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ON THE HÄRTIG-STYLE AXIOMATIZATION OF UNPROVABLE AND SATISFIABLE FORMULAS OF BERNAYS AND SCHÖNFINKEL'S CLASSES*

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Abstract

We give Härtig-style axiomatizations of unprovable and satisfiable formulas of Bernays and Schönfinkel's classes. The characterization theorem, in other words, the completeness theorem for them is proved. We propose the notion of the absolute atomic formula property which is similar to that of the atomic formula property. The proposed systems have the absolute atomic formula property as well as the atomic formula property. We show the relation among the proposed property, the atomic formula property and the subformula property. We also consider the axiomatization of provable formulas and contradictions of the classes, for which the subformula property holds. In the last section, some open problem is given as well as our methodology for further studies of axiomatizing unprovable and satisfiable formulas.

This paper contains four appendices. In Appendix A, we propose first-order opposite system without free variables in the style of M. W. Bunder. In Appendix B, we present Härtig-style axiomatizations of unprovable and satisfiable quantifier-free formulas of classical first-order predicate logic with equality. The Appendix B is a preparation for the next Appendix C. In Appendix C, we show that Bullock and Schneider's calculi for finitely satisfiable formulas have the atomic formula property. Those calculi do not have the absolute atomic formula property. In Appendix D, we give some refutation and satisfaction calculi which have neither the subformula property, nor the atomic formula property, nor the absolute atomic formula property.

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1. *Introduction.*

Since Słupecki's celebrated work on the (Ł-)decision problem of the Aristotelian syllogistics (see [36]), quite a few logicians working outside Poland have been interested in refutation systems, that is, ones deriving unprovable formulas. The concept of satisfiablility is not only closely related to the so-called Entscheidungsproblem for a logical system in general (see e.g. [18] and [1]), but also especially to that of unprovability, since they are dual concepts with each other: that is, a formula is satisfiable if and only if the negation of it is unprovable (see also the third and the fourth sections of this paper).[1] *We shall use this meta-equivalence as the definition of satisfiability.*[2]

Let *CPL* be classical first-order predicate logic *without equality*. Let $\mathcal{F}$ be the set of all well-formed formulas of *CPL which contain neither function signs nor constants.*

Let us introduce Bernays and Schönfinkel's classes $F^{\forall}$, $F^{\exists}$ and $F^{\forall\exists}$ with which we shall be concerned thoughout this paper.

Let $F^{\forall}$ be the set of all the formulas of the form

$$\forall x_1 \ldots \forall x_n A(x_1, \ldots, x_n) \ (n \geq 0),$$

say B, such that (1) $A(x_1, \ldots, x_n) \in \mathcal{F}$ contains no quantifiers and (in B, free variables may appear).

Let $F^{\exists}$ be the set of all the formulas of the form

$$\exists x_1 \ldots \exists x_n A(x_1, \ldots, x_n) \ (n \geq 1),$$

say B, such that (1) $A(x_1, \ldots, x_n) \in \mathcal{F}$ contains no quantifiers and (2) no free individual variables appear in B.

Let $F^{\forall\exists}$ be the set of all the formulas of the form

$$\forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m A(x_1, \ldots, x_n, y_1, \ldots, y_m) \ (n \geq 1 \text{ and } m \geq 1),$$

[1] Also we can say that validity (Allgemeingültigkeit in German) and satisfiability (Erfürbarkeit in German) are in a dual relation with each other, since a formula is valid if and only if the negation of it is not satisfiable.

[2] See also Appendix C.

say B, such that (1) $A(x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathcal{F}$ contains no quantifiers and (2) no free individual variables appear in B.[3]

Bernays and Schönfinkel [2] gave a solution of the decision problem for provability of a formula of $F^{\forall}$, $F^{\exists}$ and $F^{\forall\exists}$, respectively (see e.g. [18]).

In this paper, we shall propose a Härtig-style axiomatization $HC^{\forall}$ of a set $\{A \in F^{\forall} : A \text{ is not provable in } CPL\}$, a similar axiomatization $HC^{\exists}$ of a set $\{A \in F^{\exists} : A \text{ is not provable in } CPL\}$, a similar one $HC^{\forall\exists}$ of a set $\{A \in F^{\forall\exists} : A \text{ is not provable in } CPL\}$, a similar one $HCS^{\forall}$ of a set $\{A \in F^{\forall} : \neg A \text{ is not provable in } CPL\ (= A \text{ is satisfiable in } CPL)\}$ a similar one $HCS^{\exists}$ of a set $\{A \in F^{\exists} : \neg A \text{ is not provable in } CPL\}$.

The characterization theorem, in other words, the completeness theorem for them is proved. Further we shall propose the notion of the absolute atomic formula property (for short, abs-a.f.p.), which is similar to that of the atomic formula property (for short, a.f.p.) introduced in [25]. (The notion of the so-called subformula property (for short, s.f.p.) is stronger than that of a.f.p..) We shall show that the proposed calculi have the absolute atomic formula property as well as the atomic formula property.

Every logician knows that conjunctive normal form of a formula is used for a decision procedure for provability of a formula of classical propositional logic CP. Härtig [17] employed the idea for the decision procedure to give an axiomatization of formulas which are not provable in CP (see also [22, 25]). In order to derive an unprovable formula, we may conversely follow the decision procedure of the given formula. This is the very idea for the axiomatization. In this paper, the same idea will be used with the results of [2] and [17].

To the best of my knowledge, except for two papers[4], there have so far been no literature on Hilbert-style axiomatizations for a proper subset of the set of all the unprovable (satisfiable) formulas of CPL. So, the result of this paper could mean one of first steps toward an uncultivated area of logic, i.e. axiomatizing unprovable and satisfiable first-order prenex formu-

[3] In the terminology of [11], Bernays and Schönfinkel's class $F^{\forall\exists}$ is that of prenex formulas with prefixes $\exists \ldots \exists \forall \ldots \forall$. But the essence is the same. For we see that Bernays-Schönfinkel Class, say BS-Class, in the sense of [11] is solvable if and only if BS-Class has a decision procedure for satisfiability if and only if $F^{\forall\exists}$ has a decision procedure for provability. That is, 4

[4] For finite satisfiability, see Bullock and Schneider [6]; for formulas invalid in some finite domain, see Hailperin [16]. There is a generalization of [6] to the case with equality by the same authors with the same idea (see [5]).

las (note that the decision problem for the full predicate logic is unsolvable (Church [10] and Turing [44])). There must be still large room to investigate axiomatic systems in the direction of this paper ([1], [3] and [11], for example, will give us some basics and information for our further study).

In the following second section, we shall review the decision procedures for $F^{\forall}$, $F^{\exists}$ and $F^{\forall\exists}$. In the third section, Härtig's refutation calculus *HC* for *CP* will be recalled with the characterization theorem and the atomic formula property of it. In the fourth section, we shall present a satisfaction calculus *HCS* for satisfiable formulas of *CP* as a dual calculus of *HC*.

In the fifth section, we shall propose the Härtig-style axiomatization of the mentioned sets. In the sixth section, we shall give the characterization theorem for the proposed systems. In the seventh section, we shall propose the notion of the absolute atomic formula property (for short, abs - a.f.p.) which is similar to that of the atomic formula property (for short, a.f.p.) introduced in [25]. Then, it will be shown that the proposed systems have the absolute atomic formula property as well as the atomic formula property. In particular, we shall in the same section give the relation among the proposed property, the atomic formula property and the subformula property (for short, s.f.p.).

In the eighth section, we shall give the axiomatizations $P^{\forall}$, $P^{\exists}$ and $P^{\forall\exists}$ for provable formulas of $F^{\forall}$, $F^{\exists}$ and $F^{\forall\exists}$, respectively, and the axiomatizations $C^{\forall}$ and $C^{\exists}$ for contradictions of $F^{\forall}$ and $F^{\exists}$, respectively. We shall easily see that all the system proposed in this section have the subformula property. In addition, it will be shown that $P^{\forall}$ and $C^{\exists}$ have the absolute atomic formula property.

In the last section, we shall review the methodology of this paper for axiomatization, which would be useful for further studies in the direction of the paper, as well as we shall give some general comments and some open problem.

This paper contains four appendices A, B, C and D. In Appendix A, as a little curiosity, we shall propose first-order opposite system without free variables in the style of M. W. Bunder.[5] In Appendix B, we present Härtig-style axiomatizations of unprovable and satisfiable quantifier-free formulas of *CPL* argumented with equality. The Appendix B is a preparation for the next Appendix C.

In Appendix C, we shall show that Bullock and Schneider's calculi for finitely satisfiable formulas of *CPL* (*CPL* argumented with equality) have the atomic formula property. Those calculi do not have the absolute atomic formula property. This means that the notion of the atomic formula prop-

[5] This material is so included that a similar idea used in the sixth section below will be employed in order to prove the main theorem of the Appendix.

erty is not stronger than that of the absolute atomic formula property. In Appendix D, we shall give some refutation and satisfaction calculi which have neither s.f.p., nor a.f.p., nor abs-a.f.p..

2. *The decision procedures for* $F^{\forall}$, $F^{\exists}$ *and* $F^{\forall\exists}$.

In this section, we shall review the decision procedures for $F^{\forall}$, $F^{\exists}$ and $F^{\forall\exists}$, which were obtained by Bernays and Schönfinkel [2]. For avoiding possible misunderstandings below, we shall first define the following concept, *absolute distinctness*. (We shall below follow the terminologies in Kleene [34], that is, *proposition letter, predicate letter, predicate letter with attached variables, predicate letter formula.*)

Definition 2.1. Let $F_1(x_1, \ldots, x_n)$ be an arbitrary predicate letter with n attached variables of *CPL* $(n \geq 0)$ and $F_2(y_1, \ldots, y_m)$ an arbitrary predicate letter with m attached variables of *CPL* $(m \geq 0)$. The atomic predicate letter formulas $F_1(t_1, \ldots, t_n)$ and $F_2(s_1, \ldots, s_m)$ with terms $t_1, \ldots, t_n, s_1, \ldots, s_m$ are said to be *absolutely distinct* if they satisfy the following condition: *if* F_1 *and* F_2 *are the same predicate letters and* $n = m$, *then there is at least one term* $t_i (1 \leq i \leq n)$ *such that* t_i *and* s_i *are different*, otherwise they are said to be *absolutely the same*.

We shall give some concrete examples for absolute distinctness. Let F and G be different predicate letters. Let x, y, z, u and v be mutually distinct variables. Let c and d be different individual constants. For example, the following pairs of predicate letter formulas are those of absolutely distinct ones:

$$\{F, G\}, \{F, F(x)\}, \{F, F(c)\}, \{F(x), F(c)\}, \{F(c), F(d)\}, \{F(x), F(x, x)\},$$
$$\{F(x), G(x, y)\}, \{F(x, y), G(x, y)\}, \{F(x, y), F(u, v)\}, \{F(x, y), F(x, z)\},$$
$$\{F(c, y), F(c, z)\}, \{F(x, y), F(y, x)\}, \{F(c, d), F(d, c)\}, \{G(c), G(c, c, c)\},$$
$$\{F(x, y, z), F(x, y, x)\}, \{F(x, x, x), G(x, x, x)\}, \{F(x, y, c), F(x, y, x)\},$$
$$\{F(x, y, c), F(x, z, x)\}.$$[6]

[6] F and G, that is, predicate letters with 0 attached variable, are regarded as proposition letters as usual.

That is, in other words, absolutely distinct formulas are different as formulas. *Throughout this paper, "distinct (or different) formulas" are understood as "absolutely different formulas".*

Now, here are the decision procedures for $F^{\forall}$, $F^{\exists}$ and $F^{\forall\exists}$:

(I) The case of $F^{\forall}$. Let A be a formula of $F^{\forall}$, say $\forall x_1 \dots \forall x_n\, B(x_1, \dots, x_n)$ $(n \geq 0)$. (A may contain free variables.) A is provable (or valid) in *CPL* if and only if $B(x_1, \dots, x_n)$ is provable in *CP* (or valid, or an instance of tautology), where all absolutely distinct atomic predicate letter formulas in $B(x_1, \dots, x_n)$ are regarded as distinct proposition letters, respectively.

(II) The case of $F^{\exists}$. Let A be a formula of $F^{\exists}$, say $\exists x_1 \dots \exists x_n\, B(x_1, \dots, x_n)$ $(n \geq 1)$. (A may not contain free variables.) A is provable (or valid) in *CPL* if and only if for some variable y, $B(y, \dots, y)$ is provable in *CP* (or valid, or an instance of tautology), where all absolutely distinct atomic predicate letter formulas in $B(y, \dots, y)$ are regarded as distinct proposition letters, respectively.

(III) The case of $F^{\forall\exists}$. Let A be a formula of $F^{\forall\exists}$, say $\forall x_1 \dots \forall x_n \exists y_1 \dots \exists y_m$ $B(x_1, \dots, x_n, y_1, \dots, y_m)$ $(n \geq 1$ and $m \geq 1)$, (A may not contain free variables.) A is provable (or valid) in *CPL* if and only if for some finite sequence $z_1, \dots, z_m$ of variables with length m and $\{z_1, \dots, z_m\} \subseteq \{x_1, \dots, x_n\}$, $B(x_1, \dots, x_n, z_1, \dots, z_m)$ is provable in *CP* (or valid, or an instance of tautology), where all absolutely distinct atomic predicate letter formulas in $B(x_1, \dots, x_n, z_1, \dots, z_m)$ are regarded as distinct proposition letters, respectively.

For the justification of the above decision procedures, the reader may consult the excellent exposition in Hilbert-Ackermann [18], which adopts a certain smart cut-free Gentzen-(Schütte-)style calculus for *CPL* so that all the arguments for the justification of (I), (II) and (III) are kept very simple and elegant.[7]

Every formula of $F^{\forall} \cup F^{\exists} \cup F^{\forall\exists}$ contains neither function signs nor constants.[8] This restriction is essential to the application of the above de-

[7] For another type of exposition on it, see e.g. [39].

cision procedures (II) and (III).[9] We shall explain it now. Let $F(a_1)$ be a predicate letter with one attached variable and $F(a_1, a_2)$ one with two attached variables. Let x and y be distinct variables and c a constant. Without the restriction,

$$\exists x(F(c) \vee \neg F(x)) \; (\dagger)$$

and

$$\forall x \exists y(F(c, x) \vee \neg F(y, x)) \; (\dagger\dagger)$$

belong to $F^{\exists}$ and $F^{\forall\exists}$, respectively. By applying the decision procedures (II) and (III) to (†) and (††), respectively, then we immediately know that they are not provable in *CPL*. This is not true. The formulas (†) and (††) are surely theorems of *CPL*.

3. *The Härtig's refutation calculus HC and the atomic formula property of it.*

Let *CP* be classical propositional logic. We take an arbitrary formulation for *CP* and fix it. We shall use $\vee$ (disjunction), $\wedge$ (conjunction), $\supset$ (implication) and $\neg$ (negation) as the logical symbols of *CP*. We here take the same language of the system to be recalled bellow as that of *CP*.

We shall adopt some notational convention.

Definition 3.1. For any formal system X and any formula A of X, $\vdash_X A(\dashv_X A)$ means that A is (not) provable in X. For any formula A, $< A >$ stands for the set of all atomic subformulas of A. For any formula A, we shall call $< A >$ *the atomic subformula set of* A.

Definition 3.2. We inductively define '*subformula*' of a given formula of *CP* by the following: (1) If A is a formula, A is a subformula of A; (2) If A and B are formulas, the subformulas of A and the subformulas of B are subformulas of $A*B$ for every binary boolean connective $*$; (3) If A is a formula, the subformulas of A are subformulas of $\neg A$.

8 In a certain formulation of *CPL*, constants are regarded as function signs with 0 argument. So the distinction between function sign and constant are not necessary in such a formulation.

9 For the decision procedure (I), the restriction is not necessary. But, for the proof of Theorem 8.1.(4) of this paper, we also take the restriction on the formulas of $F^{\forall}$.

The Härtig-style calculus *HC* which axiomatizes a set $\{A\colon \dashv_{CP} A\}$, consists of the following axioms and rules (see [17] ([22, 25])):

Axioms:

$(HC1) \vdash_{HC} r$ for any proposition letter *r*.

$(HC2) \vdash_{HC} \neg r$ for any proposition letter *r*.

Rules:

$(HC3) \vdash_{HC} A,\ \vdash_{HC} B,\ <A> \cap <B> = \varnothing \Rightarrow \vdash_{HC} A \vee B.$

$(HC4) \vdash_{CP} A \supset B,\ \vdash_{HC} B \Rightarrow \vdash_{HC} A.$

Theorem 3.1. ([17]) *For any formula A of HC,* $\vdash_{HC} A \Leftrightarrow \dashv_{CP} A$.

Here is the definition of the atomic formula property introduced in [25].

Definition 3.3. Let *X* be a formal system. The system *X* has *the atomic formula property* (for short, a.f.p.), if for any formula *A* provable in *X*, there is a proof of *A* in *X* which contains only such formulas, say $B_1, B_2, \ldots, B_n$ that $<B_i> \subseteq <A>$ holds for any $1 \le i \le n$ (we say that such a proof has the atomic formula property).[10]

We know the following theorem.

Theorem 3.2. *HC has the atomic formula property*.

4. *The satisfaction calculus HCS and its a.f.p.*.

The idea of the axiomatization for *HC* is based on conjunctive normal form of a formula. The same idea with *disjunctive* normal form can be used to give a Härtig-style axiomatization for satisfiable formulas of *CP*. Let us carry it out with the idea. *A* satisfaction calculus *HCS* which axiomatizes a

[10] This definition is implicitely for proofs in a Hilbert-style system. But for another type of systems, e.g. a Gentzen-syle one, we can easily adapt the definition to that of them.

set $\{ A: \dashv_{CP} \neg A\,(= A$ is satisfiable in $CP) \}$, consists of the following axioms and rules:[11]

Axioms:

$(HCS1) \vdash_{HCS} s$ for any proposition letter s.

$(HCS2) \vdash_{HCS} \neg s$ for any proposition letter s.

Rules:

$(HCS3) \vdash_{HCS} A,\ \vdash_{HCS} B,\ <A> \cap <B> = \varnothing \Rightarrow \vdash_{HCS} A \wedge B.$

$(HCS4) \vdash_{HCS} A,\ \vdash_{CP} A \supset B \Rightarrow \vdash_{HCS} B.$

The rest of this section will be devoted to the proof of the characterization theorem for HCS and to the atomic formula property of it.

For the notational convenience and the descriptive precision, it is recalled that OP is a Gentzen-style sequent calculus for contradictions of CP (see [23]). (Of course, we may here assume that the language of OP is the same as that of CP. For a one-sided calculus equivalent to OP, see e.g. [38, p. 13].)

Further we shall also recall the following to make sure.

Theorem 4.1. ([23]) *For any formula A of OP*, $\vdash_{OP} A \iff \vdash_{CP} \neg A$.

The following theorem tells that HCS exactly axiomatizes a set $\{A : A$ is satisfiable in $CP\}$.

Theorem 4.2. *For any formula A of HCS*, $\vdash_{HCS} A \iff \dashv_{OP} A.$

Proof. ($\Leftarrow$): Let A be a formula of HCS. Suppose $\dashv_{OP} A$. Take a disjunctive normal form of A, say $B_1 \vee B_2 \vee \ldots \vee B_k (k \geq 1)$ (cf. e.g. [18], [45] and so on). In view of Theorem 4.1 and the completeness theorem for CP, one of the disjuncts of it is of the following form:

[11] The same calculus as HCS with a slightly different form has already used in Bullock and Schneider [6] as the quantifier-free part of their calculus for finitely satisfiable formulas of CPL. It is based on the fact that for any quantifier-free formula, it is finitely satisfiable if and only if it is satisfiable.

$$\bigwedge_{\mu=1}^{m} p_{i_\mu} \wedge \bigwedge_{\nu=1}^{n} \neg p_{j_\nu} \left(m \geq 0, n \geq 0, m+n \geq 1\right),$$

where every number i_μ is different from every number j_ν. Let B_I be the disjunct. It is obvious that

$$\vdash_{CP} B_I \supset A,$$

since A is logically equivalent to the disjunctive normal form of it. (Note that

$$< B_I > \subseteq < A >$$

holds.) It is easy to prove $\vdash_{HCS} B_I$. (Furthermore we can easily have a proof of B_I in *HCS* with the atomic formula property.) Now we apply (*HSC*4) to $\vdash_{HCS} B_I$ and $\vdash_{CP} B_I \supset A$. Then we obtain $\vdash_{HCS} A$.[12]

($\Rightarrow$): By induction on derivations, it is easy to prove that for any formula A of *HCS*, if A is provable in *HSC*, then A is satisfiable in *CP*, in other words, $\dashv_{OP} A$ by the completeness theorem. □

Corollary 4.1. *For any formula A of HCS*, $\vdash_{HCS} A \Leftrightarrow \dashv_{CP} \neg A$.

By Theorem 3.1 and Corollary 4.1, we immediately have the following.

Corollary 4.2. *For any formula A of HCS*, $\vdash_{HCS} A \Leftrightarrow \vdash_{HC} \neg A$.

We can also prove the above meta-equivalence purely syntactically, for example with a similar strategy in [22], using a suitable tableau method. But we do not here carry it out because it is lengthy.

In [42], [33] and [24], other axiomatizations for satisfiable formulas of *CP* are proposed.

Theorem 4.3. *The Härtig-style satisfaction calculus HCS has the atomic formula property*.

Proof. Observe carefully the proof of $\Leftarrow$ of Theorem 4.2. □

We remark that my calculus, which is equivalent to *HSC*, with d-Hintikka formulas as axioms ([24]) (cf. [32])) also has the atomic formula property.

[12] The proof of Theorem 3 of Bullock and Schneider [6, p. 375] is also interesting.

5. *The refutation calculi and the satisfaction culculi.*

Let

$$F_u^{\forall} = \{A \in F^{\forall} : A \text{ is not provable in } CPL\},$$
$$F_u^{\exists} = \{A \in F^{\exists} : A \text{ is not provable in } CPL\},$$
$$F_u^{\forall\exists} = \{A \in F^{\forall\exists} : A \text{ is not provable in } CPL\},$$
$$F_s^{\forall} = \{A \in F^{\forall} : \neg A \text{ is not provable in } CPL\},$$
$$F_s^{\exists} = \{A \in F^{\exists} : \neg A \text{ is not provable in } CPL\}.$$ [13]

In this section, we shall propose the Härtig-style axiomatizations $HC^{\forall}$, $HC^{\exists}$, $HC^{\forall\exists}$, $HCS^{\forall}$ and $HCS^{\exists}$ for $F_u^{\forall}$, $F_u^{\exists}$, $F_s^{\forall}$ and $F_s^{\exists}$, respectively.

We shall first write down axioms common to all the system to be proposed.

Axioms:

$(A1)$ $\vdash F(x_1, ..., x_n)$ for any predicate letter $F(a_1, ..., a_n)$ with n attached variables and for any variables $x_1, ..., x_n$ $(n \geq 0)$.

$(A2)$ $\vdash \neg F(x_1, ..., x_n)$ for any predicate letter $F(a_1, ..., a_n)$ with n attached variables and for any variables $x_1, ..., x_n$ $(n \geq 0)$.

We need some definition for further formulations. First we shall define 'subformula' of a given formula of CPL exactly.

Definition 5.1. ([34]) We inductively define '*subformula*' of a given formula of CPL by the following: (1) If A is a formula, A is a subformula of A; (2) If A and B are formulas, the subformulas of A and the subformulas of B are subformulas of $A * B$ for every binary boolean connective $*$; (3) If A is a formula, the subformulas of A are subformulas of $\neg A$; (4) If x is a vari-

[13] For these sets, we assume that no dummy (or vacuous) quantifiers are prefixed for any formula. This restriction is not an essential one. But it makes our argument below simpler.

able, $A(x)$ is a formula and t is a term free for x in $A(x)$, the subformulas of $A(t)$ are subformulas of $\forall x A(x)$ and $\exists x A(x)$.[14]

We need the following notion stronger than that of subformula, which will play an important role in this paper.

Definition 5.2. We inductively define '*absolute subformula*' of a given formula of *CPL* by the following: (1) If A is a formula, A is an absolute subformula of A; (2) If A and B are formulas, the absolute subformulas of A and the absolute subformulas of B are absolute subformulas of $A * B$ for every binary boolean connective $*$; (3) If A is a formula, the absolute subformulas of A are absolute subformulas of $\neg A$; (4) If x is a variable, $A(x)$ is a formula, the absolute subformulas of $A(x)$ are absolute subformulas of $\forall x A(x)$ and $\exists x A(x)$.[15]

Definition 5.3. For any formula A of *CPL*, $\ll A \gg$ stands for the set of all atomic absolute subformulas of A. For any formula A of *CPL*, we shall call $\ll A \gg$ *the absolute atomic formula set of* A.

We shall give some examples ($Ex1$)-($Ex7$) for the notation $\ll\ \gg$ in order to avoid possible misunderstandings. Let F and G be distinct predicate letters. Let x, y and z be different variables and c, d distinct constants.

($Ex1$) Let A be $F(x, y)$. Then $\ll A \gg$ is $\{F(x, y)\}$.

($Ex2$) Let A be $\forall x \forall y F(x, y)$. Then $\ll A \gg$ is $\{F(x, y)\}$.

($Ex3$) Let A be $F(x, y, z) \vee F(y, z, x) \vee F(z, x, y)$.
Then $\ll A \gg$ is $\{F(x, y, z), F(y, z, x), F(z, x, y)\}$.

($Ex4$) Let A be $\forall x \forall y \forall z (F(x, y, z) \vee F(y, z, x) \vee F(z, x, y))$.
Then $\ll A \gg$ is $\{F(x, y, z), F(y, z, x), F(z, x, y)\}$.

[14] The reader must notice that there are several different definition of subformula in the literature. Some of them are not suitable to define the subformula property. This definition is good for it.

[15] The definition of an absolute subformula of a given formula of *CP* of course coincides with that of a subformula of a given formula of *CP*.

$(Ex5)$ Let A be $\forall x \forall y(F \vee G \vee F(x) . \supset . F(x, x, x) \wedge \neg F(y))$.
Then $\ll A \gg$ is $\{F, G, F(x), F(x, x, x), F(y)\}$.

$(Ex6)$ Let A be $\exists z(\forall x(\neg\neg G(x, c) \supset G(x, z)) \supset . \exists y G(y, c) \vee G(d, z))$.
Then $\ll A \gg$ is $\{G(x, c), G(x, z), G(y, c), G(d, z)\}$.

$(Ex7)$ Let A be $\forall x(\neg F(x) \wedge G(y, z) \wedge (F(y) \vee F(c, d))$
$\wedge \neg(G(d, x) \supset \neg F(y)))$.
Then $\ll A \gg$ is $\{F(x), G(y, z), F(y), F(c, d), G(d, x)\}$.

Here we remark that for some formula A of CPL, $\ll A \gg \, = \, < A >$ does not hold, whereas for any formula A of CP, $\ll A \gg \, = \, < A >$ holds. For example, for the A of $(Ex1)$, $\ll A \gg \, = \, < A >$ well holds, but for the A of $(Ex2)$, $\ll A \gg \, = \, < A >$ does not hold, since $< A >$ is

$$\{F(t, s) : t \text{ and } s \text{ are terms such that } t \text{ is free for } x \text{ in } \forall y F(x, y)\}.$$

However, the following proposition holds.

Proposition 5.1. *For any formula A of CPL, $\ll A \gg$ is a finite subset of $< A >$.*

Proof. The inclusion $\ll A \gg \subseteq < A >$ is obvious, since for any formula of the forms $\forall x A(x)$ or $\exists x A(x)$, $A(x)$ is a subformula of them. The finiteness of $\ll A \gg$ is obvious, too. □

Definition 5.4. A formula A of CPL is said to be *quantifier-free* if it contains no quantifiers (in notation, $Qf(A)$).

Let us now propose some common rules of inference. The following rules are common to $HC^{\forall}$ and $HC^{\exists}$.

$(R3) \vdash A, \vdash B, Qf(A), Qf(B), \ll A \gg \cap \ll B \gg = \varnothing \Rightarrow \vdash A \vee B.$[16]

$(R4) \vdash_{CP} A \supset B, \vdash B, Qf(A), Qf(B) \Rightarrow \vdash A.$[17]

[16] The criterion to determine whether or not $\ll A \gg \cap \ll B \gg = \varnothing$ holds is based on the absolute distinctness (recall Definition 2.1).

The following rules are common to $HCS^{\forall}$ and $HCS^{\exists}$.

$$(RS3)\ \vdash A, \vdash B, Qf(A), Qf(B), \ll A \gg \cap \ll B \gg = \varnothing \Rightarrow \vdash A \wedge B.$$

$$(RS4)\ \vdash A, \vdash_{CP} A \supset B, Qf(A), Qf(B) \Rightarrow \vdash B.$$

Here we need some preparations for further formulations of the systems.

Definition 5.5. Let A be a formula of CPL. By $FV(A)$ we denote the set of all free individual variables occurring in A. For any formula A of CPL, by $FV(A) = \{x_1, \ldots, x_n\}^d$ we mean $FV(A) = \{x_1, \ldots, x_n\}$ such that $x_1, \ldots, x_n$ are distinct. By $\#\, FV(A)$, we denote the number of distinct free individual variable occurring in A.

Definition 5.6. Let A be a quantifier-free formula of CPL such that $FV(A) = \{x_1, \ldots, x_n\}^d$ $(n \geq 1)$. Say $A = B(x_1, \ldots, x_n)$. Then for any individual variable z, we define a formula $U^z(A)$ of CPL as follows:

$$U^z(A) = B(z, \ldots, z).$$

Definition 5.7. Let A be a quantifier-free formula of CPL such that $FV(A) = \{x_1, \ldots, x_n, y_1, \ldots, y_m\}^d (n \geq 1, m \geq 1)$. Say $A = B(x_1, \ldots, x_n, y_1, \ldots, y_m)$. Then, for any finite sequence $z_1, \ldots, z_m$ of individual variables with length m and $\{z_1, \ldots, z_m\} \subseteq \{x_1, \ldots, x_n\}$, we define a formula $Spec[y_1 \to z_1, \ldots, y_m \to z_m](A)$ (for short, $Spec[\overline{y \to z}](A)$) of CPL as follows:

$$Spec[\overline{y \to z}](A) = B(x_1, \ldots, x_n, z_1, \ldots, z_m).$$

We are now in a position to give the rules specific to each of the systems.

[17] In $(R4)$, we use "$\vdash_{CP} A \supset B$" by abuse of notation. This properly means that $A \supset B$ is provable in the propostional part of classical predicate logic. We shall often make use of such an abuse of notation for the simplicity of descriptions.

We need the following rule for $HC^{\forall}$ and $HCS^{\forall}$.

$(R^{\forall}) \vdash_{HC^{\forall}} A(x), \# FV(A(x)) \geq 1 \Rightarrow \vdash_{HC^{\forall}} \forall x A(x)$, provided that $Qf(A(x))$ holds or $A(x)$ is a prenex formula with prefix $\forall \ldots \forall$.

We need the following rule for $HC^{\exists}$.

$(R^{\exists}) \vdash_{HC^{\exists}} A, Qf(A), \# FV(A) \geq 1 \Rightarrow \vdash_{HC^{\exists}} Cl^{\exists}(A)$, provided $\vdash_{HC^{\exists}} U^{z}(A)$ for some variable z, where $Cl^{\exists}(A)$ is the existential closure of A (that is, if $FV(A) = \{x_1, \ldots, x_n\}^d$ $(n \geq 1)$, then $Cl^{\exists}(A) = \exists x_1 \ldots \exists x_n A$).

We need the following rule for $HC^{\forall\exists}$.

$(R^{\forall\exists}) \vdash_{HC^{\forall\exists}} A, Qf(A), \# FV(A) \geq 2 \Rightarrow \vdash_{HC^{\forall\exists}} \forall x_1 \ldots \forall x_n \exists y_1 \ldots y_m A$, provided $\vdash_{HC^{\forall\exists}} Spec[y_1 \to z_1, \ldots, y_m \to z_m](A)$ for some finite sequence $z_1, \ldots, z_m$ of variables with length m and $\{z_1, \ldots, z_m\} \subseteq \{x_1, \ldots, x_n\}$, where $FV(A)$ = $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}^d (n \geq 1, m \geq 1)$.

We need the following rule for $HCS^{\forall}$.

$(RS^{\forall}1) \vdash_{HCS^{\forall}} A, Qf(A), \# FV(A) \geq 1 \Rightarrow \vdash_{HCS^{\exists}} Cl^{\forall}(A)$, provided $\vdash_{HCS^{\forall}} U^{z}$ (A) for some variable z.

$(RS^{\forall}2) \vdash_{HCS^{\forall}} A, Qf(A), \# FV(A) \geq 2 \Rightarrow \vdash_{HCS^{\forall}} \forall x_1 \ldots \forall x_n A$, provided $\vdash_{HCS^{\forall\exists}} Spec[y_1 \to z_1, \ldots, y_m \to z_m](A)$ for some sequence $z_1, \ldots, z_m$ of variables with length m and $\{z_1, \ldots, z_m\} \subseteq \{x_1, \ldots, x_n\}$, where $FV(A)$ = $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}^d (n \geq 1, m \geq 1)$.

We need the following rule for $HCS^{\exists}$.

$(RS^{\exists}) \vdash_{HCS^{\exists}} A, Qf(A), \# FV(A) \geq 1 \Rightarrow \vdash_{HCS^{\exists}} Cl^{\exists}(A)$.

To make sure, we shall repeat the formulation of the above defined systems.

A refutation calculus $HC^{\forall}$ for $F_u^{\forall}$ consists of $(A1)$, $(A2)$, $(R3)$, $(R4)$ and $(R^{\forall})$.

A refutation calculus $HC^{\exists}$ for $F_u^{\exists}$ consists of $(A1)$, $(A2)$, $(R3)$, $(R4)$ and $(R^{\exists})$.

A refutation calculus $HC^{\forall\exists}$ for $F_u^{\forall\exists}$ consists of $(A1)$, $(A2)$, $(R3)$, $(R4)$ and $(R^{\forall\exists})$.

A satisfaction calculus $HCS^{\forall}$ for $F_s^{\forall}$ consists of $(A1)$, $(A2)$, $(RS3)$, $(RS4)$, $(RS^{\forall}1)$ and $(RS^{\forall}2)$.

A satisfaction calculus $HCS^{\exists}$ for $F_s^{\exists}$ consists of $(A1)$, $(A2)$, $(RS3)$, $(RS4)$ and $(RS^{\exists})$.

It is remarked that a *proof* of a formula in the systems $HC^{\exists}$, $HC^{\forall\exists}$, $HCS^{\exists}$ should contain exactly one application of the rules $(R^{\exists})$, $(R^{\forall\exists})$, $(RS^{\exists})$, respectively. For a proof of a formula in $HC^{\forall}$ and $HCS^{\forall}$, we shall understand it as usual, respectively.

We note that every proof in them, thus, has the quantifier-free part and the quantificational part as a proof in "extended Hauptsatz" of Gentzen ([13]) (cf. [34, p. 460]).

Here, we shall explain with an example why the rule $(R^{\exists})$ requires the condition "$\vdash_{HC^{\exists}} U^z(A)$ for some variable z", although it is clear from the decision procedure (II) in the second section.

Suppose that the condition is dropped in the rule $(R^{\exists})$. Let F be a predicate letter. Let x and y be distinct variables. So, $F(x)$ and $F(y)$ are absolutely different. Thus we can easily conclude $\vdash_{HC^{\exists}} F(x) \vee \neg F(y)$. So from it, we can get by the new $(R^{\exists})$, $\vdash_{HC^{\exists}} \exists x \exists y (F(x) \vee \neg F(y))$.

However, this inference is not correct with respect to $HC^{\exists}$, since we have $\vdash_{CPL} \exists x \exists y (F(x) \vee \neg F(y))$: that is, $\exists x \exists y (F(x) \vee \neg F(y))$ is reduced to $F(z) \vee \neg F(z)$ for any variable z. This is not our intention for $HC^{\exists}$. That is why the condition should be added. With the same reason, $(R^{\forall\exists})$ requires the similar condition (for example, consider $\forall x \exists y (F(x) \vee \neg F(y))$).

In addition, for the propositional part of the proposed systems $HC^{\forall}$, $HC^{\exists}$, $HC^{\forall\exists}$, $HCS^{\forall}$ and $HCS^{\exists}$ in this paper, we may also *mutatis mutandis* take another type of refutation (satisfaction) calculus, e.g. a system with Hintikka formulas as axioms in [22, 24] (cf. [32]).

6. *The characterization theorem.*

In this section, we shall prove the following characterization theorem.

Theorem 6.1.

(1) *For any formula A of* $F^{\forall}$, *we have*: $\vdash_{HC^{\forall}} A \iff \dashv_{CPL} A$.

(2) *For any formula A of* $F^{\exists}$, *we have*: $\vdash_{HC^{\exists}} A \iff \dashv_{CPL} A$.

(3) *For any formula A of* $F^{\forall\exists}$, *we have*: $\vdash_{HC^{\forall\exists}} A \Leftrightarrow \dashv_{CPL} A$.
(4) *For any formula A of* $F^{\forall}$, *we have*: $\vdash_{HCS^{\forall}} A \Leftrightarrow \dashv_{CPL} \neg A$.
(5) *For any formula A of* $F^{\exists}$, *we have*: $\vdash_{HCS^{\exists}} A \Leftrightarrow \dashv_{CPL} \neg A$.

Just before we prove it, we shall prove the following simple lemma.

Lemma 6.1. *Let A be a quantifier-free formula of CPL such that FV(A) =* $\{x_1, \dots, x_n, y_1, \dots, y_m\}^d$ $(n \geq 0, m \geq 1)$ *holds, say* $A = A(x_1, \dots, x_n, y_1, \dots, y_m)$. *For any sequence* $z_1, \dots, z_m$ *of variables with length m, we then have*:

$$\dashv_{CPL} A(x_1, \dots, x_n, z_1, \dots, z_m) \Rightarrow \dashv_{CPL} A(x_1, \dots, x_n, y_1, \dots, y_m).$$

Proof. Let A be such a formula of CPL. We write $A(\bar{x}, y_1, \dots, y_m)$ for $A(x_1, \dots, x_n, y_1, \dots, y_m)$ and so on. It is sufficient to show that

$$\vdash_{CPL} A(\bar{x}, y_1, \dots, y_m) \Rightarrow \vdash_{CPL} A(\bar{x}, z_1, \dots, z_m).$$

Suppose $\vdash_{CPL} A(\bar{x}, y_1, \dots, y_m)$. Since $x_1, \dots, x_n, y_1, \dots, y_m$ are all distinct free variables occurring in A, from $\vdash_{CPL} A(\bar{x}, \bar{y})$, we immediately obtain

$$\vdash_{CPL} \forall y_1 \dots \forall y_m A(\bar{x}, y_1, \dots, y_m). \quad (*)$$

Since

$$\forall y_1 \dots \forall y_m A(\bar{x}, \bar{y}) \supset A(\bar{x}, z_1, \dots, z_m)$$

is a theorem of CPL, we get $\vdash_{CPL} A(\bar{x}, z_1, \dots, z_m)$ by modus ponens from the formula and (*). □

Proof of Theorem 6.1. Below we shall freely (with making no mention of it) make use of the completeness theorem for the logics in question, if we need it for the sake of the simplicity of arguments. All the decision procedures to be cited below are those in the second section.

(i) Let $A \in F^{\forall}$, say $A = \forall x_1 \dots \forall x_n B(x_1, \dots, x_n)$ $(n \geq 0)$.

The $\Rightarrow$ of (1): Suppose $\vdash_{HC^\forall} A$. We shall prove this by induction on derivations.[18] The basis of the induction is trivial. Let us proceed to the induction steps. Theorem 3.1 can take care of the propositional part of the induction steps. (That is, $B(x_1, \ldots, x_n)$ is not provable in CP, thus not so in CPL, either.) If $n = 0$, then we need to do nothing further. So suppose $n \geq 1$. Suppose that $\forall x_n B(x_1, \ldots, x_n)$ is obtained by the rule $(R^\forall)$. By induction hypothesis, we have $\dashv_{CPL} B(x_1, \ldots, x_n)$. From it by CPL, we obtain $\dashv_{CPL} A$.

The $\Leftarrow$ of (1): Suppose $\dashv_{CPL} A$. By the decision procedure (I), $B(x_1, \ldots, x_n)$ is not provable in CP. By Theorem 3.1, we get $\vdash_{HC^\forall} B(x_1, \ldots, x_n)$. Applying $(R^\forall)$ n times to the formula, we obtain $\vdash_{HC^\forall} A$.

(ii) Let $A \in F^\exists$, say $A = \exists x_1 \ldots \exists x_n B(x_1, \ldots, x_n)$ $(n \geq 1)$.

The $\Rightarrow$ of (2): Suppose $\vdash_{HC^\exists} A$. We shall prove this by induction on derivations. By Theorem 3.1, the propositional part of the induction gives no difficulty.

Suppose that A is obtained by the rule $(R^\exists)$. By induction hypothesis, we then have $\dashv_{CPL} B(x_1, \ldots, x_n)$ and $\dashv_{CPL} U^z(B(x_1, \ldots, x_n))$ for some variable z, that is, $\dashv_{CPL} B(z, \ldots, z)$. Hence, by the decision procedure (II), we obtain $\dashv_{CPL} \exists x_1 \ldots \exists x_n B(x_1, \ldots, x_n)$.

The $\Leftarrow$ of (2): Suppose $\dashv_{CPL} A$. By the decision procedure (II), we know that there are no variable y such that $\vdash_{CPL} B(y, \ldots, y)$ holds. Thus, $\dashv_{CPL} B(z, \ldots, z)$ hold for an arbitrarily chosen variable z. By Lemma 6.1, we get $\dashv_{CPL} B(x_1, \ldots, x_n)$ from $\dashv_{CPL} B(z, \ldots, z)$. By Theorem 3.1, we have from them, $\vdash_{HC^\exists} B(x_1, \ldots, x_n)$ and $\vdash_{HC^\exists} U^z(B(x_1, \ldots, x_n))$. Then we can apply $(R^\exists)$ to it to obtain $\vdash_{HC^\exists} \exists x_1 \ldots \exists x_n B(x_1, \ldots, x_n)$.

(iii) Let $A \in F^{\forall\exists}$, say $A = \forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m B(x_1, \ldots, x_n, y_1, \ldots, y_m)$ $(n \geq 1$ and $m \geq 1)$,

such that $x_1, \ldots, x_n, y_1, \ldots, y_m$ are all distinct free variables occurring in the quantifier-free part of A. We shall write $B(\overline{x}, y_1, \ldots, y_m)$ or $B(\overline{x}, \overline{y})$, for $B(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and so on.

[18] Below we shall often take a similar proof with induction on derivations. But we shall omit the details of it, since it is routine as this proof of the $\Rightarrow$ of (1).

The $\Rightarrow$ of (3): The proof of it is similar of that of the $\Rightarrow$ of (2), using the decision procedure (III) and Lemma 6.1.

The $\Leftarrow$ of (3): Suppose $\dashv_{CPL} A$. By the decision procedure (III), there is no finite sequence $z_1, \ldots, z_m$ of varaibles with length m and $\{z_1, \ldots, z_m\} \subseteq \{x_1, \ldots, x_n\}$ such that $\dashv_{CPL} B(\bar{x}, z_1, \ldots, z_m)$ holds. Take such a sequence $z_1, \ldots, z_m$. Then we have

$$\dashv_{CPL} B(\bar{x}, z_1, \ldots, z_m). \quad (+)$$

By Lemma 6.1, we obatin from (+),

$$\dashv_{CPL} B(\bar{x}, y_1, \ldots, y_m). \quad (++)$$

From (+) and (++), we get by Theorem 3.1,

$$\vdash_{HC^{\forall\exists}} B(\bar{x}, y_1, \ldots, y_m) \text{ and } \vdash_{HC^{\forall\exists}} B(\bar{x}, z_1, \ldots, z_m),$$

since $B(\bar{x}, \bar{y})$, and $B(\bar{x}, \bar{z})$ are quantifier-free. By applying the rule $(R^{\forall\exists})$ to them, we obatin $\vdash_{HC^{\forall\exists}} A$.

(iv) Let $A \in F^{\forall}$, say $A = \forall x_1 \ldots \forall x_n B(x_1, \ldots, x_n, y_1, \ldots, y_m)$ $(n \geq 0$ and $m \geq 0)$,

such that $x_1, \ldots, x_n, y_1, \ldots, y_m$ are all distinct free variables occurring in the quantifier-free part of A. We shall again write $B(\bar{x}, y_1, \ldots, y_m)$ or $B(\bar{x}, \bar{y})$ for $B(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and so on.

The $\Rightarrow$ of (4): Suppose $\vdash_{HCS^{\forall}} A$. We shall prove this by induction on derivations. By Corollary 4.1, we can easily deal with the propositional part of the induction. If $n = 0$, then we do not need to prove further. So assume $n \geq 1$. First we shall treat the case of $m = 0$. In this case, $\vdash_{HCS^{\forall}} A$ is obtained by the rule $(RS^{\forall}1)$. Then, by Corollary 4.1, we have $\dashv_{CPL} U^{z}(\neg B(\bar{x}))$ for some variable z. By Lemma 6.1, we get $\dashv_{CPL} U^{w}(\neg B(\bar{x}))$ for any variable w. From this we have $\dashv_{CPL} Cl^{\exists}(\neg B(\bar{x}))$ by the decision procedure (II). This is nothing but $\dashv_{CPL} \neg A$. Now we shall deal with the case of $m \geq 1$. In this case, $\vdash_{HCS^{\forall}} A$ is obtained by the rule $(RS^{\forall}2)$. Then, by Corollary 4.1, we get

$$\dashv_{CPL} Spec[x_1 \to z_1, \ldots, y_n \to z_n]\ (\neg B(x_1, \ldots, x_n, y_1, \ldots, y_m))$$

for some finite sequence $z_1, \ldots, z_n$ of variables with length n and $\{z_1, \ldots, z_n\} \subseteq \{y_1, \ldots, y_n\}$. From it, by Lemma 6.1, we have

$$\dashv_{CPL} \neg B(w_1, \ldots, w_n, y_1, \ldots, y_m)$$

for any finite sequence $w_1, \ldots, w_n$ of variables with length n and $\{w_1, \ldots, w_n\} \subseteq \{y_1, \ldots, y_m\}$. Then, it follows by the decision procedure (III) that

$$\dashv_{CPL} \forall y_1 \ldots \forall y_m \exists x_1 \ldots \exists x_n \neg B\left(x_1, \ldots, x_n, y_1, \ldots, y_m\right)$$

holds. Since $x_1, \ldots, x_n, y_1, \ldots, y_m$ are all distinct free variables occurring in $B(\bar{x}, \bar{y})$, by CPL, we immediately obtain $\dashv_{CPL} \exists x_1 \ldots \exists x_n \neg B(\bar{x}, \bar{y})$, from which $\dashv_{CPL} \neg A$ follows.

The $\Leftarrow$ of (4): Suppose $\dashv_{CPL} \neg A$. If $n = 0$, then we can directly apply Corollary 4.1 to it. So we shall assume $n \geq 1$. Suppose $m = 0$. Then, by CPL,

$$\dashv_{CPL} \exists x_1 \ldots \exists x_n \neg B\left(x_1, \ldots, x_n\right)$$

holds. By the decision procedure (II), we get $\dashv_{CPL} U^{z}(\neg B(\bar{x}))$ for any variable z. Take a variable y and fix it. So we have

$$\dashv_{CPL} U^{y}(\neg B(\bar{x})). \quad (\rhd)$$

from the above. By Lemma 6.1, from $(\rhd)$,

$$\dashv_{CPL} \neg B(x_1, \ldots, x_n). \quad (\rhd \rhd)$$

So from ($\rhd$) and ($\rhd$ $\rhd$), it follows by Corollary 4.1, that $U^y(B(\bar{x}))$ and $B(\bar{x})$ are theorems of $HCS^\forall$. Then we can apply $(RS^\forall 1)$ to them to get $\vdash_{HCS^\forall} A$. Next we shall suppose $m \geq 1$. Then by *CPL*, we have

$$\dashv_{CPL} \forall y_1 \ldots \forall y_m \exists x_1 \ldots \exists x_n \neg B\ (x_1, \ldots, x_n, y_1, \ldots, y_m).$$

By the decision procedure (III), we get

$$\dashv_{CPL} \neg B\ (z_1, \ldots, z_n, y_1, \ldots, y_m)$$

for any finite sequence $z_1, \ldots, z_n$ of variables with length n and $\{z_1, \ldots, z_n\} \subseteq \{y_1, \ldots, y_m\}$. Then, from this it follows by Corollary 4.1 and Lemma 6.1 that

$$\vdash_{HCS^\forall} B\,(x_1, \ldots, x_n, y_1, \ldots, y_m) \text{ and } \vdash_{HCS^\forall} B\ (w_1, \ldots, w_n, y_1, \ldots, y_m)$$

for some finite sequence $w_1, \ldots, w_n$ of variables with length n and $\{w_1, \ldots, w_n\} \subseteq \{y_1, \ldots, y_m\}$. Then apply ($RS^\forall 2$) to them.

(v) Let $A \in F^\exists$, say $A = \exists x_1 \ldots \exists x_n B(x_1, \ldots, x_n)\,(n \geq 1)$.

The $\Rightarrow$ of (5): Suppose $\vdash_{HCS^\exists} A$. We shall prove this by induction on derivations. As above, the propositional part of the induction is easily treated. Suppose that A is obtained by the rule $(RS^\exists)$. By induction hypothesis, $\neg B(x_1, \ldots, x_n)$ is not provable in *CPL*. By *CPL*, we have $\dashv_{CPL} \forall x_1 \ldots \forall x_n \neg B(x_1, \ldots, x_n)$, which is logically equivalent to $\neg \exists x_1 \ldots \exists x_n B(x_1, \ldots, x_n)$ in *CPL*. So $\dashv_{CPL} \neg A$ holds.

The $\Leftarrow$ of (5): Suppose $\dashv_{CPL} \neg A$. By *CPL*, $\dashv_{CPL} \forall x_1 \ldots \forall x_n \neg B(x_1, \ldots, x_n)$ holds. By the decision procedure (I), we know that $\neg B(x_1, \ldots, x_n)$ is not provable in CP. So we have $\vdash_{HCS^{\exists}} B(x_1, \dots x_n)$ by Corollary 4.1. Then, applying $(RS^{\exists})$ to it, we obtain the desired $\vdash_{HCS^{\exists}} A$. □

7. *The absolute atomic formula property of the systems.*

We shall propose the notion of *the absolute atomic formula property* (of CPL), which is similar to that of the atomic formula property introduced in [25].[19] In this section, we shall see that the proposed systems have the absolute atomic formula property as well as the atomic formula property.

Definition 7.1. Let X be a formal system. The system X has *the absolute atomic formula property* (for short, abs-a.f.p.), if for any formula A provable in X, there is a proof of A in X which contains only such formulas, say $B_1, B_2, \dots, B_n$ that $\ll B_i \gg \subseteq \ll A \gg$ holds for any $1 \leq i \leq n$ (we say that such a proof has the absolute atomic formula property).

So from the above definition and the fact that for any formula A of CP, $\ll A \gg = < A >$ holds, we immediately have:

Corollary 7.1. *The Härtig-style calculi HC and HCS have the absolute atomic formula property.*

Proof. By Theorems 3.2 and 4.3. □

Our observations in this section are the following theorems.

Theorem 7.1. *The systems* $HC^{\forall}$, $HC^{\exists}$, $HC^{\forall\exists}$, $HCS^{\forall}$ *and* $HCS^{\exists}$ *have the absolute atomic formula property.*[20]

Proof. It would be enough to give only an outline of the proof. Suppose that we are given a proof π of A in the system in question. The proof π consists of a propositional π_p and a predicate-part π_q, say $\pi_p = A_1, \dots, A_n (n \geq 1)$ and $\pi_q = B_1, \dots, B_m (m \geq 0)$. (Note that the whole

[19] In [28], a similar notion, i.e. the name variable property (for short, n.v.p.) is introduced.

[20] In a strict sense, $HC^{\exists}$, $HC^{\forall\exists}$ and $HCS^{\forall}$ do not have abs-a.f.p. since $(R^{\exists})$, $(R^{\forall\exists})$, $(RS^{\forall}1)$ and $(RS^{\forall}2)$ have the special conditions, respectively, whereas they have a.f.p. even in the strict sense. It depends on the definition of a proof whether or not they have abs-a.f.p..

proof π is thus of the form $A_1, \ldots, A_n, B_1, \ldots, B_m$ with $A = B_m$ if $m \geq 1$, otherwise $A = A_n$.)

In view of Corollary 7.1., we can obtain a new proof $\tilde{\pi}_p$ of A_n. with the absolute atomic formula property. If $m = 0$, the proof $\tilde{\pi}_p$ is a desired one of A with abs-a.f.p.. Suppose $m \geq 1$. Then we easily see that $\tilde{\pi}_p, B_1, \ldots, B_m$ is a proof of A in the system in question with the absolute atomic formula property. □

Theorem 7.2. *The systems* $HC^{\forall}$, $HC^{\exists}$, $HC^{\forall\exists}$, $HCS^{\forall}$ *and* $HCS^{\exists}$ *have the atomic formula property.*

Proof. The proof of Theorem 7.1 can also be *mutatis mutandis* used in this case. □

We have just seen that the proposed systems have abs-a.f.p. as well as a.f.p.. In the Appendix B, the reader will find such systems that they have a.f.p. but not abs-a.f.p.. We also have to remark that the following statement does not hold: for any formulas A and B of CPL, $\ll A \gg \subseteq \ll B \gg \Rightarrow < A > \subseteq < B >$.

A counterexample of it is for example that $A = \forall x \forall y F(x, y)$ and $B = \forall y \forall x F(x, y)$, where $F(x, y)$ is a predicate letter of CPL with 2 attached variables. That is, we easily see that $\ll A \gg = \ll B \gg = \{F(x, y)\}$, $F(z, x) \in < A >$ and $F(z, x) \notin < B >$ where z is a variable with $z \neq y$.

So we cannot, in a straightforward way, get such a meta-implication as "X has abs-a.f.p" $\Rightarrow$ "X has a.f.p", where X is a quantificational system. Thus, we shall give the following open problem.

Open problem. Is there a quantificational system that has abs-a.f.p. but not a.f.p. ?

For the purpose of comparison, we shall define the notion of "has the subformula property" as follows.

Definition 7.2. Let X be a formal system. The system X has *the subformula property* (for short, s.f.p.), if for any formula A provable in X, there is a proof of A in X such that every formula occurring in the proof is a subformula of A (we say that such a proof has the subformula property).[21]

Here we shall show the relation among s.f.p., a.f.p. and abs-a.f.p. as follows.

[21] This definition is good for proofs both in a Hilbert-style system and in a Gentzen-style one.

Theorem 7.3. *Let X be a formal system such that Definitions* 5.1 *and* 5.2 *are meaningful in X. Then we have*:

(1) *If Definition 3.3 is meaningful in X and if X has s.f.p., then X has a.f.p..*
(2) *If Definition 7.1 is meaningful in X, then "if X has s.f.p., then X has abs-a.f.p." does not hold.*
(3) *If Definition 7.2 is meaningful in X, then "if X has a.f.p., then X has s.f.p." does not hold.*
(4) *If Definition 7.1 is meaningful in X, then "if X has a.f.p., then X has abs-a.f.p." does not hold.*
(5) *If Definition 7.2 is meaningful in X, then "if X has abs-a.f.p., then X has s.f.p." does not hold.*
(6) *If X is a propositional system and if Definitions 3.3 and 7.1 are meaningful in X, then we have: X has abs-a.f.p. if and only if X has a.f.p..*

Proof. The statements (1) and (6) are obvious from the definitions.[22]

A counterexample of "a.f.p. $\Rightarrow$ s.f.p." is the Härtig's refutation calculus *HC*, since the rule (*HC*4) of *HC* causes a problem for s.f.p..[23] We shall give an example for that below.

Let p be a proposition letter. Then, $\vdash_{HC} \neg\neg p$ holds obviously. For example, a finite sequence $p, \neg\neg p \supset p, \neg\neg p$ is a proof of $\neg\neg p$ in *HC* with a.f.p.. We can say that there is no proof of $\neg\neg p$ in *HC* with s.f.p.. All different subformulas of $\neg\neg p$ are p, $\neg p$ and $\neg\neg p$. Suppose that a proof of $\neg\neg p$ in *HC* with s.f.p., say $A_1, \ldots, A_n (n \geq 1)$ with $A_n = \neg\neg p$. Because of the form of $\neg\neg p$, there are indices $1 \leq i < n$ and $1 \leq j < n$ with $i \neq j$ such that A_n is an immediate consequence of A_i and A_j by (*HC*4) with $A_i = A_n \supset A_j$ or $A_j = A_n \supset A_i$. Such A_i or A_j are not subformulas of $\neg\neg p$. So we have a contradiction. Thus we have (3).

A counterexample of "abs-a.f.p. $\Rightarrow$ s.f.p." is, for example, the refutation calculus $HC^{\forall}$, since the rule (R4) of $HC^{\forall}$ similarly causes a problem for s.f.p..[24] So we may conclude (5).

[22] In order to conclude the same statement for formal systems beyond first-order logic, and for e.g. Gentzen-style systems, we have to extend and modify the definition of subformula and absolute subformula, and that of a.f.p. and abs-a.f.p.. But it will easily and positively be done.

[23] The satisfaction caluclus *HCS* does not have s.f.p. because of the rule (*HCS*4).

[24] The systems $HC^{\exists}$, $HC^{\forall\exists}$, $HCS^{\forall}$ and $HCS^{\exists}$ do not have s.f.p. with a similar reason.

A counterexample of "a.f.p. ⇒ abs-a.f.p." is the Bullock and Schneider's *BS* (see the Appendix C of this paper). Thus, (4) holds.

A counterexample of "s.f.p. ⇒ abs-a.f.p." is LK^{-}_{ω}, one of ω-logics (for LK^{-}_{ω}, see [14, p. 349 and p. 362]). The cut elimination theorem holds for it (see [14, p. 364] or [40, Chapter VIII, pp. 203-204, pp. 207-209]) so that it has s.f.p., and if we look at the quantificational rules for the logic, it is obvious to see that it does not have abs-a.f.p.. Hence, we have (2).[25] □

Hailperin [16] gave an axiomatization of formulas of *CPL* which are invalid in some finite domain. The quantifier-free part of the system is that "if A is quantifier-free and non-tautologous, then $\vdash A$" (see [16, p. 90]). From the syntactical viewpoint of the present paper, I am not satisfied with the quantifier-free part of Hailperin's formulation. For such a rule can tell nothing about syntax and such a property as the atomic formula one.

8. *On provable formulas and contradictions of* $F^{\forall}$, $F^{\exists}$ *and* $F^{\forall\exists}$.

With the same idea as the above, axiomatizations of provable formulas of $F^{\forall}$, $F^{\exists}$ and $F^{\forall\exists}$ can be taken care of by the decision procedures (I), (II) and (III), respectively. By $P^{\forall}$, $P^{\exists}$ and $P^{\forall\exists}$, we denote the systems for $F^{\forall}$, $F^{\exists}$ and $F^{\forall\exists}$, respectively.

For the common propositional part of $F^{\forall}$, $F^{\exists}$ and $F^{\forall\exists}$, we may *mutatis mutandis* adopt an arbitrary, well-known calculus[26] for *CP*. However, we shall choose a cut-free Gentzen-style system LK^{pr} for it, since it has s.f.p.. So we shall also formulate their quantificational rules in a Gentzen-style so that the whole systems $F^{\forall}$, $F^{\exists}$ and $F^{\forall\exists}$ are Gentzen-style ones. The propositional system LK^{pr} consists of the following axioms and rules (for simplicity, we shall take $\vee$ and $\neg$ as primitive):

Axioms:

(*GPA*)

$$\Gamma, A \to \Theta, A.$$

Rules:

[25] We assume that for this argument for (2), Definitions 5.1, 5.2, 7.1 and 7.2 have been extended and modified.

[26] The applications of the rules of it should be restricted to formulas of *CPL* which contain neither function signs nor constants.

$(GP \to \vee)$

$$\frac{\Gamma \to \Theta, A, B}{\Gamma \to \Theta, A \vee B.}$$

$(GP \vee \to)$

$$\frac{\Gamma, A \to \Theta \quad \Gamma, B \to \Theta}{\Gamma, A \vee B \to \Theta.}$$

$(GP \to \neg)$

$$\frac{\Gamma, A \to \Theta}{\Gamma \to \Theta, \neg A.}$$

$(GP \neg \to)$ For any formula A with $Qf(A)$,

$$\frac{\Gamma \to \Theta, A}{\Gamma, \neg A \to \Theta.}$$

(All formulas occurring in every axiom of LK^{pr} should contain neither quantifiers, nor function signs, nor constants. We note that the above Γ and Θ are finite sets of formulas (they may be empty). We also note that for example, a sequent $\Gamma \to \Theta, A, B$ is understood as the abriviation of $\Gamma \to \Theta \cup \{A, B\}$.)

Theorem 8.1. (Gentzen) *For any formula A of CPL with* $Qf(A)$ *such that it contains neither function signs nor constants, we have*:

$$\vdash_{LK^{pr}} A \Leftrightarrow \vdash_{CPL} A.$$

Let us now specify quantificational rules for the systems.

We need the following rule for $P^{\forall}$.

$\left(GP^{\forall}\right)$ For any formula $A(x)$ with $\# FV(A(x)) \geq 1$,

$$\frac{\to A(x)}{\to \forall x A(x),}$$

provided that $Qf(A(x))$ holds or $A(x)$ is a prenex formula with prefix $\forall \ldots \forall$.

We need the following rule for $P^{\exists}$.

$(GP^{\exists})$ For any formula A with $Qf(A)$ and $FV(A)=\{x_1, \ldots, x_n\}^d$ $(n \geq 1)$, if $\rightarrow U^z(A)$ for some variable z, then $\rightarrow \exists x_1 \ldots \exists x_n A$.

We need the following rule for $P^{\forall\exists}$.

$(GP^{\forall\exists})$ For any formula A with $Qf(A)$ and $FV(A)=\{x_1, \ldots, x_n, y_1, \ldots, y_m\}^d$ $(n \geq 1, m \geq 1)$, if $\rightarrow Spec[\overline{y \rightarrow z}](A)$ for some sequence $z_1, \ldots, z_m$ of variables with length m and $\{z_1, \ldots, z_m\} \subseteq \{x_1, \ldots, x_n\}$, then $\rightarrow \forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m A$.

Next we shall axiomatize the set of all the contradictions of $F^{\forall}$ and $F^{\exists}$, respectively. Again the decision procedures will play an essential role for that. By $C^{\forall}$ and $C^{\exists}$, we denote the axiomatic systems for $F^{\forall}$ and $F^{\exists}$, respectively.

For the common propositional part of $C^{\forall}$ and $C^{\exists}$, we shall choose a cut-free Gentzen-style system OP^{pr} (see §4 of this paper and [23]) for the opposite system for CP, since it has s.f.p.. So we shall again formulate $C^{\forall}$ and $C^{\exists}$ as Gentzen-style systems. The propositional system OP^{pr} consists of the following axioms and rules (for simplicity, we shall again take $\vee$ and $\neg$ as primitive):[27]

Axioms:

(GCA)

$$\Gamma, A \rightarrow \Theta, A.$$

Rules:

$(GC \rightarrow \vee)$ For any formulas A and B with $Qf(A)$ and $Qf(B)$,

$$\frac{\Gamma \rightarrow \Theta, A \quad \Gamma \rightarrow \Theta, B}{\Gamma \rightarrow \Theta, A \vee B.}$$

27 The calculus OP^{pr} is a version of OP in [23].

$(GC \vee \rightarrow)$

$$\frac{\Gamma, A, B \rightarrow \Theta}{\Gamma, A \vee B \rightarrow \Theta.}$$

$(GC \rightarrow \neg)$

$$\frac{\Gamma, A \rightarrow \Theta}{\Gamma \rightarrow \Theta, \neg A.}$$

$(GC \neg \rightarrow)$ For any formula A with $Qf(A)$,

$$\frac{\Gamma \rightarrow \Theta, A}{\Gamma, \neg A \rightarrow \Theta.}$$

(All formulas occurring in every axiom of OP^{pr} should contain neither quantifiers, nor function signs, nor constants.)

We note the following.

Theorem 8.2. ([23]) *For any formula A of CPL with Qf(A) such that it contains neither function signs nor constants, we have*:

$$\vdash_{OP^{pr}} A \Leftrightarrow \dashv_{CPL} A.$$

(For OP^{pr}, we have the same constraints mentioned above for LK^{pr}.)

We shall specify quantificational rules for $C^{\forall}$ and $C^{\exists}$.

We need the following rule for $C^{\forall}$.

$(GC^{\forall}1)$ For any formula A with $Qf(A)$ and $FV(A) = \{x_1, \ldots, x_n\}^d$ $(n \geq 1)$, if $\rightarrow U^z(A)$ for some variable z, then $\rightarrow \forall x_1 \ldots \forall x_n A$.

$(GC^{\forall}2)$ For any formula A with $Qf(A)$ and $FV(A) = \{x_1, \ldots, x_n, y_1, \ldots, y_m\}^d$ $(n \geq 1, m \geq 1)$, if $\rightarrow Spec[\overline{x \rightarrow z}](A)$ for some sequence $z_1, \ldots, z_n$ of variables with length n and $\{z_1, \ldots, z_n\} \subseteq \{y_1, \ldots, y_m\}$, then $\rightarrow \forall x_1 \ldots \forall x_n A$.

We need the following rule for $C^{\exists}$.

$(GC^{\exists})$ For any formula A with $Qf(A)$ and $FV(A) = \{x_1, \ldots, x_n\}^d$ $(n \geq 1)$,

$$\frac{\to A}{\to \exists x_1 \ldots \exists x_n A.}$$

Summing up, we shall repeat the formulation of the above defined systems.

A calculus $P^{\forall}$ consists of $LK^{pr} + (GP^{\forall})$.
A calculus $P^{\exists}$ consists of $LK^{pr} + (GP^{\exists})$.
A calculus $P^{\forall\exists}$ consists of $LK^{pr} + (GP^{\forall\exists})$.
A calculus $C^{\forall}$ consists of $OP^{pr} + (GC^{\forall}1) + (GC^{\forall}2)$.
A calculus $C^{\exists}$ consists of $OP^{pr} + (GC^{\exists})$.

We shall prove the characterization theorem for them as in the sixth section.

Theorem 8.3.

(1) *For any formula A of* $F^{\forall}$, *we have*: $\vdash_{P^{\forall}} A \Leftrightarrow \vdash_{CPL} A.$
(2) *For any formula A of* $F^{\exists}$, *we have*: $\vdash_{P^{\exists}} A \Leftrightarrow \vdash_{CPL} A.$
(3) *For any formula A of* $F^{\forall\exists}$, *we have*: $\vdash_{P^{\forall\exists}} A \Leftrightarrow \vdash_{CPL} A.$
(4) *For any formula A of* $F^{\forall}$, *we have*: $\vdash_{C^{\forall}} A \Leftrightarrow \vdash_{CPL} \neg A.$
(5) *For any formula A of* $F^{\exists}$, *we have*: $\vdash_{C^{\exists}} A \Leftrightarrow \vdash_{CPL} \neg A.$

Proof of Theorem 8.3. Below as in the proof of Theorem 6.1, we shall freely (with making no mention of it) make use of the completeness theorem for *CPL* (actually *CP*), if we need it to simplify our arguments. All the decision procedures to be used below are those in the second section.

(i) Let $A \in F^{\forall}$, say $A = \forall x_1 \ldots \forall x_n B(x_1, \ldots, x_n)$ $(n \geq 0)$.

The $\Rightarrow$ of (1): Suppose $\vdash_{P^\forall} A$. We shall prove this by induction on derivations. The propositional part of the induction is obvious by Theorem 8.1. Suppose $n \geq 1$. Suppose that $\forall x_n B(x_1, \ldots, x_n)$ is obtained by the rule ($GP^\forall$). By induction hypothesis, we have $\vdash_{CPL} B(x_1, \ldots, x_n)$. By the decision procedure (I), we obtain $\vdash_{CPL} A$ from it.

The $\Leftarrow$ of (1): Suppose $\vdash_{CPL} A$. By the decision procedure (I), $B(x_1, \ldots, x_n)$ is provable in CP. So we have $\vdash_{LK^{pr}} B(x_1, \ldots, x_n)$. Thus, $\vdash_{P^\forall} B(x_1, \ldots, x_n)$ holds. Applying ($GP^\forall$) n times to the sequent, we obtain $\vdash_{P^\forall} A$.

(ii) Let $A \in F^\exists$, say $A = \exists x_1 \ldots \exists x_n B(x_1, \ldots, x_n)$ $(n \geq 1)$.

The $\Rightarrow$ of (2): Suppose $\vdash_{P^\exists} A$. We shall prove this by induction on derivations. Again, we do not have any difficulties for the propositional part of the induction by Theorem 8.1.

Suppose that A is obtained by the rule $GP^\exists$. By induction hypothesis, we then have $\vdash_{CPL} U^z(B(x_1, \ldots, x_n))$ for some variable z, that is, $\vdash_{CPL} B(z, \ldots, z)$. Thus, by the decision procedure (II), we get $\vdash_{CPL} \exists x_1 \ldots \exists x_n B(x_1, \ldots, x_n)$.

The $\Leftarrow$ of (2): Suppose $\vdash_{CPL} A$. By the decision procedure (II), we see that $\vdash_{CPL} B(z, \ldots, z)$ hold for some variable z. Then, by applying ($GP^\exists$) to it, we obtain $\vdash_{P^\exists} A$.

(iii) The proof of (3): The proof of it is similar of the proof of (2), using the decision procedure (III).

(iv) Let $A \in F^\forall$, say $A = \forall x_1 \ldots \forall x_n B(x_1, \ldots, x_n, y_1, \ldots, y_m)$ $(n \geq 0$ and $m \geq 0)$,

such that $x_1, \ldots, x_n, y_1, \ldots, y_m$ are all distinct free variables occurring in the quantifier-free part of A. We shall write $B(\bar{x}, y_1, \ldots, y_m)$ or $B(x_1, \ldots, x_n, \bar{y})$ or $B(\bar{x}, \bar{y})$ for $B(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and so on.

The $\Rightarrow$ of (4): Suppose $\vdash_{C^\forall} A$. We shall prove this by induction on derivations. The propositional part of the induction is easy by Theorem 8.2. If $n = 0$, then the case belongs to the propositional one, with which we can easily deal by Theorem 8.2. So we shall assume $n \geq 1$.

First we shall deal with the case of $m = 0$. Then A is obtained only by the rule ($GC^\forall 1$). So we have $\rightarrow B(z, \ldots, z)$ for some variable z. By induction hypothesis, we get $\vdash_{CPL} \neg B(z, \ldots, z)$. By decision procedure (II) (or by *CPL*), $\vdash_{CPL} \exists x_1 \ldots \exists x_n \neg B(x_1, \ldots, x_n)$. This is nothing but $\vdash_{CPL} \neg A$.

Let us now treat the case of $m \geq 1$. Then A is obtained only by the rule $(GC^{\forall}2)$. Then $\vdash_{C^{\forall}} Spec[\overline{x \rightarrow z}](B(\bar{x}, \bar{y}))$ holds for some sequence $z_1, \ldots, z_n$ of variables with length n and $\{z_1, \ldots, z_n\} \subseteq \{y_1, \ldots, y_m\}$. By induction hypothsis, we have $\vdash_{CPL} \neg B(\bar{z}, \bar{y})$. By the decision procedure (III), we have $\vdash_{CPL} \forall y_1 \ldots \forall y_n \exists x_1 \ldots \exists x_n \neg B(\bar{x}, \bar{y})$. By *CPL*, we obtain $\vdash_{CPL} \exists x_1 \ldots \exists x_n \neg B(\bar{x}, \bar{y})$. This is nothing but $\vdash_{CPL} \neg A$.

The $\Leftarrow$ of (4): Suppose $\vdash_{CPL} \neg A$. If $n = 0$, then the case is the propositional one, which is easily dealt with by Theorem 8.2. So we shall assume $n \geq 1$.

We shall first take care of the case of $m = 0$. By *CPL*, we have $\vdash_{CPL} \exists x_1 \ldots \exists x_n \neg B(x_1, \ldots, x_n)$. By the decision procedure (II), $\vdash_{CPL} \neg B(z, \ldots, z)$ holds for some variable z. Since it is quantifier-free, we easily get $\vdash_{C^{\forall}} B(z, \ldots, z)$ by Theorem 8.2. Apply then the rule $(GC^{\forall}1)$ to it.

Let us prove the case of $m \geq 1$. As in the case of $m = 0$, $\vdash_{CPL} \exists x_1 \ldots \exists x_n \neg B(\bar{x}, \bar{y})$. Since $y_1, \ldots, y_m$ are all distinct free variables occurring in $\exists x_1 \ldots \exists x_n \neg B(\bar{x}, \bar{y})$, by *CPL*, we get $\vdash_{CPL} \forall y_1 \ldots \forall y_m \exists x_1 \ldots \exists x_n \neg B(\bar{x}, \bar{y})$. From this, by the decision procedure (III), we have $\vdash_{CPL} \neg B(z_1, \ldots, z_n, \bar{y})$ for some sequence $z_1, \ldots, z_n$ of variables with length n and $\{z_1, \ldots, z_n\} \subseteq \{y_1, \ldots, y_m\}$. By Theorem 8.2, we obtain $\vdash_{C^{\forall}} Spec[\overline{x \rightarrow z}](B(\bar{x}, \bar{y}))$. Then we can apply the rule $(GC^{\forall}2)$ to it.

(v) Let $A \in F^{\exists}$, say $A = \exists x_1 \ldots \exists x_n B(x_1, \ldots, x_n)$ $(n \geq 1)$.

The $\Rightarrow$ of (5): Suppose $\vdash_{C^{\exists}} A$. We shall prove this by induction on derivations. The propositional part of the induction can easily be verified by Theorem 8.2. So suppose that A is derived by the rule $(GC^{\exists})$. Then we have $\vdash_{C^{\exists}} B(x_1, \ldots, x_n)$. By Theorem 8.2, we get $\vdash_{CPL} \neg B(x_1, \ldots, x_n)$. By *CPL*, we obtain $\vdash_{CPL} \forall x_1 \ldots \forall x_n \neg B(x_1, \ldots, x_n)$, from which we immediately see $\vdash_{CPL} \neg A$.

The $\Leftarrow$ of (5): Suppose $\vdash_{CPL} \neg A$. By *CPL*, $\vdash_{CPL} \forall x_1 \ldots \forall x_n \neg B(x_1, \ldots, x_n)$ holds. By *CPL* (or by the decision procedure (I)), $\vdash_{CPL} \neg B(x_1, \ldots, x_n)$ follows immediately from it. Then, we get $\vdash_{C^\exists} B(x_1, \ldots, x_n)$ by Theorem 8.2. Apply then the rule $(GC^\exists)$ to the last formula. □

We have to note the following theorem.

Theorem 8.4.

(1) *The systems* $P^\forall$, $P^\exists$, $P^{\forall\exists}$, $C^\forall$ *and* $C^\exists$ *have the subformula property.*

(2) *The systems* $P^\forall$ and $C^\exists$ *have the absolute atomic formula property.*

Proof. It is easy to see it since LK^{pr} and OP^{pr} have the subformula property. □

Corollary 8.1. *The systems* $P^\forall$, $P^\exists$, $P^{\forall\exists}$, $C^\forall$ *and* $C^\exists$ *have the atomic formula property.*

Proof. Immediate from Theorem 8.4.(1). □

9. *Some concluding remarks.*

The classes treated in this paper would be of the simplest cases of first-order prenex formulas being subject to axiomatization of their provable formulas, contradictions, unprovable formulas and satisfiable ones. Our methodology for the axiomatization was the following: (1) *find a system with a decision procedure for it*; (2) *check whether we can conversely follow the decision procedure for a possible axiomatization of unprovable and satisfiable formulas (provable formulas and contradictions) of it.* It seems to me that this methodology would be effective enough in making our further study successful.[28]

The so-called subformula property (for short, s.f.p.) is a typical one of cut-free proofs of Gentzen's sequent calculus *LK*. I think that the property is in a sense an index of normalization of a proof.[29] It seems to me that a.f.p. and abs-a.f.p. are also related to normalization of proofs, though,

[28] Certain refutation and satisfaction systems for monadic predicate logic are proposed in [26, 27, 29] as an application of this methodology.

[29] Another interesting thought is found in [15, p. 259].

probably to a weaker notion of that. We wish to discuss this relation between a.f.p. and normalization further on another occasion.

The real significance of this paper is, I believe, that we proved Theorem 7.3 which shows the complete relation among s.f.p., a.f.p and abs-a.f.p., proposing $HC^{\forall}$, $HC^{\exists}$, $HC^{\forall\exists}$, $HCS^{\forall}$ and $HCS^{\exists}$, as well that we carried out a case study of the above mentioned methodology for predicate logic. We also proposed $P^{\forall}$, $P^{\exists}$, $P^{\forall\exists}$, $C^{\forall}$ and $C^{\exists}$ which have s.f.p. in the same line.

We shall here give an open problem which will naturally arise from this paper. Let us prepare some notations for that. Let

$$F_s^{\forall\exists} = \{A \in F^{\forall\exists} : \neg A \text{ is not provable in } CPL\},$$
$$F_c^{\forall\exists} = \{A \in F^{\forall\exists} : \neg A \text{ is provable in } CPL\}.$$

Here is the open problem.

Open problem. Axiomatize $F_s^{\forall\exists}$ and $F_c^{\forall\exists}$ and their subclasses, if possible.

We hope that this paper will somewhat stimulate the reader to further studies on the axiomatization of unprovable and satisfiable formulas. In particular, we know little about axiomatizations of satisfiable formulas of modal logics.

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APPENDIX

A. Closed opposite system.

The content of this Appendix A is just a little curiosity. Since a similar operation used in the proof of Theorem 6.1, that is, $\neg\forall\ldots\forall \rightarrow \exists\ldots\exists\neg$, will be employed in order to prove the main theorem (Theorem A.4) of this Appendix, this material is included in the present paper. The second reason of the inclusion of it is that the system for contradiction is the extreme case of *paraconsistent logic,*[30] which was the main theme of the Conference at which this paper was presented.

Here we shall propose a first-order opposite system (or first-order contradiction calculus) *without free variables* in the style of M. W. Bunder. In [7], Bunder proposed a certain classical first-order predicate calculus without free variables, that is, a calculus in which all well-formed formulas are (universally) closed and which therefore requires no generalization rule. By *BCL*, we shall denote his system, which is called *closed predicate calculus*. The calculus *BCL* consists of the following axioms and rule (we assume that the language of all the logics in this Appendix is that of *CPL*):

Axioms:

$(BCL1) \vdash_{BCL} \forall x_1 \ldots \forall x_n (A \supset . B \supset A)$, where $x_1, \ldots, x_n$ include the free variables occurring in A and B.

$(BCL2) \vdash_{BCL} \forall x_1 \ldots \forall x_n (A \supset . B \supset C : \supset : A \supset B . \supset . A \supset C)$, where $x_1, \ldots, x_n$ include the free variables occurring in A, B and C.

$(BCL3) \vdash_{BCL} \forall x_1 \ldots \forall x_n (\neg A \supset \neg B . \supset . B \supset A)$, where $x_1, \ldots, x_n$ include the free variables occurring in A and B.

$(BCL4) \vdash_{BCL} \forall x_1 \ldots \forall x_n \left(\forall x A \supset A^*\right)$, where $x_1, \ldots, x_n$, x include the free variables occurring in A, and A^* is the result obtained from A by replacing all free occurrences of x in A by a term having some or all of $x_1, \ldots, x_n$ as free variables, x in A however must not be in the scope of one of the quantifiers $\forall x_1, \ldots, \forall x_n$.[31]

[30] See [8, p. 57].

[31] The propositional part of *BCL*, i.e. (*BCL*1), (*BCL*2), (*BCL*3) and (*BCL*7) are based on Łukasiewicz's axiomatization for *CP* (see, e.g. [30]). Unfortunately, the description about the axiomatic system is not found in [37].

(*BCL5*) $\vdash_{BCL} \forall x_1 \ldots \forall x_n (\forall x(A \supset B) \supset . A \supset \forall x B)$, where $x_1, \ldots, x_n, x$ include the free variables occurring in $A \supset B$ with x being not free in A.

(*BCL6*) $\vdash_{BCL} \forall x_1 \ldots \forall x_n (\forall x \forall y A \supset \forall y \forall x A)$, where $x_1, \ldots, x_n, x, y$ include the free variables occurring in A.

Rule:

(*BCL7*) If $\vdash_{BCL} \forall x_1 \ldots \forall x_n (A \supset B)$, where $x_1, \ldots, x_n$ include the free variables occurring in $A \supset B$, and $\vdash_{BCL} \forall x_i \ldots \forall x_j A$, where $x_i, \ldots, x_j$ include the free variables occurring in A, then $\vdash_{BCL} \forall x_k \ldots \forall x_l B$, where $x_k, \ldots, x_l$ include the free variables occurring in B.

We easily see the following theorem.

Theorem A.1. ([7]) *For any formula A of CPL, if* $\vdash_{CPL} A$ *holds, where* $x_1, \ldots, x_n$ *are all the free variables occurring in A, then* $\vdash_{BCL} \forall x_1 \ldots \forall x_n A$ *holds.*

So immediately we have from the above theorem,

Corollary A.1. *For any formula A of CPL, if* $\vdash_{CPL} \neg A$ *holds, then* $\vdash_{BCL} \neg \exists x_1 \ldots \exists x_n A$ *holds, where* $x_1, \ldots, x_n$ *are all the free variables occurring in A.*

We further see the following, too.

Theorem A.2. *For any formula A of BCL, say* $A = \forall x_1 \ldots \forall x_n B$, *we have*: $\vdash_{BCL} A \Leftrightarrow \vdash_{CPL} B$, *where* $x_1, \ldots, x_n$ *are all the free variables occuring in B.*

Below we shall denote the opposite system of *CPL* in [23] by *SC*. Then we know the following theorem.

Theorem A.3. ([23]) *For any formula A of SC, we have*: $\vdash_{SC} A \Leftrightarrow \vdash_{CPL} \neg A$.

On the basis of Corollary A.1 and Theorem A.2, we shall propose a closed version of first-order opposite system, which derives all contradictions of *CPL*. By *BSC*, we shall denote the calculus, which is called *closed opposite system* (or *closed contradiction calculus*). The system *BSC* consists of following axioms and rules:

Axioms:

$(BSC1)$ $\vdash_{BSC} \exists x_1 \ldots \exists x_n (A \vee A. \wedge \neg A)$, where $x_1, \ldots, x_n$ include the free variables occurring in A.

$(BSC2)$ $\vdash_{BSC} \exists x_1 \ldots \exists x_n (B \wedge \neg(A \vee B))$, where $x_1, \ldots, x_n$ include the free variables occurring in A and B.

$(BSC3)$ $\vdash_{BSC} \exists x_1 \ldots \exists x_n (A \vee B. \wedge \neg(B \vee A))$, where $x_1, \ldots, x_n$ include the free variables occurring in A and B.

$(BSC4)$ $\vdash_{BSC} \exists x_1 \ldots \exists x_n (\neg A \vee B. \wedge. (C \vee B) \wedge \neg(C \vee B))$, where $x_1, \ldots, x_n$ include the free variables occurring in A, B and C.[32]

$(BSC5)$ $\vdash_{BSC} \exists x_1 \ldots \exists x_n (\forall x A \wedge \neg A^*)$, where $x_1, \ldots, x_n, x$ include the free variables occurring in A, and A^* is the result obtained from A by replacing all free occurrences of x in A by a term having some or all of $x_1, \ldots, x_n$ as free variables, x in A however must not be in the scope of one of the quantifiers $\exists x_1, \ldots, \exists x_n$.

$(BSC6)$ $\vdash_{BSC} \exists x_1 \ldots \exists x_n (\forall x(A \supset B) \wedge. A \wedge \neg \forall x B)$, where $x_1, \ldots, x_n, x$ include the free variables occurring in $A \supset B$ with x being not free in A.

$(BSC7)$ $\vdash_{BSC} \exists x_1 \ldots \exists x_n (\forall x \forall y A \wedge \neg \forall y \forall x A)$, where $x_1, \ldots, x_n, x, y$ include the free variables occurring in A.

Rule:

$(BSC8)$ If $\vdash_{BSC} \exists x_1 \ldots \exists x_n (A \wedge B)$, where $x_1, \ldots, x_n$ include the free variables occurring in $A \wedge B$, and $\vdash_{BSC} \exists x_i \ldots \exists x_j \neg A$, where $x_i, \ldots, x_j$ include the free variables occurring in A, then $\vdash_{BSC} \exists x_k \ldots \exists x_l B$, where $x_k, \ldots, x_l$ include the free variables occurring in B.

Now we can show the analogue of Theorem A.1 for *BSC* as follows.

Theorem A.4. *For any formula A of SC, if* $\vdash_{SC} A$ *holds, then* $\vdash_{BSC} \exists x_1 \ldots \exists x_n A$ *holds, where* $x_1, \ldots, x_n$ *are all the free variables occurring in A.*

[32] The propositional part of *BSC*, i.e. $(BSC1)$, $(BSC2)$, $(BSC3)$, $(BSC4)$ and $(BSC8)$ are based on Stahl's axiomatization for $\{A : \vdash_{CP} \neg A\}$ (see [42]).

Proof. The proof of it is similar to that of Theorem A.1. □

Then we immediately see the following.

Theorem A.5. ([23]) *For any formula A of BSC, say* $A = \exists x_1 \ldots \exists x_n B$, *we have:* $\vdash_{BSC} A \Leftrightarrow \vdash_{CPL} \neg B$, *where* $x_1, \ldots, x_n$ *are all the variables occurring in* B.

In the rest of Appendix A, we shall propose another calculus equivalent to *BSC*, which is a much more Bunder-like one.

Bunder [8] axiomatized the set of contradictions of *CPL* with negation, $\exists$ and the binary connective $\otimes$,[33] which is specified by the following truth table:

A	B	$A \otimes B$
T	T	F
T	F	F
F	T	T
F	F	F

We can define $A \otimes B$ as $\neg A \wedge B$ in terms of $\neg$ and $\wedge$.

The Bunder's system N for contradictions of *CPL* consists of the following axioms and rules:

Axioms:

$(N1) \vdash_N A \otimes . B \otimes A.$

$(N2) \vdash_N A \otimes . B \otimes C : \otimes : A \otimes B . \otimes . A \otimes C.$

$(N3) \vdash_N \neg B \otimes \neg A . \otimes . (\neg B \otimes A) \otimes B.$

$(N4) \vdash_N \exists x A(x) \otimes A(t)$, where t is a term of N free for x in $A(x)$.

$(N5) \vdash_N \exists x (A \otimes B) \otimes . A \otimes \exists x B$, where A contains no free occurrences of x.

$(N6) \vdash_N A,\ \vdash_N A \otimes B \ \Rightarrow\ \vdash_N B.$

$(N7) \vdash_N A \ \Rightarrow\ \vdash_N \exists x A.$

[33] The original notation in [8] for $\otimes$ is x, but we shall here use $\otimes$ in place of it.

For any formula A of N, let $T^*(A)$ be the result from A by rewriting all subformulas of the form $B \otimes C$ in A by $\neg B \wedge C$. Then we easily see the following theorem.

Theorem A.6. *For any formula A of N, we have*: $\vdash_N A \Leftrightarrow \vdash_{CPL} \neg T^*(A)$.

Proof. ($\Rightarrow$): Easy by induction on derivations.

($\Leftarrow$): We shall only give an outline of the proof. First formulate and prove the completeness theorem for N on the basis of an usual technique for CPL (e.g. Henkin's method). Then, by means of the theorem, we can easily prove the meta-implication

$$\dashv_N A \Rightarrow \dashv_{CPL} \neg T^*(A).$$

That is, if we have a model $\mathcal{M}$ falsifying A, then in view of the structure of the translation T^*, we can easily construct a model $\mathcal{M}^*$ which makes $\neg T^*(A)$ false, making use of $\mathcal{M}$. □

Now we shall propose the mentioned system, which will be denoted by BN. The system BN consists of the following axioms and rules:

Axioms:

($BN1$) $\vdash_{BN} \exists x_1 \ldots \exists x_n (A \otimes . B \otimes A)$, where $x_1, \ldots, x_n$ include the free variables occurring in A and B.

($BN2$) $\vdash_{BN} \exists x_1 \ldots \exists x_n (A \otimes . B \otimes C : \otimes : A \otimes B . \otimes . A \otimes C)$, where $x_1, \ldots, x_n$ include the free variables occurring in A, B and C.

($BN3$) $\vdash_{BN} \exists x_1 \ldots \exists x_n (\neg A \otimes \neg B . \otimes . \mathrm{B} \otimes A)$, where $x_1, \ldots, x_n$ include the free variables occurring in A and B.

($BN4$) $\vdash_{BN} \exists x_1 \ldots \exists x_n (\exists x A \otimes A^*)$, where $x_1, \ldots, x_n, x$ include the free variables occurring in A, and A^* is the result obtained from A by replacing all free occurrences of x in A by a term having some or all of $x_1, \ldots, x_n$ as free variables, x in A however must not be in the scope of one of the quantifiers $\exists x_1, \ldots, \exists x_n$.

($BN5$) $\vdash_{BN} \exists x_1 \ldots \exists x_n (\exists x (A \otimes B) \otimes . A \otimes \exists x B)$, where $x_1, \ldots, x_n, x$ include the free variables occurring in $A \otimes B$ with x being not free in A.

($BN6$) $\vdash_{BN} \exists x_1 \ldots \exists x_n (\exists x \exists y A \otimes \exists y \exists x A)$, where $x_1, \ldots, x_n, x, y$ include the free variables occurring in A.

Rule:

(*BN7*) If $\vdash_{BN} \exists x_1 \ldots \exists x_n (A \otimes B)$, where $x_1, \ldots, x_n$ include the free variables occurring in $A \otimes B$, and $\vdash_{BN} \exists x_i \ldots \exists x_j A$, where $x_i, \ldots, x_j$ include the free variables occurring in A, then $\vdash_{BN} \exists x_k \ldots \exists x_l B$, where $x_k, \ldots, x_l$ include the free variables occurring in B.

Probably we do not need to give the proof of the following theorems.

Theorem A.7. *For any formula A of N, if* $\vdash_N A$ *holds, then* $\vdash_{BN} \exists x_1 \ldots \exists x_n A$ *holds, where* $x_1, \ldots, x_n$ *are all the free variables occurring in A.*

Theorem A.8. *For any formula A of BN, say* $A = \exists x_1 \ldots \exists x_n B$, *we have*: $\vdash_{BN} A \Leftrightarrow \vdash_N B$, *where* $x_l, \ldots, x_n$ *are all the variables occurring in B.*

B. *Härtig-style axiomatizations of unprovable and satisfiable quantifier-free formulas with equality.*

Let $CPL^=$ be classical first-order predicate calculus with equality. We shall assume that apart from equality, the language of $CPL^=$ is the same as that of *CPL*.

As a preparation for the next Appendix C, we shall propose a Härtig-style axiomatization of satisfiable quantifier-free formulas of $CPL^=$. By $HCS^=$, we shall denote the proposed system. As above, the well-formed formulas of $HCS^=$ are quantifier-free ones of $CPL^=$.

Definition B.1. For any formula *A*, if it contains no equalities, then it is said to be *equality-free* (in notation: *Ef*(*A*)).

The system $HCS^=$ consists of the following axioms and rules:

Axioms:

($HCS1^=$) $\vdash_{HCS^=} F(t_1, \ldots, t_n)$ for any predicate letter $F(a_1, \ldots, a_n)$ with n attached variables and for any terms $t_1, \ldots, t_n (n \geq 0)$.

($HCS2^=$) $\vdash_{HCS^=} \neg F(t_1, \ldots, t_n)$ for any predicate letter $F(a_1, \ldots, a_n)$ with n attached variables and for any terms $t_1, \ldots, t_n (n \geq 0)$.

$(HCS3^=)$ $\vdash_{HCS^=} t = t$ for any term t.

$(HCS4^=)$ $\vdash_{HCS^=} \neg t = s$ for any distinct terms t and s.

Rules:

$(HCS5^=)$ $\vdash_{HCS^=} A$, $\vdash_{HCS^=} B$, $\ll A \gg \cap \ll B \gg = \varnothing$, $Ef(A)$, $Ef(B) \Rightarrow \vdash_{HCS^=} A \wedge B$.

$(HCS6^=)$ If $\vdash_{HCS^=} A$ and t, s are distinct terms, then $\vdash_{HCS^=} A \wedge t = t$ and $\vdash_{HCS^=} A \wedge \neg t = s$.

$(HCS7^=)$ $\vdash_{HCS^=} A$, $\vdash_{CP} A \supset B \;\Rightarrow\; \vdash_{HCS^=} B$.

Theorem B.1. *For any formula A of* $HCS^=$, $\vdash_{HCS^=} A \Leftrightarrow \dashv_{CPL^=} \neg A$.

Proof. Similar to the proof of Theorem 4.2. □

Theorem B.2. *The Härtig-style satisfaction calculus* $HCS^=$ *has the absolute atomic formula property.*

Proof. Regarding equality as a binary predicate letter in Definition 5.3, the proof is similar to that of Theorem 4.3. □

Theorem B.3. *The Härtig-style satisfaction calculus* $HCS^=$ *has the atomic formula property.*

Proof. Similar to the proof of Theorem B.2. □

Now we shall propose a Härtig-style axiomatization of unprovable quantifier-free formulas of $CPL^=$. By $HC^=$, we shall denote the proposed system. The well-formed formulas of $HC^=$ are quantifier-free ones of $CPL^=$. The system $HC^=$ consists of the following axioms and rules:

Axioms:

$(HC1^=)$ $\vdash_{HC^=} F(t_1, \ldots, t_n)$ for any predicate letter $F(a_1, \ldots, a_n)$ with n attached variables and for any terms $t_1, \ldots, t_n (n \geq 0)$.

$(HC2^=)$ $\vdash_{HC^=} \neg F(t_1, ..., t_n)$ for any predicate letter $F(a_1, ..., a_n)$ with n attached variables and for any terms $t_1, ..., t_n (n \geq 0)$.

$(HC3^=)$ $\vdash_{HC^=} \neg t = t$ for any term t.

$(HC4^=)$ $\vdash_{HC^=} t = s$ for any distinct terms t and s.

Rules:

$(HC5^=)$ $\vdash_{HC^=} A, \vdash_{HC^=} B, \ll A \gg \cap \ll B \gg = \emptyset, Ef(A), Ef(B) \Rightarrow \vdash_{HC^=} A \vee B.$

$(HC6^=)$ If $\vdash_{HC^=} A$ and t, s are distinct terms, then $\vdash_{HC^=} A \vee \neg t = t$ and $\vdash_{HC^=} A \vee t = s$.

$(HC7^=)$ $\vdash_{CP} A \supset B, \vdash_{HC^=} B \Rightarrow \vdash_{HC^=} A$.

Since we can similarly prove them as above, we shall omit the proof of the following theorems.

Theorem B.4. *For any formula A of $HC^=$,* $\vdash_{HC^=} A \Leftrightarrow \dashv_{CPL^=} A$.

Theorem B.5. *The Härtig-style refutation calculus $HC^=$ has the absolute atomic formula property.*

Theorem B.6. *The Härtig-style refutation calculus $HC^=$ has the atomic formula property.*

The systems $HC^=$ and $HCS^=$ do not have the subformula property, since they have the rules $(HC7^=)$ and $(HCS7^=)$, respectively.

C. *Bullock and Schneider's calculi for finitely satisfiable formulas and their a.f.p..*

In this Appendix C, we shall recall Bullock and Schneider's calculus for finitely satisfiable formulas with or without equality (see [6, 5]).

By *CPL* we shall here again denote classical first-order predicate calculus without equality. Let $CPL^=$ be classical first-order predicate calculus with equality. Here we assume that the language of all the calculi treated in this

Appendix contain no function symbols. However, we assume that it may contain constants.

For the convenience of the reader, we shall repeat some model-theoretic terminologies and notations which were used in [6]. If $\mathcal{M}$ is a model structure, by $|\mathcal{M}|$ we denote the universe of $\mathcal{M}$. For any formula A of CPL ($CPL^=$) we write $\mathcal{M} \models A(a_1, \ldots, a_n)$ if the assignment of $a_i \in |\mathcal{M}|$ to variable x_i satisfies A, where $x_1, \ldots, x_n$ are all the free variables coccurring in A. If there is a model structure $\mathcal{M}$ and $a_1, \ldots, a_n \in |\mathcal{M}|$ such that $\mathcal{M} \models A(a_1, \ldots, a_n)$, then A is said to be *satisfiable*,[34] and if $|\mathcal{M}|$ has exactly k elements, A is said to be *k-satisfiable*.

A formula A is said to be *finitely satisfiable* if it is k-satisfiable for some positive integer k.

Now we shall recall a series of definitions due to [6].

Definition C.1. Let A be a formula of CPL ($CPL^=$). If $t_1, \ldots, t_n$ are terms (that is, variables or constants) and $s_1, \ldots, s_n$ are terms not occurring in A, then we denote by $A[t_1/s_1, \ldots, t_n/s_n]$ the formula obtained from A by replacing each free occurrence of t_i in A by s_i for any $1 \le i \le n$.

Definition C.2. Let A be a formula of CPL ($CPL^=$). Let $x_1, \ldots, x_n$ be the sequence of all variables occurring free in A, let $y_1, y_2, \ldots$ be the sequences containing all variables not occurring in A, and let $d_1, \ldots, d_p$ be the sequences of all constants occurring in A. Then the *closure* $A^*$of A is defined by:

$$A^* = \exists x_1 \ldots \exists x_n \exists y_1 \ldots \exists y_p A[d_1/y_1, \ldots, d_p/y_p].$$

Clearly, the closure of a formula has neither constants nor free occurrences of variables.

Definition C.3. Let A be a formula of CPL ($CPL^=$). The *k-transform* A^k of A is inductively defined as follows.

(1) $A^k = A$ for any atomic A.

(2) $(\neg B)^k = \neg B^k$.

(3) $(B \wedge C)^k = B^k \wedge C^k$.

[34] In classical logic, this definition of satisfiability is equivalent to the definition adopted in the Introduction for satisfiability.

(4) $(\forall xB)^k = \left(B[x/c_1]\right)^k \wedge \ldots \wedge \left(B[x/c_k]\right)^k$, where $c_1, \ldots, c_k$ are distinct constants.

The Bullock and Schneider's calculus BS for finitely satisfiable formulas without equality consists of the following axioms and rules:

Axioms:

($BS1$) $\vdash_{BS} A$ for any atomic A.

Rules:

($BS2$) For any atomic B, if $\vdash_{BS} A$, $Qf(A)$ and $B \notin \ll A \gg$, then $\vdash_{BS} A \wedge B$ and $\vdash_{BS} A \wedge \neg B$.

($BS3$) $\vdash_{BS} A, \vdash_{CP} A \supset B \Rightarrow \vdash_{BS} B$.

($BS4$) $\vdash_{BS} A^{*k}$ for some positive integer $k \Rightarrow \vdash_{BS} A$.

Theorem C.1. ([6]) *For any formula A of CPL, A is finitely satisfiable if and only if* $\vdash_{BS} A$ *holds.*

The Bullock and Schneider's calculus $BS^{=}$ for finitely satisfiable formulas with equality in [5] consists of the following axioms and rules:

Axioms:

($BS1^{=}$) $\vdash_{BS^{=}} A$ for any equality-free atomic A.

Rules:

($BS2^{=}$) For any atomic B with $Ef(B)$, if $\vdash_{BS^{=}} A$, $Qf(A)$ and $B \notin \ll A \gg$, then $\vdash_{BS^{=}} A \wedge B$ and $\vdash_{BS^{=}} A \wedge \neg B$.

($BS3^{=}$) For any distinct terms t and s, if $\vdash_{BS^{=}} A$ and $Qf(A)$, then $\vdash_{BS^{=}} A \wedge t = t$ and $\vdash_{BS^{=}} A \wedge \neg t = s$.

($BS4^{=}$) $\vdash_{BS^{=}} A, \vdash_{CP} A \supset B \Rightarrow \vdash_{BS^{=}} B$.

($BS5^{=}$) $\vdash_{BS^{=}} A^{*k}$ for some positive integer $k \Rightarrow \vdash_{BS^{=}} A$.

We should here remark that the rule ($BS5^{=}$) must be used at most once in a proof $B_1, \ldots, B_n$ of a formula $A(= B_n)$ $(n \geq 1)$ of $BS^{=}$ and if it is used in the case of $n \geq 2$, then A must be the immediate consequence from $B_i (1 \leq i \leq n-1)$ by ($BS5^{=}$).

Theorem C.2. ([5]) *For any formula A of* $CPL^{=}$, *A is finitely satisfiable if and only if* $\vdash_{BS^{=}} A$ *holds*.

The purpose of the Appendix C is to prove the following theorem.

Theorem C.3. *The systems BS and* $BS^{=}$ *have the atomic formula property.*

We need some preparation for the proof of Theorem C.3.

Let HCS^{dQ} be the propositional part of $HCS^{\forall}$. So the well-formed formulas of HCS^{dQ} are assumed to be quantifier-free ones of *CPL*. Then, by a similar consideration as in Appendix B, we have

Theorem C.4. *For any formula A of* HCS^{dQ}, $\vdash_{HCS^{dQ}} A \Leftrightarrow \dashv_{CPL^{=}} \neg A$.

Theorem C.5. *The Härtig-style satisfaction calculus* HCS^{dQ} *has the absolute atomic formula property.*

Theorem C.6. *The Härtig-style satisfaction calculus* HCS^{dQ} *has the atomic formula property*.

Let BS^{dQ} be the propositional part of *BS* and $BS^{dQ=}$ the propositional part of $BS^{=}$. Now it is easy to prove the following theorem by induction on derivations. (The well-formed formulas of BS^{dQ} ($BS^{dQ=}$) are assumed to be quantifier-free ones of *CPL* ($CPL^{=}$).)

Theorem C.7.

(1) *For any formula A of* BS^{dQ}, $\vdash_{BS^{dQ}} A \Leftrightarrow \vdash_{HCS^{dQ}} A$.

(2) *For any formula A of* $BS^{dQ=}$, $\vdash_{BS^{dQ=}} A \Rightarrow \vdash_{HCS^{=}} A$.

Proof. We shall only prove (1). In a similar way, (2) is also easily proved.

The proof of $\Rightarrow$ of (1): It is obvious since the rule (*RS*4) of HCS^{dQ} is a generalization of the rule (*BS*2) of BS^{dQ}.

The proof of $\Leftarrow$ of (1): Let *A* be a formula of HCS^{dQ}. Suppose that we have a similar proof of *A* in HCS^{dQ} as in the proof of *A* in the proof of $\Leftarrow$

of Theorem 4.2. Then we observe that the proof of A in HCS^{dQ} can easily be transformed to that in BS^{dQ}. □

Theorem C.8.
(1) *The system* BS^{dQ} *has the atomic formula property.*

(2) *The system* $BS^{dQ=}$ *has the atomic formula property.*

Proof. The proof of (1): Let A be a formula of BS^{dQ}. Suppose $\vdash_{BS^{dQ}} A$. Then by Theorem C.7, we have $\vdash_{HCS^{dQ}} A$. Then there is a proof π of A in HCS^{dQ} with a.f.p. by Theorem C.7. Without loss of generality, we may assume that the proof π is obtained similarly as the proof of A in the proof of $\Leftarrow$ of Theorem 4.2. Then we easily see that π can be transformed to that in BS^{dQ} keeping the atomic formula property.

The proof of (2): For (2), we have a similar proof as the proof of (1) on the basis of Theorems B.3, 4.2 and C.7. □

Now let us prove the main theorem of this Appendix.

Proof of Theorem C.3. For (1), it is immediate from Theorem C.8, since every proof of A in which ($BS5^{=}$) is at least once used can be easily transformed to a proof of A in which the rule is employed only once at the end of the proof. For (2), we may just apply Theorem C.8 for it. □

It is obvious that BS and $BS^{=}$ do not have the absolute atomic formula property, looking at their rules for quantifiers. This means that the notion of the atomic formula property is not stronger than that of the absolute atomic formula property. We also mention that BS and $BS^{=}$ do not have s.f.p. because of ($BS3$) and ($BS3^{=}$), respectively.

D. *Some refutation and satisfaction calculi which have neither s.f.p., nor a.f.p., nor abs-a.f.p..*

In this Appendix D, we shall give some refutation and satisfaction calculi which have neither s.f.p., nor a.f.p., nor abs-a.f.p.. Let p be a proposition letter of CP and fix it. The language of the calculi treated below is that of CP.

The refutation calculus CPR for CP in [36] consists of the following axiom and rules:

Axiom:

$(CPR1) \vdash_{CPR} p$ for the fixed proposition letter p.

Rules:

$(CPR2) \vdash_{CPR} \sigma(A) \Rightarrow \vdash_{CPR} A$, where σ is a uniform substitution.

$(CPR3) \vdash_{CP} A \supset B, \vdash_{CPR} B \Rightarrow \vdash_{CPR} A$.

Theorem D.1. ([22]) *For any formula A of CP, $\vdash_{HC} A \Leftrightarrow \vdash_{CPR} A$.*

We shall show a satisfaction calculus CPS for CP in [33] consisting of the following axiom and rules:

Axiom:

$(CPS1) \vdash_{CPS} p$ for the fixed proposition letter p.

Rules:

$(CPS2) \vdash_{CPS} \sigma(A) \Rightarrow \vdash_{CPS} A$, where σ is a uniform substitution.

$(CPS3) \vdash_{CPS} A, \vdash_{CP} A \supset B \Rightarrow \vdash_{CPS} B$.

Theorem D.2. *For any formula A of CP, $\vdash_{CPS} A \Leftrightarrow \vdash_{CPR} \neg A$.*

Proof. Easy by induction on derivations. (It is easy to give a purely syntactical proof of it.) □

We note that CPR and CPS are finite axiomatizations of the set of unprovable and satisfiable formulas of CP, respectively.

Theorem D.3. *The systems CPR and CPS have neither s.f.p., nor a.f.p., nor abs-a.f.p..*

Proof. We shall first show that they do not have a.f.p.. Take a proposition letter q with $q \neq p$. It is obvious that $\vdash_{CPR} q$ and $\vdash_{CPS} q$ hold. Every proof of q in CPR must contain the axiom p and so is every proof of q in CPS. Hence, CPR and CPS do not have a.f.p.. So, they do not have s.f.p.. For classical propositional logic, the notion of a.f.p. coincides with that of abs-a.f.p. so that CPR and CPS do not have abs-a.f.p.. □

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