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PROOF-THEORETICAL CONSIDERATIONS ABOUT THE LOGIC OF EPISTEMIC INCONSISTENCY*

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Abstract

The *Logic of Epistemic Inconsistency* (*LEI*) provides an interesting example of a paraconsistent formal system from the proof theoretical point of view. Although *cut elimination* theorem for its sequent calculus presentation can only be proved with a restriction, this does not damage its main desirable outcomes, that are still attained. Therefore, consistency, subformula property and conservativeness of logical constants definitions are well preserved. The motivation for *LEI* designing, its sequent calculus presentation and most remarkable theorems are presented. Instances of the unremovable cut-proof pattern are analysed.

1. *Introduction*

The *Logic of Epistemic Inconsistency* (*LEI*) is a paraconsistent logic designed to support reasoning under conditions of incomplete or unaccurate knowledge. In these circumstances, conclusions must be grasped on the basis of partial evidences. In the course of reasoning, partially supported conflicting conclusions may emerge. This kind of situation asks for a criterious consideration of the total evidence available in order to solve these conflicts but, it may eventually happen, the available knowledge not being able to do so, making some of these pairs of conflicting partial conclusions turn out as unremovable conflicts, being this the best that could be possibly done under these circumstances.

It is our opinion that, when the term reasoning is used to express a more general thinking process than mere deduction, as in the situation just described, its role is not just the grasping of conclusions, as for deduction, but to perform a more comprehensive analysis of the available knowledge.

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As a consequence of this disposition, the whole set of emerging conclusions, which includes the unresovable conflicting pairs, should be assimilated and reasoned out by a suitable logic, able to do so without provoking a logical collapse. The kind of inconsistency so arising should be called *epistemic*, by reflecting not an inconsistency in the state of affairs itself but a deficiency in our knowledge about it [Pequeno & Buchsbaum 91].

Incomplete knowledge reasoning has been the topic of study of the so called *nonmonotonic reasoning* in artificial intelligence, for more than a decade now. The current attitude in the field has been to avoid the hazadous of epistemic contradiction by taking a choice between two alternatives. One, which has been called *credulous*, consists in accepting all conflicting conclusions by splitting them into self consistent subsets [Reiter 80]. The other, sometimes called *skeptical,* consists in simply rejecting all those conclusions [McCarthy 80]. In [Pequeno 90], it is suggested to keep the whole set of conclusions, no matter they possibly being contradictory, in a single theory, treating them paraconsistently. *LEI* has been designed for this purpose and it is intended to be used in combination with a nonmonotonic logic called *IDL* (for *Inconsistent Default Logic* [Pequeno 90]) which takes care of the dynamics of this kind of reasoning (the incoming of knowledge and the dismissing of no longer supported conclusions).

LEI logical structure also fits a general pattern of situations in which a plurality of views on a same subject is involved. This is the case, for instance, when a same phenomenon is reported by many observers or, in general, when it is observed in experiments under slightly varying conditions, being these variations undetected and/or unrecorded along the experiment, but anyway enough to affect it. Then, disagreement may also occur, and again, by a lack of knowledge. This fact make of these situations instances of epistemic inconsistencies. It is curious that, in the first case, epistemic inconsistencies seem to arise from the *underdetermination* of conclusions while, in the second one, they seem to come from their *overdetermination.* The designing of *LEI* semantics has been based on the intuition coming from this last scenario, while its Hilbert style axiomatics is intended to match the pattern of reasoning upon incomplete knowledge described before. *LEI* soundness and completeness establish an equivalence between these two perspectives [Pequeno & Buchsbaum 91].

Reasoning upon partial/multiple knowledge is a central issue in many Artificial Intelligence applications and its automatization is certainly profitable. Tableaux systems for *LEI* have been developed in [Corrêa & Buchsbaum & Pequeno 93] and [Buchsbaum 91], and a sequent calculus in [Martins & Pequeno 94]. Our aim here, is to discuss proof-theoretical issues through *LEI* sequent calculus by stressing *LEI* metalogical properties. In section 2, *LEI* sequent calculus — proved correct and complete with respect to the given *LEI* axiomatics — and main *LEI* theorems are presented. Proof-theoretical considerations are detailed in section 3 and conclusions, as well as further developments, are pointed out at the end.

2. *LEI sequent calculus and main theorems*

LEI has been conceived as a maximal approximation of classical logic able to support, without trivializing, epistemic contradictions arising from the application of *IDL* (default) rules, which introduce defeasible conclusions. To distinguish these conclusions from those irrevocable ones, a ? sign marking the first ones is adopted in *LEI*. The intuitive meaning of $\alpha?$ is 'there are evidences to α' or 'α is plausible'. For the sake of meta references, irrefutable, monotonic, formulas will be denoted by roman capital letters, ?-formulas by greek lower letters and sequence of formulas by greek capital letters.

Paraconsistency is attained in *LEI* by rejecting *ex falso sequitur quodlibet*, $\alpha \to (\neg\alpha \to \beta)$, to formulas suffixed by ?. Technically, *LEI* differs from da Costa paraconsistent calculi *Ci* [daCosta 74] in some key aspects, mainly: first, *LEI* retains more classical theorems than *Ci* — a remarkable example of this is the schema $\neg(\alpha \wedge \neg\alpha)$ usually taken as an inner expression of the *non-contradiction law*. Although rejected in *Ci,* it is still a theorem in *LEI*. Second, a recursive semantics has been provided for *LEI*. Whereas $\neg$ denotes negation in *LEI*, a paraconsistent negation, $\sim\alpha$ is introduced as a handy abbreviation for $\alpha \to p \wedge \neg p$, being 'p' a propositional letter. ~ can be shown to work as a classical, also called strong negation in LEI.

LEI sequent rules are:

Structural Rules

Exchange

$$LX\frac{\Gamma,\ \alpha,\ \beta,\ \Gamma' \vdash \Delta}{\Gamma,\ \beta,\ \alpha,\ \Gamma' \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta,\ \alpha,\ \beta,\ \Delta'}{\Gamma \vdash \Delta,\ \beta,\ \alpha,\ \Delta'}RX$$

Weakening

$$LW\frac{\Gamma \vdash \Delta}{\Gamma,\ \alpha,\ \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \alpha,\ \Delta}RW$$

Contradiction

$$LC\frac{\Gamma,\ \alpha,\ \alpha,\ \vdash \Delta}{\Gamma,\ \alpha,\ \vdash \Delta} \qquad \frac{\Gamma \vdash \alpha,\ \alpha,\ \Delta}{\Gamma \vdash \alpha,\ \Delta}RC$$

$$\frac{\dfrac{\Gamma \vdash \alpha \to \beta,\ \Delta}{\Gamma \vdash \alpha? \to \beta?,\ \Delta}}{\Gamma \vdash \alpha \to \beta,\ \Delta}\ ?\text{–}RC$$

Identity Rules

$$\alpha \vdash \alpha \ \ \textit{Identity}$$

$$\frac{\Delta \vdash \alpha,\ \Delta\ \ \Gamma',\ \alpha \vdash \Delta'}{\Gamma,\ \ \Gamma' \vdash \Delta,\ \Delta'}\ Cut$$

$$\frac{\Delta \vdash \alpha,\ \Delta\ \ \Gamma',\ \alpha? \vdash \Delta'}{\Gamma,\ \ \Gamma' \vdash \Delta,\ \Delta'}\ ?\text{–}Cut$$

Operational Rules

Question mark

$$L_1?\frac{\Gamma,\ \alpha? \vdash \Delta}{\Gamma,\ \alpha?? \vdash \Delta} \qquad \frac{\Gamma \vdash \alpha,\ \Delta}{\Gamma \vdash \alpha?,\ \Delta}R_1?$$

$$L_2?\frac{\Gamma,\ \alpha? \to \beta? \vdash \Delta}{\Gamma,\ (\alpha? \to \beta?)? \vdash \Delta} \qquad \frac{\Gamma \vdash \alpha \to \beta,\ \Delta}{\Gamma \vdash \alpha? \to \beta?,\ \Delta}R_2?$$

Negation

$$L\neg\ \frac{\Gamma \vdash A,\ \Delta}{\Gamma,\ \neg A \vdash \Delta} \qquad \frac{\Gamma,\ \alpha,\ \vdash \Delta}{\Gamma \vdash \neg\alpha,\ \Delta}\ R\neg$$

Conjunction

$$L_1\wedge\ \frac{\Gamma,\ \alpha\ \vdash\ \Delta}{\Gamma,\ \alpha\wedge\beta\ \vdash\ \Delta}\qquad\qquad \frac{\Gamma,\ \beta\ \vdash\ \Delta}{\Gamma,\ \alpha\wedge\beta\ \vdash\ \Delta}\ L_2\wedge$$

$$\frac{\Gamma\ \vdash\ \alpha,\ \Delta\quad \Gamma'\ \vdash\ \beta,\ \Delta'}{\Gamma,\ \Gamma'\ \vdash\ \alpha\wedge\beta,\ \Delta,\ \Delta'}\ R_\wedge$$

Disjunction

$$\frac{\Gamma,\ \alpha\ \vdash\ \Delta\quad \Gamma',\ \beta\ \vdash\ \Delta'}{\Gamma,\ \Gamma',\ \alpha\vee\beta\ \vdash\ \Delta,\ \Delta'}\ L\vee$$

$$R_1\vee\ \frac{\Gamma\ \vdash\ \alpha,\ \Delta}{\Gamma\ \vdash\ \alpha\vee\beta,\ \Delta}\qquad\qquad \frac{\Gamma\ \vdash\ \beta,\ \Delta}{\Gamma\ \vdash\ \alpha\vee\beta,\ \Delta}\ R_2\vee$$

Implication

$$L\rightarrow\ \frac{\Gamma\ \vdash\ \alpha,\ \Delta\quad \Gamma',\ \beta\ \vdash\ \Delta'}{\Gamma,\ \Gamma',\ \alpha\rightarrow\beta\ \vdash\ \Delta,\ \Delta'}\qquad\qquad \frac{\Gamma,\ \alpha^1\ \vdash\ \beta,\ \Delta}{\Gamma\ \vdash\ \alpha\rightarrow\beta,\ \Delta}\ R\rightarrow$$

1 all variables are held constants and, either α is ?-closed or R2 ? and R~ are not applied after the first time α occurs justified by being a premise.

Double Negation

$$L\neg\neg\ \frac{\Gamma,\ \alpha\ \vdash\ \Delta}{\Gamma,\ \neg\neg\alpha\ \vdash\ \Delta}\qquad\qquad \frac{\Gamma\ \vdash\ \alpha,\ \Delta}{\Gamma\ \vdash\ \neg\neg\alpha,\ \Delta}\ R\neg\neg$$

Distribution of '$\neg$' over '$\vee$'

$$LD^{\neg}_{\vee}\ \frac{\Gamma,\ \neg\alpha\wedge\neg\beta\ \vdash\ \Delta}{\Gamma,\ \neg(\alpha\vee\beta)\ \vdash\ \Delta}\qquad\qquad \frac{\Gamma\ \vdash\ \neg\alpha\wedge\neg\beta,\ \Delta}{\Gamma\ \vdash\ \neg(\alpha\vee\beta),\ \Delta}\ RD^{\neg}_{\vee}$$

Distribution of '$\neg$' over '$\wedge$'

$$LD^{\neg}_{\wedge}\ \frac{\Gamma,\ \neg\alpha\vee\neg\beta\ \vdash\ \Delta}{\Gamma,\ \neg(\alpha\wedge\beta)\ \vdash\ \Delta}\qquad\qquad \frac{\Gamma\ \vdash\ \neg\alpha\vee\neg\beta,\ \Delta}{\Gamma\ \vdash\ \neg(\alpha\wedge\beta),\ \Delta}\ RD^{\neg}_{\wedge}$$

Distribution of '$\neg$' over '$\rightarrow$'

$$LD^{\neg}_{\rightarrow}\ \frac{\Gamma,\ \alpha\wedge\neg\beta\ \vdash\ \Delta}{\Gamma,\ \neg(\alpha\rightarrow\beta)\ \vdash\ \Delta}\qquad\qquad \frac{\Gamma\ \vdash\ \alpha\wedge\neg\beta,\ \Delta}{\Gamma\ \vdash\ \neg(\alpha\rightarrow\beta),\ \Delta}\ RD^{\neg}_{\rightarrow}$$

Distribution of '$\neg$' over '?'

$$LD_?^{\neg}\ \frac{\Gamma,\ (\neg\alpha)?\vdash\ \Delta}{\Gamma,\ \neg(\alpha?)\ \vdash\ \Delta}\qquad\qquad \frac{\Gamma\ \vdash\ (\neg\alpha)?,\ \Delta}{\Gamma\ \vdash\ \neg(\alpha?),\ \Delta}\ RD_?^{\neg}$$

Distribution of '?' over '$\vee$'

$$LD_\vee^?\ \frac{\Gamma,\ \alpha?\vee\beta?\vdash\ \Delta}{\Gamma,\ (\alpha\vee\beta)?\vdash\ \Delta}$$

Classical Negation

$$\frac{\Gamma\ \vdash\ \alpha,\ \Delta}{\Gamma\ \vdash\ \sim((\sim\alpha)?),\ \Delta}\ R\sim$$

First Order Rules

Universal Quantification

$$L\forall\ \frac{\Gamma,\ \alpha(t)^1\ \vdash\ \Delta}{\Gamma,\ \forall x\alpha(x)\ \vdash\ \Delta}\qquad\qquad \frac{\Gamma\ \vdash\ \alpha(x)^2,\ \Delta}{\Gamma\ \vdash\ \forall x\alpha(x),\ \Delta}\ R\forall$$

1 t is free for x in $\alpha(x)$
2 x must not occur free in $\Gamma,\ \Delta$

Existential Quantification

$$L\exists\ \frac{\Gamma,\ \alpha(x)^1\ \vdash\ \Delta}{\Gamma,\ \exists x\alpha(x)\ \vdash\ \Delta}\qquad\qquad \frac{\Gamma\ \vdash\ \alpha(t)^2,\ \Delta}{\Gamma\ \vdash\ \exists x\alpha(x),\ \Delta}\ R\exists$$

1 x must not occur free in $\Gamma,\ \Delta$
2 t is free for x in $\alpha(x)$

Quantifiers Equivalence

$$L\neg\forall\ \frac{\Gamma,\ \exists x(\neg\alpha)\ \vdash\ \Delta}{\Gamma,\ \neg(\forall x\alpha)\ \vdash\ \Delta}\qquad\qquad \frac{\Gamma\ \vdash\ \exists x(\neg\alpha),\ \Delta}{\Gamma\ \vdash\ \neg(\forall x\alpha),\ \Delta}\ R\neg\forall$$

$$L\neg\exists\ \frac{\Gamma,\ \forall x(\neg\alpha)\ \vdash\ \Delta}{\Gamma,\ \neg(\exists x\alpha)\ \vdash\ \Delta}\qquad\qquad \frac{\Gamma\ \vdash\ \forall x(\neg\alpha),\ \Delta}{\Gamma\ \vdash\ \neg(\exists x\alpha),\ \Delta}\ R\neg\exists$$

Extended Rule for Classical Negation

Classical Negation

$$L_e \sim \frac{\Gamma \vdash \alpha,\ \Delta}{\Gamma,\ \sim\alpha \vdash \Delta} \qquad \frac{\Gamma,\ \alpha \vdash \Delta}{\Gamma \vdash\ \sim\alpha,\ \Delta}\ R_e \sim$$

LEI sequent rules are closely related to *LEI* axiomatics presented in [Pequeno & Buchsbaum 91]. Paraconsistency has been captured by restricting $L\neg$ to classical, ?-free formulas. Double negation rule, distribution rules of $\neg$ over $\wedge$, $\vee$ and $\rightarrow$ and quantifier equivalences are derivable in classical logic but not in *LEI* and must be explicitly stated. Central focus has been devoted to the new and peculiar ? operator. It has been defined by operational rules but its structural aspects are also expressed by two new rules: ?-*Cut* and ?-*RC*. The former states that: 'if α comes from β and γ comes from α?, then certainly γ also comes from the *stronger* α, or from its proof using β'. The latter is closely related to the operational rule $R_2?$. ?$-RC$ restores the same context $\Gamma \vdash \alpha \rightarrow \beta,\ \Delta$ if it was given as premise a deduction where $R_2?$ was applied.

Question mark rules reveal operational ?-features. $R_1?$ states that from α you can extract its plausibility α?. The opposite is obviously not valid. Nevertheless, a weak converse of $R_1?$ — $L_1?$ — is valid. $L_1?$ states that additional ?'s are superfluous. $L_2?$ expresses that external ?'s are irrelevant to an implication when both antecedent and succedent are ?-closed (see definition 2.2 below). $R_2?$ provides the propagation of ? along inferences when used in conjunction with $L \rightarrow$ to simulate a sort of ?-Modus Ponens $(\alpha?,\ \alpha \rightarrow \beta / \beta?)$ as in the proof below:

$$\dfrac{\Gamma_1 \vdash \alpha? \qquad \dfrac{\dfrac{\Gamma_2 \vdash \alpha \rightarrow \beta}{\Gamma_2 \vdash \alpha? \rightarrow \beta?} \qquad \dfrac{\alpha? \vdash \alpha? \quad \beta? \vdash \beta?}{\alpha? \rightarrow \beta?,\ \alpha? \vdash \beta?}}{\Gamma_2,\ \alpha? \vdash \beta?}}{\Gamma_1,\ \Gamma_2 \vdash \beta?}$$

The same is achieved in LEI axiomatics using Modus Ponens and rule $\dfrac{\alpha \rightarrow \beta}{\alpha? \rightarrow \beta?}$ as in the following:

$$\cfrac{\alpha? \quad \cfrac{\alpha \to \beta}{\alpha? \to \beta?}}{\beta?}$$

Distribution of ? over $\vee$ is assured by $LD^?_\vee$. Its symmetrical $RD^?_\vee$ is derived. $R\sim$ states that it is impossible to be sure about α and, at the same time, to have an evidence to its classical negation $(\sim\alpha)?$ A theory including α and $(\sim\alpha)?$ is trivial in *LEI*. Notice the double line at rules $R_2?$ and $R\sim$ It marks a restriction over $R\to$ application provided $R_2?$ and $R\sim$ do not hold constant the variables over evidences implicitly quantified by ?. On the other hand, the implicative forms of $R_2?$ and $R\sim$, i.e. $(\alpha\to\beta)\to(\alpha?\to\beta?)$ and $\alpha\to\sim((\sim\alpha)?)$ would carry out $\alpha?\to\alpha$ which is not reasonable.

Main *LEI* theorems are:

Theorem 2.1. *all thesis of classical logic hold to ?-free formulas in LEI.*

Corollary 2.1. *the negation symbol* $\neg$ *behaves classically for ?-free formulas in LEI:*

$\vdash (A\to B)\to((A\to\neg B)\to\neg A)$

$\vdash \neg\neg A\to A$

Theorem 2.2. *The defined symbol* $\sim$ *really behaves as classical negation*:

$\vdash (\alpha\to\beta)\to((\alpha\to\sim\beta)\to\sim\alpha)$

$\vdash \sim\sim\alpha\to\alpha$

A form of replacement theorem can be recovered with the help of a special implication introduced by definition [Pequeno & Buchsbaum 91].

Definition 2.1. *(strong implication)* $\alpha\Rightarrow\beta$ *is a short form for* $(\alpha\to\beta)\wedge(\neg\beta\to\neg\alpha)$ *and (strong double implication)* $\alpha\Leftrightarrow\beta$ *is a short form for* $(\alpha\Rightarrow\beta)\wedge(\neg\beta\Rightarrow\neg\alpha)$

Theorem 2.3. *(replacement) Let* α' *be a formula obtained from* α *by substituting* β' *for some occurrences (not necessarily all) of* β. *Then* $\Gamma\vdash\beta\Leftrightarrow\beta'$ *entails* $\Gamma\vdash\alpha\Leftrightarrow\alpha'$

Definition 2.2. *(?-closed formula) A formula is ?-closed, i.e., it is under the scope of ?, if it has one of the forms* $\alpha?$, $\neg\beta$, $\sim\beta$, $\beta\#\gamma$, *where* β *and* γ *are ?-closed formulas and* $\#\in\{\to,\ \wedge,\ \vee\}$.

Theorem 2.4. *If* α *is a ?-closed formula, then* $\vdash \alpha \Leftrightarrow \alpha?$.

A restricted version of the deduction theorem can be kept also.

Theorem 2.5. *If* $\Gamma, \alpha \vdash \beta$ *and if does not occur any free variable in the set of premisses, then* $\Gamma \vdash \alpha \rightarrow \beta$ *unless* α *is not ?-closed and the LEI rules* $R_2?$ *or* $R\sim$ *are used after the first time* α *occurs in a proof by virtue of being a premise.*

3. *Proof-theoretical issues about LEI*

Cut elimination theorem was established by Gentzen for classical and intuitionistic logics and assures that every derivation using sequent calculus can be transformed into another one with the same endsequent, in which no cuts occur [Gentzen 35]. Prawitz's Normal Form theorems assert an equivalent result but for natural deduction calculus [Prawitz 71]. Proof-theoretical issues concerned cut-elimination and normalization theorems in *LEI* will be now discussed.

Cut elimination in *LEI*, established in [Martins & Pequeno 94], was proved with a restriction: some proofs where $R_2?$ is applied aiming the propagation of ? along $\rightarrow$ to simulate ?-Modus Ponens $(\alpha?,\ \alpha \rightarrow \beta / \beta?)$. In fact, appealing to history, Gentzen transformation of his Natural Deduction calculi *NJ* and *NK*, for intuitionistic and classical logics respectively, to his correspondent *Logistic* Sequent Calculi *LJ* and *LK* had requested the use of cut rule to effectively translate elimination rules to left ones [Ungar 92]. It will enforce our argument below in favour of cut maintenance on such proofs. To illustrate this, consider, as a typical example, the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\alpha \vdash \alpha \vee \beta}}{\vdash \alpha \rightarrow (\alpha \vee \beta)}}{\vdash \alpha? \rightarrow (\alpha \vee \beta)?} \quad \dfrac{\alpha? \vdash \alpha?(\alpha \vee \beta)? \vdash (\alpha \vee \beta)?}{\alpha?, \alpha? \rightarrow (\alpha \vee \beta)? \vdash (\alpha \vee \beta)?}}{\alpha? \vdash (\alpha \vee \beta)?} \quad \dfrac{\dfrac{\dfrac{\dfrac{\beta \vdash \beta}{\beta \vdash \alpha \vee \beta}}{\vdash \beta \rightarrow (\alpha \vee \beta)}}{\vdash \beta? \rightarrow (\alpha \vee \beta)?} \quad \dfrac{\beta? \vdash \beta?(\alpha \vee \beta)? \vdash (\alpha \vee \beta)?}{\beta?, \beta? \rightarrow (\alpha \vee \beta)? \vdash (\alpha \vee \beta)?}}{\beta? \vdash (a \vee \beta?)}}{\alpha? \vee \beta? \vdash (\alpha \vee \beta)?, (\alpha \vee \beta)?}}{\alpha? \vee \beta? \vdash (\alpha \vee \beta)?}}{\vdash \alpha? \vee \beta? \rightarrow (\alpha \vee \beta)?}$$

If we translate this example in a natural deduction style we get (vide in [Martins & Pequeno 94] the straightforward derivation of *LEI* natural deduction rules from *LEI* sequent ones):

1.	$[\alpha?\vee\beta?]$	
2.	$\quad[\alpha?]$	
3.	$\quad\quad[\alpha]$	
4.	$\quad\quad\alpha\vee\beta$	
5.	$\quad\alpha\rightarrow(\alpha\vee\beta)$	$I\rightarrow(3,\ 4)$
6.	$\quad\alpha?\rightarrow(\alpha\vee\beta)?$	$I_2?(5)$
7.	$\quad(\alpha\vee\beta)?$	$E\rightarrow(2,\ 6)$
8.	$\quad[\beta?]$	
9.	$\quad\quad[\beta]$	
10.	$\quad\quad\alpha\vee\beta$	
11.	$\quad\beta\rightarrow(\alpha\vee\beta)$	
12.	$\quad\alpha?\rightarrow(\alpha\vee\beta)?$	
13.	$\quad(\alpha\vee\beta)?$	
14.	$(\alpha\vee\beta)?$	
15.	$\alpha?\vee\beta?\rightarrow(\alpha\vee\beta)?$	

In this translation, note that we do not have a characterization of a redundant inference. In fact, in step 5, for instance, we have introduced an implication that was subsequently eliminated, but only in step 7. We have an intermediate step (the application of $I_2?$ or rule $R_2?$) which cannot be permuted in order to explicit this candidate to redundancy. This proof is indeed in normal form: the analytical part corresponds to $\vee$ elimination — steps 1 to 14 — while the synthetical part is the $\rightarrow$ introduction — step 15 — and the minimum part is the conclusion of the analytical part and the premise of the synthetical one, i.e. step 14. No reduction steps $(\beta,\ \eta,\ \zeta)$ can be applied in order to get rid off redundances. In fact, Normal Form theorem was established in *LEI* [Martins & Pequeno 94] as in classical logic just adding new β and η reductions to deal with question mark rules. Nevertheless, no additional ζ reductions were necessary.

Backing to the correspondent sequent calculus proof, we can easily see that rule $R_2?$ really disallow the characterization of a redundant cut: between the elimination of $\rightarrow$ through the cut rule and the introduction of $\rightarrow$ through $R\rightarrow$ rule, we have an application of $R_2?$. Appealing to the rewriting process behind cut elimination, there is no way to be free of such intermediate rule by permuting cut upward, since this rule cannot be broken through the deduction theorem represented by $R\rightarrow$. In fact, $R\rightarrow$ is also restricted to cases analogous to the ones related to quantifiers where either the discharged premise α is ?-closed or rules $R_2?$ and $R\sim$ are not applied after the first time α appears as a premise. It is a consequence of interpreting ? as a sort of existential quantification of *hidden* variables over evidences. Hence, if the discharged premise α is not under the scope of ? (α is not ?-closed), the application of R_2? and $R\sim$ to α will not hold constant such ?-quantified hidden variables.

To end this section, we shall show that the main desirable consequences of cut-elimination will be kept even though we have maintained some special occurrences of cuts in proofs. Such cut-proofs can be framed in the following pattern:

$$\dfrac{\dfrac{\Gamma_1 \vdash \alpha \to \beta,\ \Delta_1}{\Gamma_1 \vdash \alpha? \to \beta?,\ \Delta_1} \qquad \dfrac{\Gamma_2 \vdash \alpha?,\ \Delta_2 \quad \Gamma_3,\ \beta?,\ \vdash \Delta_3}{\Gamma_2,\ \Gamma_3,\ \alpha? \to \beta? \vdash \Delta_2,\ \Delta_3}}{\Gamma_1,\ \Gamma_2,\ \Gamma_3 \vdash \Delta_1,\ \Delta_2,\ \Delta_3}$$

The desirable outcomes of cut-elimination are:

1. Consistency of this calculus: even if Γ_1 and Δ_1 were empty on the above proof, we would never prove the empty sequent $\vdash$ since it will always exist at least $\gamma \in \Gamma_2 \cup \Delta_2$ and $\delta \in \Gamma_3 \cup \Delta_3$ s.t. $\gamma \vdash \delta$ or $\vdash \gamma,\ \delta$ or $\gamma,\ \delta \vdash$ where $\gamma,\ \delta$ come from the application of operational rules to identity axioms upper to $\Gamma_2 \vdash \alpha?,\ \Delta_2$ and $\Gamma_3,\ \beta? \vdash \Delta_3$ respectively.

2. Subformula Property: observe the following example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\alpha \wedge \beta \vdash \alpha}}{\vdash (\alpha \wedge \beta) \to \alpha}}{\vdash (\alpha \wedge \beta)? \to \alpha?} \quad \dfrac{\alpha? \vdash \alpha? (\alpha \wedge \beta)? \vdash (\alpha \wedge \beta)?}{(\alpha \wedge \beta)? \to \alpha?, (\alpha \wedge \beta)? \vdash \alpha?}}{(\alpha \wedge \beta)? \vdash \alpha?} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\beta \vdash \beta}{\alpha \wedge \beta \vdash \beta}}{\vdash (\alpha \wedge \beta) \to \beta}}{\vdash (\alpha \wedge \beta)? \to \beta?} \quad \dfrac{\beta? \vdash \beta? (\alpha \wedge \beta)? \vdash (\alpha \wedge \beta)?}{(\alpha \wedge \beta)? \to \beta?, (\alpha \wedge \beta)? \vdash \beta?}}{(a \wedge \beta?)? \vdash \beta?}}{\dfrac{\dfrac{(\alpha \wedge \beta)?, (\alpha \wedge \beta)? \vdash \alpha? \wedge \beta?}{(\alpha \wedge \beta)? \vdash \alpha? \wedge \beta?}}{\vdash (\alpha \wedge \beta)? \to \alpha? \wedge \beta?}}$$

In this first example we have two occurrences of cut rule. Consider, for instance, the one where the cuted formula is $(\alpha \wedge \beta)? \to \beta?$. Note that such formula is the implicative form of the formulas which occur in the sequent conclusion (i.e., $(\alpha \wedge \beta)? \vdash \beta?$) of this rule.

$$\dfrac{\dfrac{\dfrac{\beta \vdash \beta}{\alpha \wedge \beta \vdash \beta}}{\vdash (\alpha \wedge \beta) \to \beta}}{\vdash (\alpha \wedge \beta)? \to \beta?} \qquad \dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\alpha \wedge \beta \vdash \alpha}}{\vdash (\alpha \wedge \beta) \to \alpha}}{\vdash (\alpha \wedge \beta)? \to \alpha?} \qquad \dfrac{\dfrac{(\alpha \wedge \beta)? \vdash (\alpha \wedge \beta)? \quad (\alpha \wedge \beta)? \vdash (\alpha \wedge \beta)? \quad \dfrac{\alpha? \vdash \alpha? \quad \beta? \vdash \beta?}{\alpha?,\ \beta? \vdash \alpha? \wedge \beta?}}{(\alpha \wedge \beta)? \to \alpha?,\ (\alpha \wedge \beta)? \to \beta?,\ (\alpha \wedge \beta)?,\ (\alpha \wedge \beta)? \vdash \alpha? \wedge \beta?}}{(\alpha \wedge \beta)? \to \alpha?,\ (\alpha \wedge \beta)? \to \beta?,\ (\alpha \wedge \beta)? \vdash \alpha? \wedge \beta?}$$

$$\dfrac{\vdash (\alpha \wedge \beta)? \to \alpha? \qquad (\alpha \wedge \beta)? \to \alpha?,\ (\alpha \wedge \beta)? \to \beta?,\ (\alpha \wedge \beta)? \vdash \alpha? \wedge \beta?}{(\alpha \wedge \beta)? \to \beta?,\ (\alpha \wedge \beta)? \vdash \alpha? \wedge \beta?}$$

$$\dfrac{\vdash (\alpha \wedge \beta)? \to \beta? \qquad (\alpha \wedge \beta)? \to \beta?,\ (\alpha \wedge \beta)? \vdash \alpha? \wedge \beta?}{(\alpha \wedge \beta)? \vdash \alpha? \wedge \beta?}$$

$$\dfrac{(\alpha \wedge \beta)? \vdash \alpha? \wedge \beta?}{\vdash (\alpha \wedge \beta)? \to \alpha? \wedge \beta?}$$

This second example has the same endsequent but the proof tree is somehow different. Now, the same cuted formula $(\alpha \wedge \beta)? \to \beta?$ is not the implicative form of $(\alpha \wedge \beta)? \vdash \alpha? \wedge \beta?$, the cut sequent conclusion. However, the consequent of $(\alpha \wedge \beta)? \to \beta?$, i.e. $\beta?$, can be obtained by application of $R\wedge$ over $(\alpha \wedge \beta)? \vdash \alpha? \wedge \beta?$ before applying cut. Actually, it is what happens in the first proof. Hence, whenever the cuted formula is not the implicative form of formulas in the cut sequent conclusion, you can rewrite the proof applying the rewriting steps of cut-elimination [Martins & Pequeno 94] in order to accomplish this constraint.

The last example is the following, where we consider as hypothesis $\alpha \Leftrightarrow \alpha$ and $\beta \Leftrightarrow \beta?$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha? \vdash \alpha?}{\alpha? \wedge \beta? \vdash \alpha?}}{\vdash (\alpha? \wedge \beta?) \to \alpha?}}{\vdash (\alpha? \wedge \beta?)? \to \alpha??} \qquad \dfrac{(\alpha? \wedge \beta?)? \vdash (\alpha? \wedge \beta?)? \qquad \dfrac{\alpha? \vdash \alpha?}{\alpha?? \vdash \alpha?}}{(\alpha? \wedge \beta?)? \to \alpha??,\ (\alpha? \wedge \beta?)? \vdash \alpha?}}{(\alpha? \wedge \beta?)? \vdash \alpha?} \qquad \dfrac{\Sigma 2}{(\alpha? \wedge \beta?)? \vdash \beta?}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(\alpha? \wedge \beta?)? \vdash \alpha? \qquad (\alpha? \wedge \beta?)? \vdash \beta?}{(\alpha? \wedge \beta?)?,\ (\alpha? \wedge \beta?)? \vdash \alpha? \wedge \beta?}}{(\alpha? \wedge \beta?)?^{1} \vdash \alpha? \wedge \beta?^{1}}}{(\alpha \wedge \beta)? \vdash \alpha \wedge \beta}}{\vdash (\alpha \wedge \beta)? \to (\alpha \wedge \beta)}$$

1. using replacement theorem (see theorem 2.3), and the hypothesis '$\alpha \Leftrightarrow \alpha?$' and '$\beta \Leftrightarrow \beta?$'
2. the same proof of $(\alpha? \wedge \beta?)? \vdash \alpha?$

In this example, the cuted formula is $(\alpha? \wedge \beta?)? \to \alpha??$ and the cut sequent conclusion is $(\alpha? \wedge \beta?)? \vdash \alpha?$. Although the former is not the implicative form of the latter, you can apply the replacement theorem to satisfy this restriction since we have $\alpha?? \Leftrightarrow \alpha?$ (see theorem 2.4).

To sum up, subformula property is somehow modified to the following: 'all formulas that appear in proofs where the only remaining possible cuts are those pointed above are either subformulas (or rewriting formulas) of those which occur in the endsequent or are implicative forms of subformulas (or rewriting formulas) of them. Rewriting formulas are those obtained through distribution rules or by the replacement theorem applied to strong equivalences'.

3. Inversion Principle: $R_2?$ is an anomalous rule in the sense that it does not have a left (operational) rule to confront with. Indeed, from $\alpha ? \rightarrow \beta ?$, it does not make sense to eliminate ? in order to get $\alpha \rightarrow \beta$. Therefore, we really do not want an elimination rule to deal with such case. Thus, the inversion principle, as stated in classical logic, holds to all *LEI* logical constants but to ?. It does not carry out any dangerous results. The conservative property, the main related result of the inversion principle, still remains by the use of *?-RC* (see below) which could be thought as the 'symmetrical' rule of $R_2?$.

4. Conservative Property: *prima facie*, it would seem that ? is not conservative since we have an ?-rule ($R_2?$) without any symmetrical one. In fact, we do not have an operational ?-rule 'to destroy' the effect of introducing ? over $\rightarrow$, i.e.,

$$L_3? \quad \frac{\Gamma,\ \alpha \rightarrow \beta \vdash \Delta}{\Gamma,\ \alpha ? \rightarrow \beta ? \vdash \Delta}$$

In truth, this rule will allow to prove $(\alpha ? \rightarrow \beta ?) \rightarrow (\alpha \rightarrow \beta)$, and it does not make sense. Nevertheless, ? is still a conservative operator but characterized somehow differently from Gentzen operators. In Gentzen (see [Gentzen 35] and [Prawitz 71]), each logical constant is defined locally by introduction and elimination rules in his natural deduction calculi *NJ* and *NK* or by right and left operational rules in his logistic sequent calculi *LJ* and *LK*. The introduction of any logical symbol does not presume a premise with a fixed pattern as the implicative formula $\alpha \rightarrow \beta$ in the case of $R_2?$ and $\alpha ? \rightarrow \beta ?$ in the case of $L_2?$ Therefore, all *LEI* logical symbols, with the exception of ?, obey Gentzen style.

It urges the necessity of characterizing ? properly. We have done it not only by operational ?-rules but also by *?-RC* and *?-Cut* rule. *?-RC* and *?-Cut* reflect a sort of structural property, an ordering of credibility or plausibility, associated to *?*-formulas. Thus, $\alpha \rightarrow \beta$ may be considered as stronger than $\alpha ? \rightarrow \beta ?$ in *?-RC* and α stronger than $\alpha ?$ in *?-Cut*. *Ergo,* the definition of the new logical constant *?* is characterized not only by (question mark) operational rules as in standard Gentzen logical operators, but also by the structural rule *?-RC* and the identity rule ?-*Cut.*

To see why it is still a conservative definition, we have only to analyse if the case where cut is preserved in the proof, i.e., when we use $R_2?$, still keep the conservative property of *?*, i.e., all we have introduced by means of application of $R_2?$ can be discharged by another *?*-rule.

Consider this rule:

$$\frac{\Gamma \vdash \alpha \to \beta,\ \Delta}{\Gamma \vdash \alpha? \to \beta?,\ \Delta}\ R_2?$$

As we have already said, it has not and could not have an operational symmetrical left rule 'to cancel' the introduction of *?* on the right hand side of $\vdash$. This cancellation will be done through the structural rule *?-RC.*

$$\frac{\dfrac{\Gamma \vdash \alpha \to \beta,\ \Delta}{\Gamma \vdash \alpha? \to \beta?,\ \Delta}}{\Gamma \vdash \alpha \to \beta,\ \Delta}\ ?\text{-}RC$$

Thus, the operator introduced by $R_2?$ can be eliminated by *?-RC.* We restore on the third line the same formula on the first line which was object of $R_2?$ application.

4. *Conclusions and Further Works*

The *Logic of Epistemic Inconsistency* is here presented through sequent calculus rules. It is a remarkable example of a formal system where a normal form theorem can be proved but not unrestricted cut elimination. It has been shown that a special cut-proof pattern cannot be eliminated in proofs where $R_2?$ is applied. In such circumstances, there is no way of rewriting the proof in order to become free off cut application since $R_2?$ cannot be 'split' into $R\to$ (the deduction theorem) due to existential interpretation of *?* over evidences. Fortunately, the preservation of this cut-proof does not damage the desirable outcomes of cut-elimination: 1) the calculus consistency has still been maintained; 2) subformula property, although slightly modified: the cuted remaining formula has an implicational form of endsequent subformulas. Thus, formulas in a (quasi) cut-free proof is still predictable; 3) the inversion principle has not been attested to *?* since a symmetrical left operational rule to $R_2?$ does not make sense, but 4) the conservativeness of all logical constants has been assured, even for *?* where *?-RC* plays the role of symmetrical rule to R_2?

As future work, we aim to extend sequent calculus treatment to those dynamic aspects of incomplete knowledge reasoning covered by *IDL*. The idea is to get an uniform formalization encompassing all the aspects of this kind of reasoning.

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