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MOSTLY MEYER MODAL MODELS

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Abstract
In this paper, it is shown that several modal relevant logics are complete over a semantics in which the truth conditions for the modal operators are the standard Kripkean truth conditions, viz., $a \models \Box A$ iff $\forall x(Sab \Rightarrow b \models A)$ and $a \models \Diamond A$ iff $\exists x(Sax \,\&\, x \models A)$. This proof confirms *most* of a conjecture made by Meyer in Routley and his "Semantics of Entailment II" (*Journal of Philosophical Logic* 1 (1973)).

1. *Introduction*

In [4], Meyer suggests adding the postulate

$$Sab \Rightarrow Sa{*}b{*} \tag{1}$$

to the specification of frames for *NR* to satisfy the following scheme:

$$(K\vee)\Box(A \vee B) \rightarrow (\Diamond A \vee \Box B)$$

where S is a binary accessibility relation on worlds used to give a truth condition for necessity in the standard Kripkean manner. But completeness of *R4* (= *NR* + $K\vee$) over the resulting semantics has not yet been shown. In [3], however, Mares and Meyer show that *R4* is complete over the class of *NR*-model structures that satisfy (2) below.

$$Sab \Rightarrow \exists x(Sax \,\&\, Sa{*}x{*} \,\&\, x \leq b). \tag{2}$$

Although (2) has some virtues (see [3]), it does not yield the standard Kripkean relationship between possibility and the accessibility relation. That is,

1 I wish to thank Greg Restall and Bob Meyer for useful conversations pertaining to the topic of this paper.

$$a \models \Diamond A \text{ iff } \exists x(Sax \,\&\, x \models A)$$

does not always hold in Mares and Meyer's models.

In this paper I show that several modal relevant logics with $(K\vee)$ are complete over a class of modified *NR*-model structures that satisfy (1). The modification to *NR*-model structures consists in a change to the postulate that relates the binary accessibility relation to $\leq$. In [4], *NR*-model structures are stipulated to satisfy postulate (3) below.

$$(a \leq b \,\&\, Sbc) \Rightarrow Sac \tag{3}$$

The difficulty is that when (1) is combined with (3) (and the other postulates defining *NR*-model structures), it is possible to derive the following:

$$(a \models \Box A \,\&\, b \leq a) \Rightarrow b \models \Box A \tag{4}$$

Usually we think of $\leq$ as something like a part-whole relation on worlds. Propositions are closed upward under $\leq$; if $a \leq b$ and 'A' holds at a, then 'A' holds at b as well. It is counterintuitive that, in these model structures, necessitive propositions are also closed downwards under $\leq$. In addition, (4) makes it impossible to treat $\leq$ in the canonical model as the subset relation between theories as is usually done in completeness proofs for relevant logics. This fact has until now foiled all attempts to prove completeness for *R*4 over Meyer's models. I avoid this difficulty by replacing (3) with (5) below.

$$(a \leq b \,\&\, Sbc) \Rightarrow \exists x(Sax \,\&\, x \leq c) \tag{5}$$

I call the class of *R*-model structures that satisfy (1) and (5) "Mostly Meyer model structures", because they are so similar to the class of model structures specified by Meyer on page 70 of [4].

The condition (5) recapitulates a condition used by Božić and Došen, in [1] in their model theory for intuitionistic modal logic. They express this condition using the notation of relational products. For two binary relations P and Q, let us write '$PQac$' when there is some b such that Pab and Qbc. Then we can rewrite (5) as

$$\leq Sac \Rightarrow S\leq ac$$

or more elegantly as

$$\leq S \subseteq S \leq.$$

It is interesting to note, then, that the techniques of [1] can be extended beyond intuitionistic modal logic to relevant modal logic. There is nothing about these techniques, furthermore, that makes them particular to modal logics based on R. Similar modal logics based on other relevant logics, from B to E, are ameniable to the same treatment.

In what follows, I show that the "base" logic, RK^-, is complete over the class of Mostly Meyer RK^--model structures. After formulating RK^-, I present the frame theory for R, using version of the model theory due to Routley and Meyer (and first published in their [5]. I utilize "unreduced frames" (in the sense of [6] in which there may be more than one regular world. Then I define three classes of R-model structures. The first is the class of $R\Box$-model structures. This class of model structures combines (1) and (3). I show that this class of model structures has the untoward features that I claim for it. Next I introduce the notion of an RK^--model structure (RK^--*ms*). The completeness proof of [2] in effect shows that RK^- is complete over the class of RK^-*ms*. The third class of model structures is the class of Mostly Meyer ($MM\,RK^-$*ms*). In section seven below, I prove that the set of formulae valid over the class of $MM\,RK^-$*ms* is a subset of the class of formulae valid over the class of RK^-*ms*. Thus I show that RK^- is complete over the class of $MM\,RK^-$*ms*. In closing I indicate what postulates have to be added to the specification of model structures to obtain model structures for stronger modal relevant logics.

2. *The Logic $MM\,RK^-$*

I use a standard modal language with proposition letters $p_1, p_2, \ldots,$ connectives $\rightarrow, \wedge, \vee, \sim, \Box$, parentheses and the usual formation rules. As usual, possibility is defined as $\Diamond A =_{df} \sim \Box \sim A$. The logic RK^- is axiomatized as follows:

Axiom Schemes

RT All substitution instances of theorems of R.

Agg $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$

K∨ $\Box(A \vee B) \rightarrow (\Box A \vee \Diamond B)$

Rules

$$\frac{\vdash A \to B \quad \vdash A}{\vdash B}\, MP$$

$$\frac{\vdash A \quad \vdash B}{\vdash A \wedge B}\, Adj$$

$$\frac{\vdash A \to B}{\vdash \Box A \to \Box B}\, RM$$

$$\frac{\vdash A}{\vdash \Box A}\, RN$$

Note that this logic is slightly weaker than the logic *RK* of [2]; RK^{-}, unlike *RK*, does not contain every instance of the scheme $\Box(A \to B) \to (\Box A \to \Box B)$. But this loss does not signify a very significant weakness of RK^{-}, since it does contain every instance of the closely related scheme $(\Box(A \to B) \wedge \Box A) \to \Box B$.

3. *R-Frames*

An unreduced *R*-frame is a is a quadruple <*K*, *P*, *R*, *> such that *K* is a non-empty set (the set of worlds), *P* (the set of regular worlds) is a non-empty subset of *K*, *R* is a ternary relation on *K* and * is a unary operatory on *K* that satisifies the following definitions and postulates. Where *a, b, c, d,...* are worlds,

$$D\leq.\ a \leq b =_{df} \exists x(x \in P \ \&\ Rxab),$$
$$DR^2.\ R^2abcd =_{df} \exists x(Rabx \ \&\ Rxcd).$$

R1 $\leq$ is a partial order.

R2 *Raaa*.

R3 $(a \leq b \ \&\ Rbcd) \Rightarrow Racd$.

R4 $Rabc \Rightarrow Rac{*}b{*}$.

R5 $a{*}{*} = a$.

R6 $R^2abcd \Rightarrow R^2acbd$.

4. *R*□-*Frames*

An *R*□-frame is a quintuple *K* = <*K*, *P*, *R*, \$, *> such that <*K, P, R,* *> is an unreduced *R*-frame and is a binary relation on *K* frame and

R□1 $(a \leq b \;\&\; \$bc) \Rightarrow \$ac,$

R□2 $(a \in P \;\&\; \$ab) \Rightarrow b \in P,$

R□3 $\$ab \Rightarrow \$a{*}b{*}.$

An *R*□ *ms* is a triple *M* = <*K*, v, $\models_v$> where *K* =<*K, P, R,* \$, *> is an *R*□ -frame, *v* is a valuation on *K* and $\models_v$ is a satisfaction relation. A valuation on *K* is a function from proposition letters to subsets of *K* closed upwards under ≤ , that is, the following "hereditariness postulate" holds:

$$HP. (a \leq b \;\&\; a \in v(p)) \Rightarrow b \in v(p)$$

As usual, each valuation determines a relation $\models_v$ between worlds and formulae such that:

- $a \models_v p$ iff $a \in v(p)$
- $a \models_v A \wedge B$ iff $a \models_v A$ and $a \models_v B$
- $a \models_v \sim A$ iff $a{*} \not\models_v A$
- $a \models_v A \rightarrow B$ iff $\forall x \forall y (Raxy \Rightarrow (x \models_v A \Rightarrow y \models_v B))$
- $a \models_v \Box A$ iff $\forall x (\$ax \Rightarrow x \models_v A)$

A formula *A* is said to be *verified* in an *R*□ *ms M* = <<*K, P, R,* *>, *v*, $\models_v$> iff, for all $a \in P$, $a \models_v A$ and *A* is said to be *R*□-*valid* iff it is verified in all *R*□*ms*.

Now let us spend a moment showing that what I said in the introduction about *R*□3 is true.

Lemma 4.1 *If M* = <<*K, P, R,* \$, *>, *v*, $\models_v$> *is an R*□*ms, then, for any worlds a and b in K, if* $a \leq b$ *and* $b \models_v \Box A$*, then* $a \models_v \Box A$.

Proof. Suppose that $a \leq b$ and $b \models_v \Box A$. Now let *c* be an arbitrary world such that \$*ac*. Thus, by *R*□3, $\$a{*}c{*}$. It can be easily shown that, on any *Rms*, if $a \leq b$, then $b{*} \leq a{*}$. So, $b{*} \leq a{*}$. Thus, by *R*□1, $\$b{*}c{*}$. By *R*□3, $\$b{**}c{**}$, and so by R5, \$*bc*. Since $b \models_v \Box A$, $c \models_v A$. But, since *c* is an arbitrary world such that \$*ac*, it is the case that $a \models_v \Box A$, which is what we wanted to show.

An annonymous referee has pointed out to me that *prima facie* $R\square 1$ seems intuitive. Paraphrasing slightly, the referee argued that, if *a* says less than *b*, then what is possible relative to *a* must be possible relative to *b* too -- any information a had to rule a possibility out must be preserved in *b*. The referee's point is well taken in the context of Routley and Meyer's *NRms*, but not in the context of model structures that satisfy the Kripke truth condition for possibility. For, if $a \leq b$ and a $\models_v \Diamond A$, then b $\models_v \Diamond A$. Thus later worlds do not rule out any possibilitive propositions true at earlier worlds. Given Kripke's truth condition for possibility and this hereditariness condition, it appears as if later worlds have to have the same worlds accessible to them as earlier worlds. For if *b* were to have accessible to it a proper subset of the worlds accessible to *a*, it would become problematic how we could show that all the possibilitive propositions true at *a* are also true at *b*.

5. RK^--*Frames*

Thus, we are stuck with a dilemma, either we should abandon $R\square$ 1, which has some intuitive plausibility, or abandon Kripke's truth condition for possibility. In [3], we adopted the latter course. In that work, we replaced $R\square$ 3 with the following postulate:

$$RK \quad \$ab \Rightarrow \exists x(x \leq b \ \&\ Tax),$$

where

$$Tab =_{df} \$ab \ \&\ \$a{*}b{*}.$$

Let us call the resulting frames, RK^--frames, and the corresponding model structures, RK^-ms, for RK^- is sound and complete over this class of model structures. Soundness and completeness are nice features, but the definition of the class of RK^--frames does lack intuitiveness.

Let us call the class of formulae valid over the class of RK^-ms, RK^-valid.

I use the hereditariness lemma below in the completeness proof of section seven. The proof of the hereditariness lemma follows the usual pattern (see [4]).

Lemma 5.1 *(*RK^-ms *Hereditariness) In any* RK^-ms, *for any worlds a and b, if* $a \models_v A$ *and* $a \leq b$, *then* $b \models_v A$, *for any formula A.*

6. *Mostly Meyer Frames*

Although $R\Box 3$ is problematic when added to frames that satisfy $R\Box 1$ and $R\Box 2$, it does have some nice features. In particular, as I said in the introduction, it makes it possible to derive the standard Kripke truth condition for possibility. Happily, we can define a class of frames over which RK^- is complete and which do not have the untoward features of frames dealt with at the end of section four above. In this section, I define this class of frames which I call 'mostly Meyer RK^--frames'.

A mostly Meyer RK^--frame is a quintuple $K = \langle K, P, R, S, *\rangle$ such that $\langle K, P, R, *\rangle$ is an unreduced R-frame and S is a binary relation on K, such that it satisifies the following postulates:

MM1 $(a \le b \,\&\, Sbc) \Rightarrow \exists x(Sax \,\&\, x \le c)$,

MM2 $(a \in P \,\&\, Sab) \Rightarrow b \in P$,

MM3 $Sab \Rightarrow Sa{*}b{*}$.

An $MMRK^-ms$ is a triple $M = \langle K, u, \models_u^{mm}\rangle$, where u is a valuation on K and acts just like a valuation in an RK^-ms (i.e., its range is made up of closed up sets of worlds). But the satisfaction relation $\models_u^{mm}$ determined by u is slightly different from $\models_v$. The truth conditions for the various connectives remain the same (replacing $\models_v$ with $\models_u^{mm}$) except for the truth condition for necessity which now reads

$$a \models_u^{mm} \Box A \text{ iff } \forall x(Sax \Rightarrow x \models_u^{mm} A)$$

As before, a formula A is said to be *verified* in an $MMRK^-ms$ $M = \langle\langle K, P, R, S, *\rangle, u, \models_u^{mm}\rangle$ iff, for all $a \in P$, $a \models_u^{mm} A$ and A is said to be *Mostly-Meyer valid* iff it is verified in all $MMRK^-ms$.

The reason we need $MM1$ is to ensure the truth of the hereditariness lemma below. The original hereditariness condition (3) works in the context of Routley and Meyer's model structures for NR to transfer necessities from indices to (weakly) later indices (under $\le$). It says that the set of worlds accessible to a later world is a subset of the set of worlds accessible to an earlier world. The new condition (5) does the same job, but is not as strong. It uses the hereditariness relation, $\le$, to transfer truths at earlier worlds to later worlds and to transfer truths at worlds accessible to earlier worlds to worlds accessible to later worlds. Case 2 of the proof of the hereditariness lemma below makes this claim rigorous.

Lemma 6.1 (*MM Hereditariness*) *If* $a \le b$ *and* $a \models_u^{mm} A$, *then* $b \models_u^{mm} A$.

Proof. Suppose that $a \leq b$ *and* $a \vDash_u^{mm} A$
Case 1. A is p, $B \wedge C$, $\sim B$, or $B \rightarrow C$. The proof is as usual (see, e.g., [5]).
Case 2. $A = \Box B$. By hypothesis, $a \vDash_u^{mm} \Box B$. Now suppose that c is such that Sbc. I show that $c \vDash_u^{mm} B$. By $MM1$, here is a d such that Sad and $d \leq c$. By the hypothesis and the truth condition for necessity, $d \vDash_u^{mm} B$. By the inductive hypothesis, $c \vDash_u^{mm} B$, as required. Thus, generalizing, $b \vDash_u^{mm} \Box B$.
From the hereditariness theorem the soundness theorem follows in the usual way (see, e.g., [5]).

Since one of the advantages I have claimed for $MMRK^{-}ms$ is that they preserve Kripke's truth condition for possibility, I should prove it.

Lemma 6.2 $a \vDash_u^{mm} \Diamond A$ *if and only if* $\exists x(Sax \,\&\, x \vDash_u^{mm} A)$

Proof. $\Rightarrow$ Suppose that $a \vDash_u^{mm} \Diamond A$. Then $a \vDash_u^{mm} \sim\Box\sim A$. By the truth condition for negation, $a{*} \nvDash_u^{mm} \Box \sim A$. By the truth condition for necessity, there is some $b*$ such that $Sa{*}b{*}$ and $b*$ not $\vDash_u^{mm} {\sim}A$. By the truth condition for negation, $b \vDash_u^{mm} A$. But, by $MM3$, $Sa{*}{*}b{*}{*}$. By $R5$, Sab. Hence there is a world b such that Sab and $b \vDash_u^{mm} A$.
$\Leftarrow$ The argument for this direction is much like that for its converse, only backwards to it. Suppose that there is a world b such that Sab and $b \vDash_u^{mm} A$. By the truth condition for negation and $R5$, $b{*} \vDash_u^{mm} {\sim}A$. By $MM3$, $Sa{*}b{*}$. Thus, by the truth condition for necessity, $a{*} \vDash_u^{mm} \Box{\sim}A$. By the truth condition for negation, $a \vDash_u^{mm} {\sim}\Box{\sim}A$, i.e. $a \vDash_u^{mm} \Diamond A$.

7. *Completeness*

I prove completeness by showing that the set of Mostly-Meyer valid formulae is a subset of the RK^{-}-valid formulae. Since RK^{-} is complete over $RK^{-}ms$, it is therefore complete over $MMRRK^{-}ms$.

The following lemma is easy to prove by inspection..

Lemma 7.1 *If* $\langle K, P, R, \$, *\rangle$ *is an* RK^{-}*-frame, then* $\langle K, P, R, T, *\rangle$ *is a Mostly-Meyer* RK^{-}*-frame.*

Given an $RK^{-}ms$. $\langle\langle K, P, R.\$, *\rangle, v, \vDash_v\rangle$, I define a relation $\vDash_v'$ by means of the following clauses:

- $a \models'_v p$ iff $a \in v(p)$
- $a \models'_v A \wedge B$ iff $a \models'_v A$ and $a \models'_v B$
- $a \models'_v \sim A$ iff $a* \models'_v A$
- $a \models'_v A \rightarrow B$ iff $\forall x \forall y (Raxy \Rightarrow (x \models'_v A \Rightarrow y \models'_v B))$
- $a \models'_v \Box A$ iff $\forall b (Tab \Rightarrow b \models'_v A)$

Lemma 7.2 below follows directly from lemma 7.1 and the definition of $\models'_v$.

Lemma 7.2 *If <<K, P, R, \$, *>, v, $\models_v$> is an RK⁻ms, then then <<K, P, R, T, *>, v, $\models'_v$> is an MMR K⁻ms.*

Let us call <*K, P, R, T,* *> 'the Mostly Meyer *RK*⁻-frame determined by <<*K, P, R, \$,* *>' and <<*K, P, R, T,* *>, *v*, $\models'_v$> 'the *MMR K⁻ms* determined by <<*K, P, R, \$,* *>, *v*, $\models_v$>'.

Lemma 7.3 *Let K = <K, P, R, \$, *>, be an RK⁻-frame and v a valuation on K. Then, for the RK⁻ms <<K, P, R, T, *>,v, $\models'_v$> and the MMR K⁻ms K⁻, <<K, P, R, \$, *>, v, $\models_v$>,* for any $a \in K$, $a \models'_v A$ *iff* $\models_v A$.

Proof. By induction on the complexity of A. The proof of the theorem for $A = p$, $B \wedge C$, $B \rightarrow C$, $\sim B$ are easy. Suppose that $A = \Box B$. $\Rightarrow$ Assume that $a \models'_v \Box B$. Then, for all b such that Tab, $b \models'_v B$. Now suppose that c is such that $\$ac$. I show that $c \models_v B$. By *RK*, there is some $b \leq c$ such that Tab. By the hereditariness lemma for *RK*⁻ms, $c \models'_v B$. By the induction hypothesis, $c \models_v B$, as promised. Thus, by the truth condition for necessity for *R K⁻ms*, $a \models'_v \Box B$. $\Leftarrow$ is easy.

Corollary 7.4 *For any formula A, if A is Mostly-Meyer valid, then A is R K⁻-valid.*

Theorem 7.5 *R K*⁻ is complete over the class of *MMR K⁻ms*.

8. *Extensions*

Adding postulates to the definition of *MMR K⁻ms*, we can determine classes of Mostly Meyer model structures for stronger logics. The following are some correspondences between axiom schemes and semantic postulates:

Scheme	Postulate
$\Box(A \to B) \to (\Box A \to \Box B)$	$\exists x(Rabx \,\&\, Sxc) \Rightarrow \exists x \exists y(Sax \,\&\, Sby \,\&\, Rxyc)$
$\Box A \to A$	Saa
$\Box A \to \Box\Box A$	$(Sab \,\&\, Sbc) \Rightarrow Sac$
$A \to \Box\Diamond A$	$Sab \Rightarrow Sba$
$\Box A \to \Diamond A$	$\exists x Sax$

Completeness for the logics resulting from adding some or all of the above schemes to RK^- and the corresponding postulates to the definition of the class of Mostly Meyer RK^--frames can be shown either directly, through the standard canonical model construction or by adding the appropriate postulate(s) to the definition of RK^--frames (replacing S with \$) and showing that, for each resulting RK^-ms, the $MMRRK^-$ms that it determines satisfies the postulates (replacing S with T).

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