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A COMPLETENESS THEOREM IN DEONTIC LOGIC WITH SYSTEMATIC FRAME CONSTANTS*

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Contents

1. Introduction

The purpose of this paper is to give semantically sound and complete axiomatizations of all members in a certain infinite hierarchy of systems of *dyadic* deontic logic [logics of *conditional* obligation and permission, if you prefer]. In the semantics of any such system there is, in addition to a family of relations of "deontic accessibility" among possible worlds, a *weak preference relation* 'is at least as ideal as' on the set of such worlds [as in Hansson (1969)]. Using that preference relation, we are able to distinguish various "levels of perfection" in the models of our systems, and each level of perfection will be represented in the object-language of the systems by a so-called *systematic frame constant*. The truth conditions and axioms governing any such constants will be seen to play a highly important, characteristic role in our axiomatization.

The plan of the paper is as follows. After having presented the syntax, semantics and axiomatic proof theory of an infinite sequence G^*m [m = 1,2,...] of dyadic deontic logics in Section 2, we introduce the notion of a canonical G^*m-structure in Section 3, where we also prove four lemmata on such structures. These lemmata suffice to establish, in Section 4, the desired completeness of each system G^*m. In Section 5, finally, we con-

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sider two weaker logics of conditional obligation, for which the completeness problem remains open. With respect to one of those logics, however, there are excellent reasons for believing that the result of adding to it a so-called "infinitary" rule of proof [i.e. one having an infinite number of premisses] is complete relative to our proposed semantics in at least a certain weak sense.

Some technical and historical remarks to round this introductory section.

In the completeness proof we use the familiar Henkin technique of maximal consistent sets of sentences (formulas), as transferred to modal logic in Makinson (1966) and Lemmon & Scott (1966). Our method for modelling logics of conditional obligation is somewhat special in that it treats the connective for conditional obligation fundamentally as what Chellas (1975) calls a *sententially indexed* modality [see Section 2 *infra* and Chellas (1975), Note 14]. As to the technique of systematic frame constants adopted here, it seems to originate, as far as deontic logic is concerned, with my Åqvist (1984) and Åqvist (1987). In §23 of the latter contribution a not very successful attempt was made to deal with the completeness problem for the two systems discussed in Section 5 below; but, in the light of the present paper, it was certainly on the right track. Some further insights into the potentialities of this technique were later gained in two papers by the present author on discrete tense logic, Åqvist (1991) and (1992).

Finally, we observe that the need for logics of *conditional* obligation and permission was realized at quite an early stage in the development of modern deontic logic: thus, the dyadic deontic logic of von Wright (1956) was proposed as a reaction to the Prior (1954) Paradoxes of Commitment ("derived obligation"), and that of von Wright (1964) as a reaction to the Chisholm (1963) Contrary-to-Duty Imperative Paradox. In fact, the main interest of dyadic deontic logics may be said to consist in their capacity to deal with the phenomenon of *reparational* ["secondary", "contrary-to-duty"] obligations arising in cases where a primary obligation has been violated. Moreover, the topic is interestingly related to Conditional Logic as well as to Preference Theory, as witnessed by its later history. See again my Åqvist (1984) and (1987), where a number of additional useful references can be found. In Section 8 of those two contributions, however, we argued, like van Eck (1981), that the dyadic approach à la Hansson (1969) appears unable to handle certain interesting problem areas, which strongly indicate the need for a successful combination of deontic and *temporal* logic [this point was indeed made already by Spohn (1975) in his excellent examination of the Hansson (1969) dyadic deontic logic DSDL3, of which the systems dealt with in the present paper are straightforward extensions].

Nevertheless, in the opinion of the present writer, this circumstance does not in any way detract from the interest of studying Hansson-style logics of conditional obligation as such. As appears from the combined dyadic-deontic-temporal logic DARB of Åqvist & Hoepelman (1981) and Åqvist (1991a), the situation is rather the opposite one.

2. *The systems G*m: syntax, semantics and proof theory*

The *language* of the systems G^*m (m any positive integer) has, in addition to an at most denumerable set Prop of propositional variables and the usual Boolean sentential connectives (including the constants *verum* and *falsum*, i.e. $\top$ and $\bot$), the following characteristic primitive *logical connectives*: O (for conditional obligation), P (for conditional permission), N (for universal necessity), M (for universal possibility), and a family $\{Q_i\}(1 \leq i < \omega)$ of *systematic frame constants*, which are indexed by the set of positive integers. We take the Q_i to represent different "levels of perfection" in the models of our systems, as explained below.

The set Sent of well formed sentences (formulas) is then defined in the straightforward way. We note that there are no restrictions as to iterations of dyadic deontic operators or modal ones. Moreover, we write O_BA [P_BA] to render the ordinary language locution "if B, then it ought to be that A" ["if B, then it is permitted that A"]. We prefer this style of notation to the current one $O(A/B)[P(A/B)]$, because (i) it is parenthesis-free, and (ii) the reading goes from left to right, and not the other way around.

Let us next turn to the *semantics* for the systems G^*m. By a *G^*m-structure* we understand an ordered quintuple

$$\mu = (W, R, \geq, m, V)$$

where:

(i) $W \neq \varnothing$ [W is a non-empty set of "possible worlds"].
(ii) R: Sent $\rightarrow$ Pow($W \times W$) [R is a function which assigns to each sentence a binary relation of deontic accessibility on W].
(iii) $\geq \subseteq W \times W$ [$\geq$ is a binary, weak preference relation on W].
(iv) m is the positive integer under consideration.
(v) V: Prop $\rightarrow$ Pow(W) [V is a valuation function which to each propositional variable assigns a subset of W].

We can now tell what it means for any sentence A to be *true at* a point ("world") x ($\in W$) *in* a G^*m-structure μ [in symbols: $\mu, x \models A$], starting out with obvious clauses like

$\mu, x \models p$ iff $x \in V(p)$ (for any p in the set Prop)
$\mu, x \models \top$
not: $\mu, x \models \bot$

and so on for molecular sentences having Boolean connectives as their principal sign. We then handle sentences having the characteristic G^*m-connectives as their principal sign as follows:

$\mu, x \models O_B A$ iff for every y in W with $x R_B y$: $\mu, y \models A$
$\mu, x \models P_B A$ iff for some y in W with $x R_B y$: $\mu, y \models A$
$\mu, x \models NA$ iff for each y in W: $\mu, y \models A$
$\mu, x \models MA$ iff for some y in W: $\mu, y \models A$.

Finally, we have to provide truth conditions for the frame constants Q_i. In order to do so, let us first define a denumerably infinite sequence opt_i (i = 1,2,...) of subsets of W by the following recursion:

$$opt_i = \begin{cases} \{x \in W : (\text{for each } y \in W)\ x \geq y\}, & \text{if } i = 1 \\ \{x \in W - (opt_1 \cup \ldots \cup opt_{i-1}) : (\text{for each } y \in W - \\ \quad (opt_1 \cup \ldots \cup opt_{i-1}))\ x \geq y\}, & \text{if } i > 1 \end{cases}$$

Intuitively, opt_1 is the set of "best" (i.e. $\geq$-maximal) members of W as a whole, opt_2 is the set of best members of W-opt_1 (the "second best" members of W), opt_3 is the set of best members of W - ($opt_1 \cup opt_2$); and so on. The truth condition for the Q_i is then the following:

$\mu, x \models Q_i$ iff $x \in opt_i$, for all positive integers i.

We now focus our attention on a special kind of G^*m-structures called "*G^*m-models*". By a *G^*m-model* we shall mean any G^*m-structure μ, where R, $\geq$, and m satisfy the following additional conditions (for each A in Sent and any x,y in W):

$\gamma^{0.}$ $x R_A y$ iff $\mu, y \models A$ and for each z in W: if $\mu, z \models A$, then $y \geq z$.
Trans. $\geq$ is transitive in W.

LimAss. Every non-empty subset of W has at least one $\geq$-maximal element, in symbols:
$(\forall X \subseteq W)(X \neq \varnothing \supset \{x \in X : (\forall y \in X) x \geq y\} \neq \varnothing)$.

Exactly m Levels of Perfection. This condition requires the set $\{opt_1, opt_2, ..., opt_m\}$ to be a *partition* of W in the familiar sense that

(a) for all i, j with $1 \leq i \neq j \leq m : opt_i \cap opt_j = \varnothing$, and
(b) $opt_1 \cup ... \cup opt_m = W$.

Finally, we require our opt-classes to satisfy

(c) for each i with $1 \leq i \leq m$: $opt_i \neq \varnothing$, and
(d) for each positive integer i with $i > m$: $opt_i = \varnothing$.

Our definition of the notion of a G^*m-model is thereby complete. As usual, we say that a sentence A is *G^*m-valid* iff $\mu, x \models A$ for all G^*m-models μ and all points x in W. And we say that a set Γ of sentences is *G^*m-satisfiable* iff there exists a G^*m-model μ and a member x of W such that for all sentences A in Γ: $\mu, x \models A$. Clearly, for every positive integer m, A is G^*m-valid iff the singleton $\{\neg A\}$ is not G^*m-satisfiable.

It is now time to consider the *proof theory* of the systems G^*m. Thus, for any positive integer m, the *axiomatic system* G^*m is determined by the following rule of inference, rule of proof, and axiom schemata (where we use 'i', 'j' as variables over the positive integers):

Rule of inference

mp (modus ponens) $\dfrac{A, A \to B}{B}$

Rule of proof

nec (necessitation for N) $\dfrac{A}{NA}$

[For the distinction between a rule of *inference* and a rule of *proof*, see e.g. Sundholm (1983)].

Axiom schemata

a0 All tautologies over Sent
a1 $P_B A \leftrightarrow \neg O_B \neg A$
a2 $O_B(A \to C) \to (O_B A \to O_B C)$

a3 $O_B A \rightarrow NO_B A$
a4 $NA \rightarrow O_B A$
a5 S5-schemata for N,M (i.e. $MA \leftrightarrow \neg N \neg A$, $N(A \rightarrow B) \rightarrow (NA \rightarrow NB)$, $NA \rightarrow A$, $NA \rightarrow NNA$, $MNA \rightarrow A$)
$\alpha 0$ $N(A \leftrightarrow B) \rightarrow (O_A C \leftrightarrow O_B C)$
$\alpha 1$ $O_A A$
$\alpha 2$ $O_{A \wedge B} C \rightarrow O_A(B \rightarrow C)$
$\alpha 3$ $MA \rightarrow (O_A B \rightarrow P_A B)$
$\alpha 4$ $P_A B \rightarrow (O_A(B \rightarrow C) \rightarrow O_{A \wedge B} C)$
$\alpha 5$ $Q_i \rightarrow \neg Q_j$, for all $1 \leq i \neq j \leq m$
$\alpha 6$ $P_B Q_i \rightarrow ((Q_i \vee \ldots \vee Q_{i-1}) \rightarrow \neg B)$, for all i with $1 < i \leq m$
$\alpha 7$ $Q_1 \rightarrow (O_B A \rightarrow (B \rightarrow A))$
$\alpha 8$ $(Q_i \wedge O_B A \wedge B \wedge \neg A) \rightarrow P_B(Q_1 \vee \ldots \vee Q_{i-1})$, for $1 < i \leq m$
$\alpha 9$ $Q_1 \vee \ldots \vee Q_m$
$\alpha 10$ $MQ_1 \wedge \ldots \wedge MQ_m$
$\alpha 11$ $\neg Q_i$, for all i such that $m < i < \omega$.

As usual, the above axiom schemata and rules determine syntactic notions of *G*m-provability* and *G*m-deducibility* as follows. We say that a sentence A is *G*m-provable* [in symbols: $\vdash_{G*m} A$] iff A belongs to the smallest subset of Sent which (i) contains every instance of a0, ..., a5, $\alpha 0$, ..., $\alpha 11$ as its member, and which (ii) is closed under the rule of inference mp and the rule of proof nec. Again, we say that the sentence A is *G*m-deducible from* the set Γ ($\subseteq$ Sent) of *assumptions* [in symbols: $\Gamma \vdash_{G*m} A$] iff there are sentences $B_1, \ldots, B_k$ in Γ, for some natural number $k \geq 0$, such that $\vdash_{G*m} (B_1 \wedge \ldots \wedge B_k) \rightarrow A$.

Moreover, letting $\Gamma \subseteq$ Sent, we say that Γ is *G*m-inconsistent* iff $\Gamma \vdash_{G*m} \bot$, and *G*m-consistent* otherwise. Finally, we say that Γ is *maximal G*m-consistent* iff Γ is $G*m$-consistent and, for each sentence A, either $A \in \Gamma$ or $\neg A \in \Gamma$; where this latter condition is known as requiring Γ to be *negation-complete*.

We leave to the reader the task of verifying the following result, in the absence of which our axiomatic theories would be pointless:

Soundness Theorem.

Weak version: Every $G*m$-provable sentence is $G*m$-valid.

Strong version: Every $G*m$-satisfiable set of sentences is $G*m$-consistent.

Both versions are to be established for any positive integer m.

3. *Canonical G*m-structures*: *some basic results*

Definition.
For any positive integer m, let WG^*m be the set of all maximal G^*m-consistent sets of sentences. Let w be a fixed element of WG^*m. Define the *canonical G*m-structure generated by* w as the quintuple

$$\mu^w = (W, R, \geq, m, V)$$

where:

(i) $W = \{x \in WG^*m$ for each sentence A, if $NA \in w$, then $A \in x\}$.
(ii) $R =$ the function from Sent into Pow ($W \times W$) such that for each B in Sent and all x,y in W: $x R_B y$ iff for all C in Sent, if $O_B C \in x$, then $C \in y$.
(iii) $\geq$ to be defined in a moment.
(iv) m = the positive integer under consideration.
(v) V = the valuation function such that for all p in Prop: $V(p) = \{x \in W: p \in x\}$.

We still have to define $\geq$; to that purpose we appeal to the following preparatory result:

Justification Lemma.
Let W be defined as in clause (i) *supra*. Then, for each $x \in W$ there is *exactly one* positive integer i with $1 \leq i \leq m$ such that $Q_i \in x$.

Proof.

Existence. Since x is maximal G^*m-consistent [$x \in W$] and $\vdash Q_1 \vee ... \vee Q_m$ [by $\alpha 9$], the disjunction $Q_1 \vee ... \vee Q_m \in x$, so that at least one of its disjuncts $Q_i [1 \leq i \leq m]$ must be in x, as desired.

Uniqueness. Immediate by the fact that every instance of $\alpha 5$ is in x. □

On the basis of this Lemma, we define a "ranking" function r from W into the closed interval $[1, m]$ of positive integers as follows: for each x in W,

$$r(x) = \textit{the } i, \text{ with } 1 \leq i \leq m, \text{ such that } Q_i \in x.$$

We now supply the missing clause (iii) in the definition of the canonical G^*m-structure generated by w:

(iii) $\geq$ = the binary relation on W such that for all x,y in W: $x \geq y$ iff $r(x) \leq r(y)$.

Our desired completeness result for the systems G^*m ($m = 1,2,...$) can then be seen to follow from three Lemmata on canonical G^*m-structures, to which we must now pay attention.

Saturation Lemma.
Let μ^w be defined as above. Then W is such that for all A, B in Sent, all x in W, and all positive integers i with $1 \leq i \leq m$:

(i)	$NA \in x$	iff for all y in W, $A \in y$.
(ii)	$MA \in x$	iff for some y in W, $A \in y$.
(iii)	$O_BA \in x$	iff for all y in W with $x R_B y$, $A \in y$.
(iv)	$P_BA \in x$	iff for some y in W with $x R_B y$, $A \in y$.
(v)	$Q_i \in x$	iff $x \in opt_i$

Proof.
Ad (i)-(iv). The non-trivial parts of these clauses can be established by an application to G^*m of Lemma 3 in Makinson (1966) p. 382; in that application we appeal to our axiom schemata a2-a5, the rule of proof nec (for N), and the easily derived rule of proof A/O_BA.

Ad (v). We begin by verifying the *useful facts* that $Q_{r(x)} \in x$, and that $Q_i \in x$ iff $i = r(x)$ [for all x, i at issue].
The proof of (v) then proceeds by the following induction on i.

Basis. $i = 1$. We are to show that $Q_1 \in x$ iff $x \in opt_1$. Starting with the "only if" direction, we observe that the counterassumption [$Q_1 \in x$, $x \notin opt_1$] implies, by our useful facts and relevant definitions, both that $r(x) = 1$ and that $r(x) > 1$. Contradiction. As for the converse direction, the counterassumption [$x \in opt_1$, $Q_1 \notin x$] implies, by our second useful fact, that $r(x) > 1$. Again, by axiom schema $\alpha 10$, we have $MQ_1 \in x$ [x maximal consistent], so that, by clause (ii) of the present Lemma, $Q_1 \in y$ for some y in W, whence $r(y) = 1$ by our second useful fact. Hence, for some y in W, $r(x) > r(y)$, but this result contradicts the assumption that $x \in opt_1$ [see the definition of opt_1 *supra* and that of $\geq$ in the canonical structure μ^w].

Induction Step. $i > 1$. Assume the inductive hypothesis to the effect that, for all y in W and all j with $1 \leq j \leq i$-1, $Q_j \in y$ iff $y \in opt_j$.
We first deal with the left-to-right direction in clause (v), and make the counterassumption that $Q_i \in x$ whilst $x \notin opt_i$. Then, by our useful facts,

$r(x) = i$ and, by axiom schema $\alpha 5$ together with the inductive hypothesis, $Q_j \notin x$ and $x \notin opt_j$ for all j with $1 \leq j \leq i$-1. Hence, by the definition of opt_i, there must be in W a y such that y belongs to none of the sets $opt_1, ..., opt_{i-1}$, and with $r(x) > r(y)$. But the former condition implies, by the inductive hypothesis and our useful facts, that $r(y)$ cannot be among the numbers 1, ..., i-1, while the latter condition precisely implies that $r(y)$ must be among those numbers, since $r(x) = i$. Contradiction.

For the right-to-left direction in clause (v), make the counterassumption that $x \in opt_i$ whilst $Q_i \notin x$. By the definition of opt_i we obtain that $x \notin opt_1$ and ... and $x \notin opt_{i-1}$ so that, by the inductive hypothesis, $Q_1 \notin x$ and ... and $Q_{i-1} \notin x$, whence, by our useful facts, $r(x) \neq 1$ and ... and $r(x) \neq i$-1. Since $Q_i \notin x$ and $r(x) \neq i$, we conclude that $r(x) > i$. Also, by axiom schema $\alpha 10$, we have $M\,Q_i \in x$ [x maximal consistent], whence, by clause (ii) of the present Lemma, there is in W a y such that $Q_i \in y$, $r(y) = i$, and, by $\alpha 5$, $Q_j \notin y$ for $j = 1, ..., i$-1. Hence, by the inductive hypothesis, there is a y in W-($opt_1 \cup ... \cup opt_{i-1}$) with $r(x) > r(y)$, which result contradicts the initial assumption that $x \in opt_i$ [just check the relevant definitions]. □

Coincidence Lemma (to the effect that, as applied to any sentences, the notions of *truth* and *membership* coincide with respect to the points in generated canonical G^*m-structures).

Let w be any fixed maximal G^*m-consistent set of sentences, and let $\mu^w = (W, R, \geq, m, V)$ be the canonical G^*m-structure generated by w. Then, for each A in Sent and each x in W,

$$\mu^w, x \models A \text{ iff } A \in x.$$

Proof. By induction on the length of A.

The most exciting case in the induction basis is the one where A is some systematic frame constant Q_i with $1 \leq i < \omega$. Assume first that $1 \leq i \leq m$. Then we have the following chain of equivalences:

$$\mu^w, x \models Q_i \text{ iff } x \in opt_i \text{ iff } Q_i \in x.$$

Here, the first "iff" holds by virtue of the truth condition for Q_i, and the second "iff" by clause (v) of the just established Saturation Lemma, whence the desired result. Assume next that $m < i < \omega$. Then we easily verify that $x \notin opt_i$ and $Q_i \notin x$ for any such i and any x in W; use clause (v) again as well as axioms $\alpha 9$ and $\alpha 11$.

As for the interesting cases in the induction step, they go through nicely by virtue of clauses (i)-(iv) in the Saturation Lemma. □

Verification Lemma (where a number of remaining points are verified). As defined, the canonical structure μ^w is a G^*m-model.

Proof. Leaving the somewhat complicated condition γ^0 for the moment, we observe that the *transitivity* in W of $\geq$ is immediate by the definition of $\geq$ in canonical structures. By the same definition, the satisfaction of *LimAss* is immediate as well. We consider next the conditions (a)-(d) listed under *Exactly* m *Levels of Perfection.*

Ad (a). By clause (v) of the Saturation Lemma, the counterassumption to (a) implies that, for some i,j with $1 \leq i \neq j \leq m$ and some x in W, we have both $Q_i \in x$ and $Q_j \in x$. But, by axiom schema $\alpha 5$, this contradicts the consistency of x.

Ad (b). The interesting task is to show that W is a subset of $opt_1 \cup ... \cup opt_m$. Suppose it is not. Then, for some x in W, we have $x \notin opt_1$ and ... and $x \notin opt_m$, whence, by clause (v) of the Saturation Lemma, $Q_1 \notin x$ and ... and $Q_m \notin x$. Hence, by the maximal consistency of x, we get $\neg Q_1 \wedge ... \wedge \neg Q_m \in x$, which is impossible by the fact that $\alpha 9$ $[Q_1 \vee ... \vee Q_m]$ is in x.

Ad (c). Use axiom $\alpha 10$ together with clauses (ii) and (v) in the Saturation Lemma!

Ad (d). By the definition of opt_i, the counterassumption to (d) implies that W is not included in $opt_1 \cup ... \cup opt_m$. But this is impossible by our argument for (b) above.

We still have to verify that the characteristic condition γ^0, relating R to $\geq$, is satisfied by μ^w. The proof will *inter alia* illustrate the usefulness of the axiom schemata $\alpha 1$ and $\alpha 6$-$\alpha 8$. In the proof we shall use &, $\supset$, $\forall$, $\exists$, etc. as metalinguistic shorthands with their familiar meanings and use 'x', 'y', 'z' as variables over W. We are then to establish:

$$\gamma^0.\ x R_A y \text{ iff } \mu^w, y \models A \ \&\ \forall z(\mu^w, z \models A \supset y \geq z).$$

Left-to-right: Assume for any A in Sent and any x,y in W:

1. $x R_A y$	hypothesis
Then:	
2. $\forall C(O_A C \in x \supset C \in y)$	from 1 by the definition of R in μ^w
3. $O_A A \in x$	$\alpha 1$, x max cons since $x \in W$
4. $A \in y$	2, 3, universal instantiation, mp
5. $\exists z(A \in z \ \&\ r(y) > r(z))$	hypothesis (to be reduced *ad absurdum*)

6. $A \in z \,\&\, r(y) > r(z)$ hypothesis for existential instantiation ($1 \leq r(y), r(z) \leq m$, since $y,z \in W$)

Let $r(y) = i$ and $r(z) = j$ ($1 \leq i,j \leq m$):

7. $Q_i \in y \,\&\, Q_j \in z \,\&\, i > j$ useful facts, 6
8. $P_A Q_i \in y$ or $\neg P_A Q_i \in y$ y max cons since $y \in W$
9. $P_A Q_i \in y$ hypothesis (= first disjunct in 8)
10. $P_A Q_i \in z$ from 9 by the fact that $\vdash P_A B \rightarrow NP_A B$, whence $NP_A Q_i \in y$; $z \in W$
11. $P_A Q_i \rightarrow ((Q_1 \vee \ldots \vee Q_{i-1}) \rightarrow \neg A) \in z$ $\alpha 6$, $z \in$ W, $i > 1$ by 7
12. $Q_1 \vee \ldots \vee Q_{i-1}) \rightarrow \neg A \in z$ 10, 11, $z \in W$
13. $Q_1 \vee \ldots \vee Q_{i-1} \in z$ from the second and third conjuncts in 7, whence $j \in \{1, \ldots, i\text{-}1\}$
14. $\neg A \in z$ 12, 13, $z \in W$
15. Contradiction by the first conjunct in 6, 14
16. $\neg P_A Q_i \in y$ hypothesis (= second disjunct in 8)
17. $O_A \neg Q_i \in y$ 16, a1, $y \in W$
18. $NO_A \neg Q_i \in y$ 17, a3, $y \in W$
19. $O_A \neg Q_i \in x$ 18, $x \in W$
20. $\neg Q_i \in y$ 2, 19, univ inst and mp
21. Contradiction from 20 and the first conjunct in 7

Thus, discharging 16, 9 and 6, the hypothesis 5 is reduced *ad absurdum*. Hence:

22. $\forall z(A \in z \supset r(y) \leq r(z))$ by negation introduction and some trivial transformations

Then, 4 and 22 yield the desired conclusion by the Coincidence Lemma and the definition of $\geq$.

Right-to-left: Assume for any A in Sent and any x,y in W:

1. not: $x R_A y$ hypothesis

Then [we want to derive the negation of the right member of γ^0]:

2. $\exists C(O_A C \in x \,\&\, C \notin y)$ from 1 by the definition of R in μ^w
3. $O_A C \in x \,\&\, C \notin y$ hypothesis for existential instantiation

Let $r(y) = k$ [$1 \leq k \leq m$]:

4. $Q_k \in y$ useful facts
5. $k = 1$ hypothesis
6. $A \in y$ hypothesis
7. $Q_1 \in y$ 4, 5, logic of =
8. $\neg C \in y$ second conjunct in 3, y max cons since $y \in W$
9. $Q_1 \rightarrow (O_A C \rightarrow (A \rightarrow C)) \in y$ $\alpha 7$, $y \in W$
10. $C \in y$ $O_A C \in y$ by 3, a3 and clause (i) of the Saturation Lemma; 6, 7 and 9, $y \in W$

11. Contradiction 8, 10
12. $A \notin y$ from the deduction 6-11, discharging 6
13. $A \notin y$ or $\exists z(A \in z \,\&\, r(y) > r(z))$ from 12 by disjunction introduction

13 is "almost" our desired conclusion in this case of $k = 1$. Again:

14. $k > 1$ hypothesis
15. $A \in y$ hypothesis
16. $(Q_k \wedge O_A C \wedge A \wedge \neg C) \in y$ $O_A C \in y$ as in step 10; 4, 8, 15
17. $P_A(Q_1 \vee ... \vee Q_{k-1}) \in y$ 14, 16, $\alpha 8$, $y \in W$
18. $P_A Q_1 \in y$ or ... or $P_A Q_{k-1} \in y$ immediate from 17 by P_A being distributive over $\vee$

Consider any j such that $j \in \{1, ..., k\text{-}1\}$ and assume:

19. $P_A Q_j \in y$ hypothesis
20. $\exists z(y R_A z \,\&\, Q_j \in z)$ from 19 by the Saturation Lemma: (iv)
21. $\exists z(A \in z \,\&\, Q_j \in z)$ from 20 by the definition of R and $\alpha 1$
22. $\exists z(A \in z \,\&\, r(y) > r(z))$ $r(y) = k > j = r(z)$, 21

Now, since the deduction 19-22 goes through for all $j \in \{1, ..., k\text{-}1\}$, 22 follows from 18 by a step of disjunction elimination, where all the k-1 hypotheses 19 are discharged. Hence:

23. $A \notin y$ or $\exists z(A \in z \,\&\, r(y) > r(z))$ from the deduction 15-22 by conditional proof, discharging 15 etc.

Then, 13 and 23 yield the desired conclusion by the Coincidence Lemma and the definition of $\geq$: we discharge the hypotheses 5 and 14 by another step of disjunction elimination and the hypothesis 3 by existential instantiation.

This completes the proof of the right-to-left direction in the verification that γ^0 holds in μ^w, as well as that of the Verification Lemma as a whole. □

4. *Completeness of the axiomatic systems G^*m* $[m = 1,2,...]$

Weak version: Every G^*m-valid sentence is G^*m-provable.
Strong version: Every G^*m-consistent set of sentences is G^*m-satisfiable.

Proof. As the weak version is immediate from the strong one, let us concentrate on the latter.

Let Γ be any G^*m-consistent set of sentences. By Lindenbaum's Lemma Γ has a maximal G^*m-consistent extension, call it Γ_ω. Form the canonical G^*m-structure generated by Γ_ω, i.e. the structure μ^{Γ_ω} as defined *supra*. By the Verification Lemma, μ^{Γ_ω} is a G^*m-model. By the Coincidence Lemma, we obtain in particular that for each sentence A:

$$\mu^{\Gamma_\omega}, \Gamma_\omega \models A \text{ iff } A \in \Gamma_\omega$$

since Γ_ω clearly belongs to the "universe" W of μ^{Γ_ω} [S5 for N]. Hence, since $\Gamma \subseteq \Gamma_\omega$, we have $\mu^{\Gamma_\omega}, \Gamma_\omega \models A$ for every $A \in \Gamma$. In other words, assuming Γ to be any G^*m-consistent set of sentences, we have constructed a G^*m-model, viz. μ^{Γ_ω}, such that for some x in its universe W, viz. Γ_ω, $\mu^{\Gamma_\omega}, x \models A$ for each A in Γ; i.e. we have shown Γ to be G^*m-satisfiable. □

5. *Two weaker systems: G* and G*

We close the present paper by considering two axiomatic systems, for which the completeness problem remains open.

The first of these systems, called G* *simpliciter*, has the same language as any of the G^*m, the same rule of inference mp, and the same rule of proof nec (for N). Moreover, the axiom schemata a0-a5, $\alpha 0$-$\alpha 4$, and $\alpha 7$ remain untouched in G*, whilst the proviso on $\alpha 5$ now reads: "for all i,j with $1 \leq i \neq j < \omega$", and the proviso on $\alpha 6$ and $\alpha 8$ reads: "for all i with $1 < i < \omega$". [Clearly, these schemata, with provisos thus extended, were provable already in any G^*m, due to $\alpha 11$]. Finally, the axiom schemata $\alpha 9$, $\alpha 10$ and $\alpha 11$ are dropped altogether from the present axiomatic G*.

The second system, called G, is even weaker than G*, because there are no systematic frame constants Q_i in its primitive logical vocabulary at all. Hence, the axiom schemata of G are just a0-a5 and $\alpha 0$-$\alpha 4$, i.e. what remains after we have dropped every schema in any G^*m containing occurrences of frame constants. The rules of inference and proof remain in G.

The main intuition behind these weaker systems, especially G*, is the following: we don't want to assume any longer that there are *exactly m* levels of perfection ["opt-classes"] in every model of the system; instead, we want to allow for *variation* in the number of perfection-levels in such models. This leads to the following tentative semantics for G*. First of all, we drop the index m in the definition of a G*-structure. The truth definition relative to G*-structures remains as given above. In the

definition of a G*-model, the essential changes pertain to the conditions (a)-(d) listed under *Exactly m Levels of Perfection*:
we keep (a) in the form

(a') for all i,j with $1 \leq i \neq j < \omega$: $opt_i \cap opt_j = \varnothing$,

and drop (b)-(d) altogether.

Now, although we could safely assert a moment ago that the completeness problem remains open for the systems G* and G as just described, there is an additional remark concerning G* to be made here. Perhaps we dropped axiom $\alpha 10$ and the matching semantical condition (c) too hastily, i.e. without trying to find adequate substitutes for them. For, according to the semantics of G* just sketched, the following condition (c') apparently holds in any G*-model by the construction of the classes opt_i ($i = 1,2,\ldots$) together with LimAss:

$$(c')\begin{cases} opt_1 \neq \varnothing \\ opt_i \neq \varnothing \text{ implies } opt_{i-1} \neq \varnothing, \text{ for all } i \text{ with } 1 < i < \omega \end{cases}$$

This condition suggests that in the axiomatics for G* we replace schema $\alpha 10$ by the following:

$$\alpha 10' \begin{cases} MQ_1 \\ MQ_i \rightarrow MQ_{i-1}, \text{ for all } i \text{ with } 1 < i < \omega \end{cases}$$

Clearly, $\alpha 10'$ was provable already in any G^*m; in like manner, (c') held in every G^*m-model, for any $m = 1,2,\ldots$.

Given the present amended formulation of the axiomatic system G*, can we claim it to be complete with respect to the present semantics for it? I think the answer is: almost, but not quite. More precisely, I think the following is a reasonable conjecture:

Let G*inf be the result of adding this *infinitary* rule of proof to G*:

$$inf \quad \frac{\vdash Q_i \rightarrow A \text{ for all positive integers } i}{\vdash A}$$

This rule of proof may be characterized as a sort of deontic analogue of the Gabbay (1981) irreflexivity rule in tense logic. My present conjecture is then to the effect that G*inf is *weakly* complete in the sense that every G*-valid sentence is provable in G*inf. The detailed proof of this claim must be deferred to another occasion, however.

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