Γ is simultaneously satisfiable.

Theorem II above now follows immediately. Suppose that $\Gamma \Vdash A$, but that $\Gamma \nvdash A$. Then $\Gamma \cup \{\sim A\}$ is d-consistent; hence, by Lemma 6, $\Gamma \cup \{\sim A\}$ is simultaneously satisfiable; hence $\Gamma \nVdash A$, a contradiction.

This completes the demonstration of strong completeness for *DLQID*. Theorem II has the obvious corollary:

Corollary — If $\Vdash A$, then $\vdash A$.

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REFERENCES

[1] Goble, L., "'Ought' and Extensionality," *Noûs*, 30 (1996), 330–355.
[2] Goble, L., "Opacity and the Ought-to-Be," *Noûs*, 7 (1973), 407–412.
[3] Loewer, B. and Belzer, M., "Help for the Good Samaritan Paradox," *Philosophical Studies*, 50 (1986), 117–127.
[4] Quine, W. V., "Reference and Modality," in *From a Logical Point of View*, (Harvard University Press, Cambridge, MA), 1961, 139–159.
[5] Thomason, R. H., "Some Completeness Results for Modal Predicate Calculi," in *Philosophical Problems in Logic: Some Recent Developments*, Lambert, K., ed., (D. Reidel, Dordrecht), 1970, 56–76.
[6] van Fraassen, B., "The Logic of Conditional Obligation," *Journal of Philosophical Logic*, 1 (1972), 417–438.