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AN ALGEBRAIC ANALYSIS OF THE LOGICAL FORM OF PROPOSITIONS

Daniel VANDERVEKEN and Marek NOWAK

The main purpose of this paper is to use algebraic methods in order to formulate a new simple logic of propositions which is adequate for the purposes of universal grammar. Recently many logicians and linguists have used and improved the formalisms of modern logic to interpret directly or after translation important fragments of ordinary language. Thus Cresswell [CRE75], Kaplan [KAP70], Lewis [LEW72], Montague [MON74], Prior [PRI67] and Vanderveken [VAN90-1] have formulated philosophical logics of sense, denotation, time, modality or force which are contributions to the foundations of the formal semantics of English. Like Montague and others, we think that there are no important theoretical differences between natural and formal languages.

However, we do not believe that the primary units of meaning in the use and comprehension are isolated propositions with truth conditions. On the contrary, following Austin [AUS62], Searle [SEA69] and other philosophers of language, we think that the meanings of utterances are complete speech acts of the type called *illocutionary acts* like assertions, requests, and promises that speakers intend to perform and communicate in speaking. Most elementary illocutionary acts are of the form F(P): they consist of a *force* F with a propositional content P. Thus as Searle and Vanderveken [SEA85] have argued repeatedly, every elementary sentence (whenever its logical form is fully analysed) contains an illocutionary force marker in addition to a clause expressing a proposition. From a linguistic point of view, the most common syntactic features of force markers are the verbal mood, the sentencial type and other features such as the intonation contour (when the utterance is oral) or the punctuation signs (when it is written). Consequently, if we want to interpret adequately all syntactic types of sentences (and not only declarative sentences) and to formalize practical as well as theoretical inferences in formal semantics, it is better to take into account the following fact: any proposition P that is the sense of an elementary sentence in the context of a model must also be the possible content of an illocutionary act. Indeed it is part of the meaning of that sentence that it can be used literally to perform a speech act of form F(P) in that context.

On this view, a natural logic of propositions that is adequate for formal semantics must take into account their double nature. On one hand, propositions are *senses of sentences with truth values*. On the other hand, they are also the *contents of conceptual thoughts such as speech acts and attitudes*. In particular, it is necessary to formulate a logic of propositions that is compatible with the general principle of their expressibility in language use. According to that principle, any proposition that is the sense of a sentence in a context of utterance is always *expressible* in the performance of an illocutionary act. Now it is clear that the human beings who use and understand natural languages have restricted cognitive abilities. They can only utter finitely long sentences and make a finite number of acts of reference and of predication in each context. Consequently, there are cognitive constraints on an adequate analysis of the logical form of propositions. For example, a proposition must have a finite number of propositional constituents. Otherwise, we could not apprehend it in an act of thought. Most propositional logics until now have unfortunately neglected such cognitive criteria of adequacy so that formal semantic theories of natural languages often do not account for the fact that these languages are human.

A simple natural logic of propositions satisfying these requirements has already been formulated in [VAN91] and [VAND94] using axiomatic as well as model-theoretical methods. That new logic of propositions is justified by a large philosophical discussion on the foundations and formal ontology of semantics.

In this paper, we will use algebraic methods to describe the logical form of propositions advocated in this new logic. In our approach, as in Cresswell's hyperintensional logic and Parry's logic of analytic implication [PAR33], a proposition conceived as the sense of a sentence has a *structure of constituents* in addition to truth conditions. From a logical point of view, each proposition is an ordered pair, whose first element, called its *content*, is a finite set of so-called atomic propositions that represents how the propositional constituents are related by predication. The second element of a proposition represents how the truth conditions of that proposition are determined from the truth possibilities of its atomic propositions. It is the set of all truth value assignments to atomic propositions that make that proposition true.

Our new logic of propositions differs from other non classical logics of propositions under two aspects:

First, the content of a proposition is not like in Parry's analysis just the set of all senses that are propositional constituents. It is a finite set of atomic propositions that also represents how the propositional constituents are related by predication in the proposition. Thus, for example, we distinguish the proposition that John loves or does not love Mary and the proposition that Mary loves or does not love John, even if these two propositions have the same propositional constituents and the same truth values in the same possible worlds.

Second, unlike other non classical logics that assign to propositions a content in addition to the truth conditions, our propositional logic does not only consider extensionally the truth conditions which are in modal logic simple functions from possible worlds (or contexts) to truth values. On the contrary, we consider also the nature of the function by the application of which we determine the truth conditions of a proposition from the list of the truth possibilities of its atomic propositions. Thus we will distinguish propositions with different cognitive values like the proposition that arithmetic is complete and the proposition that arithmetic is both complete and incomplete, while these strictly equivalent propositions with the same content are identified in Parry's logic. Moreover, we also identify propositions with the same cognitive value like the proposition that the morning star is the evening star and the proposition that the evening star is the morning star, even if the corresponding hyperintensions of the two sentences are different contents of thought in Cresswell's logic.

The set of propositions that are expressible in the formal language of our logic obeys the laws of a model-theoretic algebra that is similar to the syntactic algebra of the language. One advantage of that algebraic approach is to enlarge the notion of "strong implication" and to formulate a new definition of the relation of logical consequence (in the tradition of Tarski [TAR36]) on the algebra of propositions. The second advantage of that algebraic approach is to conceive the notion of proposition in a new way.

1. *The algebra of propositions.*

Let U be any non-empty set of *individual objects* and I be any non-empty set of indices, which represent *possible worlds* (or possible *contexts* of utterances). In our logic, there are two different *types of propositional constituents*: first the *individual concepts* that serve to *refer* to individual objects and second the *attributes (properties and relations)* that serve to *predicate*. Following Carnap, we will use the set U^I of all functions $\mathfrak{C}$: $I \rightarrow U$ in order to represent the set of individual concepts, and for any $n = 1,2,...,$ the set $\left(\mathcal{P}\left(U^n\right)\right)^I$ of all functions $\mathcal{R}_n: I \rightarrow \mathcal{P}\left(U^n\right)$ in order to represent the set of n-ary attributes or relations in intension.

Each atomic proposition $\mathcal{A}$ has a *finite positive number of propositional constituents* namely: one attribute $\mathcal{R}_n$ of degree $n \geqslant 1$ and a number k $(1 \leq k \leq n)$ of individual concepts $\mathfrak{C}_1,....,\mathfrak{C}_k$ In an atomic proposition the attribute $\mathcal{R}_n$ is predicated of a n-ary sequence of the objects that fall under these individual concepts and that predication determines the truth conditions of that atomic proposition. For example, if the order of predication is such that the attribute $\mathcal{R}_n$ is predicated of the sequence of individuals that fall under the concepts $\mathfrak{C}_1, \ldots, \mathfrak{C}_n$. in the atomic proposition $\mathcal{A}$ then $\mathcal{A}$ is true in a world i iff $< \mathfrak{C}_1(i), \ldots, \mathfrak{C}_n(i) > \in \mathcal{R}_n(i)$.

From a logical point of view, two atomic propositions are identical in our logic iff they have the same propositional constituents and the predications that are made with their constituents are such that they are true in the same possible worlds (or contexts).

Thus we can define as follows the logical type of the so-called atomic propositions. From a logical point of view an atomic proposition $\mathcal{A}$ is an ordered pair: its first element is a set of intensions containing a single attribute of degree n, $n \geq 1$, and a number k, $(1 \leq k \leq n)$ of individual concepts; the second element of an atomic proposition $\mathcal{A}$ is a subset of the set I, that contains all possible worlds $i \in I$ where that atomic proposition is true given the predication that is made with its constituents.

Thus, by definition, the set U_a of all atomic propositions is

$$\{< \{\mathcal{R}_n, \mathfrak{C}_1, \ldots, \mathfrak{C}_n\}, \{i \in I: < \mathfrak{C}_1(i), \ldots, \mathfrak{C}_n(i) > \in \mathcal{R}_n(i)\} >:$$

$$\mathcal{R}_n \in \left(\mathcal{P}(U^n)\right)^I, \mathfrak{C}_1, \ldots, \mathfrak{C}_n \in U^I, n = 1, 2, \ldots\}.$$

Now we can define inductively the set of *propositions* as the smallest subset U_p of the set $\mathcal{P}(U_a) \, x \, \mathcal{P}(\mathcal{P}(U_a))$ satisfying the following conditions:

(i) $\{< \{\mathcal{A}\}, [\{\mathcal{A}\}) >: \mathcal{A} \in U_a\} \subseteq U_p$, where for any $W \subseteq U_a$,
$[W) = \{W' \in \mathcal{P}(U_a): W \subseteq W'\}$,

(ii) for any $P \in U_p$, $< id_1(P), \mathcal{P}(U_a) - id_2(P) > \in U_p$,

(iii) for any P, $Q \in U_p$, $< id_1(P) \cup id_1(Q), id_2(P) \cap id_2(Q) > \in U_p$,
where for any $< A, B > \in \mathcal{P}(U_a) \, x \, \mathcal{P}(\mathcal{P}(U_a))$, $id_1(< A, B >) = A$, $id_2(< A, B >) = B$.

Now, consider an algebra $\underline{U}_p = (U_p, \neg, \wedge, \vee, \rightarrow)$ generated by the set of elementary propositions $\{(\mathcal{A}): \mathcal{A} \in U_a\}$, $(\mathcal{A}) = < \{\mathcal{A}\}, [\{\mathcal{A}\}) >$, where for any $P,\ Q \in U_p$:

$$\neg P = \ < id_1(P),\ \mathcal{P}(\mathrm{U_a})\ -\ id_2(P) >,$$
$$P \wedge Q = \ < id_1(P) \cup id_1(Q),\ id_2(P) \cap id_2(Q) >,$$
$$P \vee Q = \neg(\neg P \wedge \neg Q) = \ < id_1(P) \cup id_1(Q),\ id_2(P) \cup id_2(Q) >,$$
$$P \rightarrow Q = \neg P \vee Q = \ < id_1(P) \cup id_1(Q),\ \left(\mathcal{P}(\mathrm{U_a}) - id_2(P)\right) \cup id_2(Q) >,$$

This algebra $\underline{U}_p$ will be called hereafter the *algebra of propositions.*

It is clear that any element P of that algebra of propositions is an ordered pair, whose first element is a finite non-empty subset of the set of atomic propositions U_a ; it is the *content* of that proposition P. On the other hand, the second element of the pair is a subset of $\mathcal{P}(U_a)$; it serves to determine the *truth conditions* of the proposition P.

2. *Language and its interpretations.*

As we said earlier, a proposition should be considered simultaneously as the content of possible human thoughts and as the sense of possible sentences of natural languages - which are sentences which could be used literally to express these thoughts. In section 1, we have defined the formal concept of a proposition in a language independent way. Now we will also describe propositions as the senses of sentences. For that purpose, we will present a special formal object-language which contains formulas whose senses belong to the set U_p of propositions. This formal language is a simple first-order language - without quantifiers, individual variables and functional symbols.

Let Const and Pred be respectively a set of individual constants and a set of predicate symbols. Our *formal language* is the algebra $\underline{L} = (L, \neg, \wedge, \vee, \rightarrow)$ freely generated by the set At of free generators of the form: $r_n(c_1, ..., c_n)$, where r_n is n-ary predicate symbol and $c_1, ..., c_n$ are individual constants, n = 1,2, ...

By the *interpreting function of the language L* we understand an assignment

$$s\text{: Const} \cup \text{Pred} \cup \text{At} \rightarrow U^I \cup \bigcup\left\{\left(\mathcal{P}(U^n)\right)^I : n = 1, 2 \ldots\right\} \cup U_a$$

such that for any $c, c_1, ..., c_n \in$ Const, $r_n \in$ Pred:

$s(c) \in U^I$, $s(r_n) \in (\mathcal{P}(U^n))^I$, $s(r_n(c_1, \ldots, c_n)) =$
$= < \{s(r_n), s(c_1), \ldots, s(c_n)\}, \{i \in I: < s(c_1)(i), \ldots, s(c_n)(i) > \in s(r_n)(i)\} >$.
Given the following homomorphism $h_s: \underline{L} \to \underline{U}_p$: for any $A \in At$, $h_s(A) = (s(A))$, we can say that for any $\alpha \in L$, the proposition $h_s(\alpha)$ *is the sense of sentence* α *with respect to s.*

3. *Analysis of the content and of the truth conditions of a proposition.*

In order to characterize the content of a proposition let us first make the following obvious definition of the occurrence of an atomic proposition in a proposition:

For any atomic proposition $\mathcal{B}$:
(1) $\mathcal{B}$ occurs in $(\mathcal{A})$ iff $\mathcal{B} = \mathcal{A}$,
(2) $\mathcal{B}$ occurs in $\neg P$ iff $\mathcal{B}$ occurs in P,
(3) $\mathcal{B}$ occurs in $P \wedge Q$ iff $\mathcal{B}$ occurs in P or $\mathcal{B}$ occurs in Q.

For any $P \in U_p$, $\mathrm{id}_1(P)$ is the set of all atomic propositions occurring in P. Notice that the content of any proposition is always a finite non-empty subset of U_a.
In order to characterize the truth conditions let us consider each proposition as a sense of sentences.
For any interpreting function s consider the function g_s: $\mathcal{P}(U_a) \to \{0,1\}^L$, where $\{0,1\}$ is the set of truth-values, with the following property: for any $W \in \mathcal{P}(U_a)$, $g_s(W): L \to \{0,1\}$ is the classically admissible valuation on L such that for any $A \in At$, $g_s(W)(A) = 1$ iff $s(A) \in W$.

Lemma 3.1. For any interpreting function s, for any $\alpha \in L$ and $W \subseteq U_a$: $W \in \mathrm{id}_2(h_s(\alpha))$ iff $g_s(W)(\alpha) = 1$

Proof. (By induction on the length of α) Let s be any fixed interpreting function of $\underline{L}$ and $W \subseteq U_a$.

1) Let $\alpha \in At$. Then $h_s(\alpha) = (s(\alpha))$ and consequently
$W \in \mathrm{id}_2(h_s(\alpha))$ iff $W \in [\{s(\alpha)\})$ iff $s(\alpha) \in W$ iff $g_s(W)(\alpha) = 1$.

2) Let α be of the form $\neg\beta$, where $\beta \in L$ is such that (*) $W \in \mathrm{id}_2(h_s(\beta))$ iff $g_s(W)(\beta) = 1$. Then $W \in \mathrm{id}_2(h_s(\neg\beta))$ iff $W \in \mathrm{id}_2(\neg h_s(\beta))$ iff $W \notin \mathrm{id}_2(h_s(\beta))$ iff $g_s(W)(\beta) = 0$ iff $g_s(W)(\neg\beta) = 1$ by (*) and the fact that $g_s(W)$ is classically admissible.

3) Let α be of the form $\beta \wedge \gamma$, where (*) is assumed for β and for γ. Then $W \in \mathrm{id}_2(h_s(\beta \wedge \gamma))$ iff $W \in \mathrm{id}_2(h_s(\beta) \wedge h_s(\gamma))$ iff $W \in \mathrm{id}_2(h_s(\beta)) \cap \mathrm{id}_2(h_s(\gamma))$ iff $g_s(W)(\beta) = g_s(W)(\gamma) = 1$ iff $g_s(W)(\beta \wedge \gamma) = 1$. □

Lemma 3.1 enables to give a simple characterization of the set $\mathrm{id}_2(P)$ for any proposition P. Indeed, for a given P, one can choose the formula α and the interpreting function s such that $P = h_s(\alpha)$. So if for instance we consider the proposition P of the form: $(\neg(\mathscr{A}_1) \wedge (\mathscr{A}_2)) \rightarrow (\mathscr{A}_1)$, then we should take into account the formula $(\neg A_1 \wedge A_2) \rightarrow A_1$, $A_1, A_2 \in At$, and the interpreting function s such that $s(A_i) = \mathscr{A}_i$, $i = 1, 2$. Then $\mathrm{id}_2(P)$ is the family of all $W \subseteq U_a$ such that the functions $g_s(W)$ associated with W forms the set of all classically admissible valuations on L which take the value 1 on the formula $(\neg A_1 \wedge A_2) \rightarrow A_1$.

4. Some logical properties of a proposition.

First of all, we should define when a proposition P is true or false. If we consider an elementary proposition of the form:

$$(<\{\mathscr{R}_n, \mathscr{C}_1, \ldots, \mathscr{C}_n\}, \{i \in I: <\mathscr{C}_1(i), \ldots, \mathscr{C}_n(i)> \in \mathscr{R}_n(i)\}>),$$

we can obviously say that it is true in a context $i \in I$ iff $<\mathscr{C}_1(i), \ldots, \mathscr{C}_n(i)> \in \mathscr{R}_n(i)$.

Given the nature of elementary propositions and of truth functions, we obtain the following general truth definition for complete propositions:

(i) for any $\mathscr{A} \in U_a$, $(\mathscr{A})$ is true in i iff $i \in id_2(\mathscr{A})$,
(ii) for any $P \in U_p$, $\neg P$ is true in i iff P is false in i,
(iii) for any $P, Q \in U_p$, $P \wedge Q$ is true in i iff P and Q are true in i.

However we should connect the fact that a proposition is true or false with its truth conditions. The following Lemma establishes such a connection:

Lemma 4.1. Let for any $i \in I$, U_a^i be $\{\mathscr{A} \in U_a : i \in \mathrm{id}_2(\mathscr{A})\}$. Then for any proposition P, P is true in $i \in I$ iff $U_a^i \in \mathrm{id}_2(P)$.

Proof. Straightforward by induction on the form of a proposition P. □

We can now define as follows other important properties of propositions:

> A proposition P is a *tautology* iff $\mathrm{id}_2(P) = \mathscr{P}(U_a)$,
> P is a *contradiction* iff $\mathrm{id}_2(P) = \varnothing$,
> P is *necessary* iff for each $i \in I$, P is true in i,
> P is *impossible* iff for each $i \in I$, P is false in i.

According to Lemma 4.1, it is easily seen that any tautology is a necessary proposition, but that the converse is not true. And similarly, any contradiction is an impossible proposition, although not conversely. Such distinctions are important from a philosophical point of view, because there are many necessarily true propositions which unlike tautologies are neither true a priori nor known to be true by virtue of linguistic competence.

5. Two consequence relations on the set of propositions

We will now define two relations of logical consequence on the set of propositions: the first relation, that is called the "strict" or "usual" consequence relation is expressed by the modal connective of strict implication (hence the term "strict"). Although it can be defined easily in logic, it has no systematic psychological reality, in the sense that human beings are not able to infer all strict consequences of a proposition whenever they reason from the hypothesis of the truth of that proposition. On the contrary, the second consequence relation, that we will call the "strong" consequence relation, has a psychological reality and can be taken as the formal ground of many human reasonings with simple propositions.

Let $\Gamma \subseteq U_p$ and $P \in U_p$. We will say that Γ *strictly entails* P ($\Gamma \multimap P$ in symbols) iff for any $i \in I$, P is true in i whenever each $Q \in \Gamma$, is true in i.

In that way, we have for instance the law of introduction of disjunction: $\{P\} \multimap P \vee Q$, which is often not applied in our practical reasoning.

The strong consequence relation is closely related to the algebraic structure of the set of propositions. So first let us mention some properties of the algebra $\underline{U}_p$.

Lemma 5.1. For any equality σ in the signature $(\neg, \wedge, \vee, \rightarrow)$, σ is an equality in the algebra $\underline{U}_p$ iff σ is a Boolean equality and the set of variables occurring in the left term of σ is identical with the set of variables occurring in the right term.

Proof. Assume that we have the following variables: $x_0, x_1, \ldots,$ and let σ be of the form: $f(x_{i_1}, \ldots, x_{i_n}) = g(x_{j_1}, \ldots, x_{j_m})$, where $x_{i_1}, \ldots, x_{i_n}$ $(x_{j_1}, \ldots, x_{j_m})$ are all the different variables occurring in the term $f(x_{i_1}, \ldots, x_{i_n})\,(g(x_{j_1}, \ldots, x_{j_m}))$.

$(\Rightarrow)$: Assume that σ holds in the algebra $\underline{U}_p$. First suppose that $\{x_{i_1}, \ldots, x_{i_n}\} \neq \{x_{j_1}, \ldots, x_{j_m}\}$. Let $x_{i_k} \notin \{x_{j_1}, \ldots, x_{j_m}\}$ for some $k \in \{1, \ldots, n\}$. Notice that according to the assumption, for any propositions $P_1, \ldots P_n, Q_1, \ldots, Q_m$, if $\mathrm{id}_1(f(P_1, \ldots, P_n)) = \mathrm{id}_1(g(Q_1, \ldots, Q_m))$, then $\mathrm{id}_1(P_1) \cup \ldots \cup \mathrm{id}_1(P_n) = \mathrm{id}_1(Q_1) \cup \ldots \cup \mathrm{id}_1(Q_m)$. Thus, substituting: $x_{j_l} \rightarrow P$ for any $l = 1, \ldots, m$, $x_{i_l} \rightarrow P$ for any $l = 1, \ldots, n$, $l \neq k$, where P is any proposition, and $x_{i_k} \rightarrow Q$, where Q is such that $\mathrm{id}_1(Q) \not\subseteq \mathrm{id}_1(P)$, we obtain that $\mathrm{id}_1(P) \cup \mathrm{id}_1(Q) = \mathrm{id}_1(P)$, which is impossible. Analogously if $\{x_{j_1}, \ldots, x_{j_m}\} \not\subseteq \{x_{i_1}, \ldots, x_{i_n}\}$.

In order to show that σ must be Boolean equality, notice that for any propositions $P_1, \ldots, P_n$: $\mathrm{id}_2(f(P_1, \ldots, P_n)) = f'(\mathrm{id}_2(P_1), \ldots, \mathrm{id}_2(P_n))$ for any function f of n variables in the signature $(\neg, \wedge, \vee, \rightarrow)$, where f' is set-theoretical operation corresponding to f. So the equality: $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$ holds in $\underline{U}_p$ iff it is Boolean.

$(\Leftarrow)$: by the last argument of the proof $(\Rightarrow)$. □

Following Lemma 5.1, the equalities:

$$x \wedge x = x,$$

$$x \wedge y = y \wedge x,$$

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z,$$

are satisfied in $\underline{U}_p$. So we can consider the reduct $(U_p, \wedge)$ of $\underline{U}_p$ as a meet-semilattice.

We will say that for any $\varnothing \neq \Gamma \subseteq U_p$, $P \in U_p$, Γ *strongly entails* P ($\Gamma \vdash_\epsilon P$ in symbols) iff $P \in [\Gamma)$, where $[\Gamma)$, is the filter generated in the semilattice $(U_p, \wedge)$ by the set Γ. We also put $\{P \in U_p : \varnothing \vdash_\epsilon P\} = \varnothing$.

The following obvious lemma explains the strong consequence relation in terms of content and of truth conditions:

Lemma 5.2. For any $\varnothing \neq \Gamma \subseteq U_p$, $P \in U_P$:

$\Gamma \Vdash\!\epsilon P$ iff there exists $\{P_1, ..., P_n\} \subseteq \Gamma$ such that
$\mathrm{id}_1(P) \subseteq \mathrm{id}_1(P_1) \cup ... \cup \mathrm{id}_1(P_n)$ and
$\mathrm{id}_2(P_1) \cap ... \cap \mathrm{id}_2(P_n) \subseteq \mathrm{id}_2(P)$.

Proof. Notice that for any $\varnothing \neq \Gamma \subseteq U_p$, $P \in U_p$, $P \in [\Gamma)$ iff $P_1 \wedge ... \wedge P_n \leq P$ for some $P_1, ..., P_n \in \Gamma$, where $\leq$ is the partial ordering of the semilattice $(U_p, \wedge)$, which is defined as follows: for any $P, Q \in U_p$: $P \leq Q$ iff $P \wedge Q = P$ *iff* $\mathrm{id}_1(Q) \subseteq \mathrm{id}_1(P) \& \mathrm{id}_2(P) \subseteq \mathrm{id}_2(Q)$. □

One can show using Lemmas 4.1 and 5.2 that for any $\varnothing \neq \Gamma \subseteq U_p$, $P \in U_p$, $\Gamma \Vdash\!\epsilon P$ implies that $\Gamma \dashv\!\epsilon P$. But the converse does not holds; for instance in general $\{P\} \Vdash\!\epsilon P \vee Q$ does not hold.

6. *A representation of propositions.*

Now we are going to formulate a different algebraic approach to the concept of proposition equivalential to the one that has just been presented. A proposition will be conceived less intuitively but its structure will turn out to be more simple. We will be able to identify a proposition with an ordered pair consisting of two finite sets.

Let us introduce for any non-empty and finite set $W \subseteq U_a$ the following equivalence relation on the set $\mathcal{P}(U_a)$:

for any V and $V' \in \mathcal{P}(U_a)$, $V = V'(W)$ iff $W \cap V = W \cap V'$.

We will need the following lemma about propositions:

Lemma 6.1. For any proposition P and any $W \in \mathrm{id}_2(P)$:
$[W]_{\mathrm{id}_1(P)} \subseteq \mathrm{id}_2(P)$, where for any finite $\varnothing \neq V \subseteq U_a$ and any $W \subseteq U_a$, $[W]_v = \{W' \subseteq U_a: W \equiv W'(V)\}$.

Proof. Straightforward by induction on the length of the proposition P. □

We will use Lemma 6.1 in the proof of the following:

Lemma 6.2. For any finite $\varnothing \neq W \subseteq U_a$ and any $\mathcal{W} \subseteq \mathcal{P}(W)$:
$< W, \bigcup \{[W']_W: W' \in \mathcal{W}\} > \in U_p$.

Proof. Let $W = \{\mathscr{A}_1, \ldots, \mathscr{A}_n\} \subseteq U_a$ and $\mathscr{W} = \{W_1, \ldots, W_k\}$, where $0 \leq k \leq 2^n$, be any family of subsets of the set W.
1) Suppose that $k > 0$, so that $\mathscr{W} \neq \varnothing$. For any $j = 1, \ldots, k$ let $W_j = \{\mathscr{A}^j_1, \ldots, \mathscr{A}^j_{f(j)}\}$, $W - W_j = \{\mathscr{A}^j_{f(j)+1}, \ldots, \mathscr{A}^j_n\}$, where $f(j) \in \{0, 1, \ldots, n\}$ (in case $f(j) = 0$, $W_j = \varnothing$ and similarly when $f(j) = n$, $W - W_j = \varnothing$).
We show that: $<\{\mathscr{A}_1, \ldots, \mathscr{A}_n\}, [W_1]_W \cup \ldots \cup [W_k]_W> =$
$((\mathscr{A}^1_1) \wedge \ldots \wedge (\mathscr{A}^1_{f(1)}) \wedge \neg(\mathscr{A}^1_{f(1)+1}) \wedge \ldots \wedge \neg(\mathscr{A}^1_n)) \vee \ldots$
$\vee((\mathscr{A}^k_1) \wedge \ldots \wedge (\mathscr{A}^k_{f(k)}) \wedge \neg(\mathscr{A}^k_{f(k)+1}) \wedge \ldots \wedge \neg(\mathscr{A}^k_n))$.
Let the last proposition be P_0. It is obvious that
$\mathrm{id}_1((\mathscr{A}^j_1) \wedge \ldots \wedge (\mathscr{A}^j_{f(j)}) \wedge \neg(\mathscr{A}^j_{f(j)+1}) \wedge \ldots \wedge \neg(\mathscr{A}^j_n)) = \{\mathscr{A}_1, \ldots, \mathscr{A}_n\}$, for any $j = 1, \ldots, k$. This means that $\mathrm{id}_1(P_0) = \{\mathscr{A}_1, \ldots, \mathscr{A}_n\}$. Next notice that
$W_j \in id_2((\mathscr{A}^j_1) \wedge \ldots \wedge (\mathscr{A}^j_{f(j)}) \wedge \neg(\mathscr{A}^j_{f(j)+1}) \wedge \ldots \wedge \neg(\mathscr{A}^j_n))$.

Consequently, for any $j = 1, \ldots, k$, $W_j \in \mathrm{id}_2(P_0)$. So let $V \in [W_1]_W \cup \ldots \cup [W_k]_W$. Then $W_j \equiv V(W)$ for some j. Hence, by the Lemma 6.1: $V \in \mathrm{id}_2(P_0)$. And conversely, if $V \in \mathrm{id}_2(P_0)$, then $V \in \mathrm{id}_2((\mathscr{A}^j_1) \wedge \ldots \wedge (\mathscr{A}^j_{f(j)}) \wedge \neg(\mathscr{A}^j_{f(j)+1}) \wedge \ldots \wedge \neg(\mathscr{A}^j_n))$ for some j. And this implies that $W_j \cap V = W_j$ and $(W - W_j) \cap V = \varnothing$. So $W \cap V = (W_j \cup (W - W_j)) \cap V = (W_j \cap V) \cup ((W - W_j) \cap V) = W_j = W \cap W_j$, that is $V \in [W_j]_W \subseteq [W_1]_W \cup \ldots \cup [W_k]_W$.
Finally: $[W_1]_W \cup \ldots \cup [W_k]_W = \mathrm{id}_2(P_0)$.

2) Suppose that $k = 0$, so that $\mathscr{W} = \varnothing$. It is obvious that in case:
$<\{\mathscr{A}_1, \ldots, \mathscr{A}_n\}, \varnothing> = ((\mathscr{A}_1) \wedge \neg(\mathscr{A}_1)) \wedge (\mathscr{A}_2) \wedge (\mathscr{A}_3) \wedge \ldots \wedge (\mathscr{A}_n)$. □

Now, consider the following algebra $\underline{V}_p$ similar to $\underline{U}_p$:
$\underline{V}_p = (V_p, \neg, \wedge, \vee, \rightarrow)$, where $V_p = \{<W, \mathscr{W}>: W \in \mathscr{P}_{fin}(U_a), \mathscr{W} \subseteq \mathscr{P}(W)\}$, $\mathscr{P}_{fin}(U_a)$ is the family of all non – empty and finite subsets of U_a and for

any $< W_1, \mathcal{W}_1 >, < W_2, \mathcal{W}_2 > \in V_p$:
$\neg < W_1, \mathcal{W}_1 > = < W_1, \mathcal{P}(W_1) - \mathcal{W}_1 >$, $< W_1, \mathcal{W}_1 > \wedge < W_2, \mathcal{W}_2 > =$
$= < W_1 \cup W_2, \{V_1 \cup V_2 : V_1 \in \mathcal{W}_1, V_2 \in \mathcal{W}_2, V_1 \cap W_2 = V_2 \cap W_1\} >$,
$< W_1, \mathcal{W}_1 > \vee < W_2, \mathcal{W}_2 > = \neg(\neg < W_1, \mathcal{W}_1 > \wedge \neg < W_2, \mathcal{W}_2 >)$,
$< W_1, \mathcal{W}_1 > \rightarrow < W_2, \mathcal{W}_2 > = \neg < W_1, \mathcal{W}_1 > \vee < W_2, \mathcal{W}_2 >$.

Theorem 6.3. The algebras $\underline{U}_p$ and $\underline{V}_p$ are isomorphic.

Proof. We show that the function $g: U_p \rightarrow V_p$ defined as follows: for any $P \in U_p$, $g(P) = < \mathrm{id}_1(P), \{W \cap \mathrm{id}_1(P): W \in \mathrm{id}_2(P)\} >$, is the required isomorphism.

Following Lemma 6.2, we can consider the function $f: V_p \rightarrow U_p$ defined as follows: for any $< W, \mathcal{W} > \in V_p$,
$f(< W, \mathcal{W} >) = < W, \bigcup\{[W']_W : W' \in \mathcal{W}\} > = < W, \{V \subseteq U_a : V \cap W \in \mathcal{W}\} >$.
Then for any $< W, \mathcal{W} > \in V_p$, $g(f(< W, \mathcal{W} >)) = < W, \{V \cap W : V \in \{V' \subseteq U_a : V' \cap W \in \mathcal{W}\}\} > = < W, \mathcal{W} >$. Moreover, for any $P \in U_p$, $f(g(P)) =$
$< \mathrm{id}_1(P), \bigcup\{[W']_{\mathrm{id}_1(P)}: W' \in \{W \cap \mathrm{id}_1(P): W \in id_2(P)\}\} > = < \mathrm{id}_1(P),$
$\bigcup\{[W]_{\mathrm{id}_1(P)}: W \in \mathrm{id}_2(P)\} >$. According to Lemma 6.1, $\bigcup\{[\mathrm{W}]_{\mathrm{id}_1(P)}: W \in \mathrm{id}_2(P)\} \subseteq \mathrm{id}_2(P)$, the converse inclusion is obvious, so $\mathrm{f}(\mathrm{g}(\mathrm{P})) = P$.
Thus, the function g is 1-1 and onto. In order to show that g preserves the operation $\neg$, notice that:
(1) $\mathcal{P}(\mathrm{id}_1(P)) = \{W \cap \mathrm{id}_1(P): W \in \mathrm{id}_2(P)\} \cup \{W \cap \mathrm{id}_1(P): W \notin \mathrm{id}_2(P)\}$, and
(2) $\{W \cap \mathrm{id}_1(P): W \in \mathrm{id}_2(P)\} \cap \{W \cap \mathrm{id}_1(P): W \notin \mathrm{id}_2(P)\} = \varnothing$, for any $P \in U_p$.
(One can prove (2) by a reduction *ad absurdum*. Suppose that it is false. Then there exist $W_1 \in \mathrm{id}_2(P)$ and $W_2 \notin \mathrm{id}_2(P)$ such that $W_1 \equiv W_2(\mathrm{id}_1(P))$. So, by Lemma 6.1, we obtain a contradiction).
In that way, the following law holds for any $P \in U_p$:
$g(\neg P) = < \mathrm{id}_1(\neg P), \{W \cap \mathrm{id}_1(\neg P): W \in \mathrm{id}_2(\neg P)\} > = < \mathrm{id}_1(P),$
$\{W \cap \mathrm{id}_1(P): W \notin \mathrm{id}_2(P)\} > = < \mathrm{id}_1(P), \mathcal{P}(\mathrm{id}_1(P)) -$
$\{W \cap \mathrm{id}_1(P): W \in \mathrm{id}_2(P)\} > = \neg g(P)$, due to (1) and (2).
Moreover, for any $P, Q \in U_p$, we have:

$g(P \wedge Q) = < \mathrm{id}_1(P \wedge Q), \{W \cap \mathrm{id}_1(P \wedge Q): W \in \mathrm{id}_2(P \wedge Q)\} > = < \mathrm{id}_1(P) \cup \mathrm{id}_1(Q), \{(W \cap \mathrm{id}_1(P)) \cup (W \cap \mathrm{id}_1(Q)): W \in \mathrm{id}_2(P) \& W \in \mathrm{id}_2(Q)\} >$.

But obviously the following inclusion holds:

$\{(W \cap \mathrm{id}_1(P)) \cup (W \cap \mathrm{id}_1(Q)): W \in \mathrm{id}_2(P) \& W \in \mathrm{id}_2(Q)\} \subseteq \{V_1 \cup V_2: V_1 \in \{W \cap \mathrm{id}_1(P): W \in \mathrm{id}_2(P)\} \& V_2 \in \{W \cap \mathrm{id}_1(Q): W \in \mathrm{id}_2(Q)\} \& V_1 \cap \mathrm{id}_1(Q) = V_2 \cap \mathrm{id}_1(P)\}$.

And in order to show that the converse inclusion holds, notice that for any $W_1 \in \mathrm{id}_2(P)$, $W_2 \in \mathrm{id}_2(Q)$:

(3) $W_1 \cap \mathrm{id}_1(P) = ((W_1 \cap \mathrm{id}_1(P)) \cup (W_2 \cap \mathrm{id}_1(Q))) \cap \mathrm{id}_1(P)$, and

(4) $W_2 \cap \mathrm{id}_1(Q) = ((W_1 \cap \mathrm{id}_1(P)) \cup (W_2 \cap \mathrm{id}_1(Q))) \cap \mathrm{id}_1(Q)$, whenever $(W_1 \cap \mathrm{id}_1(P)) \cap \mathrm{id}_1(Q) = (W_2 \cap \mathrm{id}_1(Q)) \cap \mathrm{id}_1(P)$.

Further put $W = (W_1 \cap \mathrm{id}_1(P)) \cup (W_2 \cap \mathrm{id}_1(Q))$. According to Lemma 6.1, from (3) and (4) we obtain that $W \in \mathrm{id}_2(P)$ and $W \in \mathrm{id}_2(Q)$. Thus $g(P \wedge Q) = < \mathrm{id}_1(P) \cup \mathrm{id}_1(Q), \{V_1 \cup V_2: V_1 \in \{W \cap \mathrm{id}_1(P): W \in \mathrm{id}_2(P)\} \& V_2 \in \{W \cap \mathrm{id}_1(Q): W \in \mathrm{id}_2(Q)\} \& V_1 \cap \mathrm{id}_1(Q) = V_2 \cap \mathrm{id}_1(P)\} > = < \mathrm{id}_1(P), \{W \cap \mathrm{id}_1(P): W \in \mathrm{id}_2(P)\} > \wedge < \mathrm{id}_1(Q), \{W \cap \mathrm{id}_1(Q): W \in \mathrm{id}_2(Q)\} > = g(P) \wedge g(Q)$. □

Obviously we can treat a proposition as an ordered pair of the form $< W, \mathcal{W} >$. One can express all the properties of propositions and of consequence relations in the new way. For instance, $< W, \mathcal{W} >$ is a tautological (contradictory) proposition iff $\mathcal{W} = \mathcal{P}(W)(\mathcal{W} = \varnothing)$; for any $< W_1, \mathcal{W}_1 >, < W_2, \mathcal{W}_2 > \in V_p$, $\{< W_1, \mathcal{W}_1 >\} \vdash < W_2, \mathcal{W}_2 >$ iff $W_2 \subseteq W_1 \&$ $\{V \cap W_2: V \in \mathcal{W}_1\} \subseteq \mathcal{W}_2$ etc.

These logical investigations on propositions have been further developped in order to analyse quantification and modalities. They will be published in Vanderveken's next book on *The logic of propositions*.

Daniel VANDERVEKEN
Université du Québec à Trois-Rivières
Marek NOWAK
University of Lodz

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