Logique & Analyse **141-142** (1993), 61-73

CAN THE BEST OF ALL POSSIBLE WORLDS BE A RANDOM STRUCTURE?

Wim MIELANTS

Abstract.
Leibniz defined the best of all possible worlds as this one which maximises symmetry and the variety of substructures. There exist many mathematical theories with a unique countable model with this property, and in many cases this is also the countable random model. So the idea that the best of all possible worlds (in the sense of Leibniz) could be a random structure is not absurd.

1. Introduction

Let A be a set of axioms on a first-order language L, and let T be the theory generated by A or the set of logical consequences of A. By the completeness theorem of Gödel this is also the set of modeltheoretic consequences of A, and if T is consistent there are models of T. Also if not all the models of T are finite, then by the theorem of Löwenheim-Skolem there are always countable models of T.

We consider the set A of axioms as a set of natural laws, and the class of all countable models of T as the class of possible worlds for which these natural laws hold.

If the language L is a finite relational language and if the finite models of T form an amalgamation class, then by a theorem of Fraïssé there exists up to an isomorphism a unique countable infinite model of T which is homogeneous and universal. This model is called the Fraïssé limit of T. We shall give arguments why this Fraïssé limit can be considered as the best of all possible worlds in the sense of Leibniz for which the natural laws in A hold, and also why the theorem of Fraïssé can be considered as the modeltheoretic analogue of the principle of composability of Leibniz. We will prove (using topological and probabilistic arguments) that some properties of these Fraïssé limits are modeltheoretic analogues of some metaphysical statements such as: "almost any possible world is isomorphic with the best of all possible worlds", and: "the best of all possible worlds is also the most probable of all possible worlds, or it is a random structure".

2. *Leibniz's principle of composability.*

In his work : "De rerum originatione radicale" Leibniz writes: "Since there is something rather than nothing, everything which is possible tends to exist" (Omne possibile habeat conatum ad Existentiam). But not everything that is possible exists, since two possibilities together are in general no longer a possibility. The principle of composability of Leibniz says that the real world is that possible world which realises the maximal number of possibilities, and this is only the case if this world has the highest possible perfection and harmony. For this reason Leibniz calls the real world the best of all possible worlds.

3. *Homogeneous models*

Let $L(R_1, R_2, \ldots, R_m)$ be a finite relational first-order language. We denote the arity of the relational symbol $R_i (1 \leq i \leq m)$ by $n(i)$. Each model of $L(R_1, \ldots, R_m)$ is called an *L*-structure (or a relational structure on the language L). If $S_1 = (\Omega^1, R_1^1, R_2^1, \ldots, R_n^1)$ and $(\Omega^2, R_1^2, R_2^2, \ldots, R_n^2)$ with $R_i^1 \subseteq cart^{n(i)}\Omega^1$ and $R_i^2 \subseteq cart^{n(i)}\Omega^2$ are two L-structures with universe Ω^1 and Ω^2, respectively then a map $f: \Omega^1 \to \Omega^2$ is called an *L*-morphism if $R_i^1(x_1, x_2, \ldots, x_{n(i)}) \leftrightarrow R_i^2(f(x_1), f(x_2), \ldots, f(x_{n(i)}))$. If f is injective we call it an embedding, if f is bijective an isomorphism. An isomorphism of an *L*-structure S to itself is called an automorphism or a symmetry of S. We denote the symmetry group (or automorphism group) of S by Aut$S$. A map $f: \Omega^1 \to \Omega^2$ so that $R_i^1(x_1, x_2, \ldots, x_{n(i)}) \to R_i^2(f(x_1), f(x_2), \ldots, f(x_{n(i)}))$ is called a weak morphism.

The class of all *L*-structures which can be embedded in a given *L*-structure S is called the age of S and we denote it by Age S.

The class of all finite *L*-structures (with a finite universe) of Age S will be denoted by F Age S.

Definition 3.1. A relational *L*-structure S is called *homogeneous* if, given any isomorphism $f: A \to B$ between two finite substructures A and B of S, there is an automorphism $g \in$ Aut$S$, whose restriction to A is f. This is the strongest symmetry condition we can impose on a *L*-structure. An easy example of a homogeneous relational structure on a countable universe is $Q_<$ where Q is the set of rational numbers and $<$ is the usual order relation. Indeed, whenever $x_1 < x_2 < \ldots < x_n$ and $y_1 < y_2 < \ldots < y_n$, there is an automorphism (or order preserving permutation of Q) π such that $\pi(x_i) = y_i (1 \leq i \leq n)$.

This can be done by linear interpolation of the intervals (x_i, x_{i+1}) and at the two ends by appropriate shifts.

Property (A). If $A, B \in F\text{Age}S$ and $f: A \to B$, $h: A \to S$ are embeddings, then there is an embedding $g: B \to S$ such that $fg=h$. It suffices to require this when B has only one point more than A.

It is easy to prove that if S is a homogeneous model, property (A) will be true. In particular, it holds in a trivial way for $Q_<$. But conversely, if S is an L-structure on a countable universe so that property (A) is true, then S is a homogeneous relational structure. This can be proved using the classical back and forth argument of model theory {4,11}

So, a relational structure on a countable universe is homogeneous if and only if property (A) holds.

Since we consider only finite substructures it is possible to translate property (A) by a countable (but recursive) axiom system in the language L {11}.

These axioms are also called "Alice's Restaurant Axioms"{16} because they assert that you can get anything you want, as we shall see later on.

Using property (A) and the back and forth argument it is also easy to prove that two homogeneous L-structures S_1 and S_2 with $F\text{Age}S_1 = F\text{Age}S_2$ are isomorphic.

4. $\aleph_0$- categorical models.

A theory T on a first order language L is called $\aleph_0$*-categorical* if and only if any two models on a countable universe are isomorphic. A countable L-structure S is called an $\aleph_0$-categorical structure if and only if its theory in L is $\aleph_0$-categorical, or if any countable L-structure S' elementary equivalent with S is isomorphic with S. For instance, Cantor has proved that $Q_<$ is an $\aleph_0$-categorical structure {5}. In 1959 Ryll-Nardzewsky, Engeler and Svevonius have proved that a countable L-structure S is $\aleph_0$-categorical if and only if Aut$S$ is an oligomorphic permutation group on the points of the universum, this means that Aut$S$ has a finite number of orbits in its natural action on the n-tuples of points and this for all $n \in \mathbb{N}$. In particular Aut $Q_\leq$ has exactly $n!$ orbits in its natural action on the set of n-tuples of rational numbers. So, there are limits for first order languages in making complete descriptions of countable structures. This is only possible if this structure is very symmetrical, more precisely if and only if its group of symmetries is oligomorphic.

Since L is a finite relational language also each homogeneous L-structure will be $\aleph_0$-categorical. Of course, if the axioms (or the natural laws) generate an $\aleph_0$-categorical theory there is (up to an isomorphism) only one possible world.

5. *Universal models.*

If D is a class of L-structures, then an L-structure S is called D-universal if and only if for each $S' \in D$ there exists an embedding $f: S' \to S$ (or if $D \subset \text{Age}S$). If $S \in D$, then S, is a *universal element* of the class D. If S is D-universal and if for each $S' \in D$ there exist an embedding $g: S' \to S$ such that each symmetry of $g(S')$ can be extended to a symmetry of S, then we call S *symmetrical D-universal.* If D is the class of all finite models or the class of all countable models of a first of order theory T on L, then we call a model S of T which is D-universal finite T-universal or countable T-universal respectively. If S' is an L-structure we say that S' is younger than S if and only if $\text{Age}S' \subset \text{Age}S$. We denote the set of all L-structures which are younger than S by Young S.

We call an L-structure S *Cameron-universal* if and only if it is (Young S) - universal. Young S is in fact the maximal set D of L-structures for which S can be D-universal. From property (A) follows that each countable homogeneous relational structure is Cameron-universal (one needs only a forth argument). One can also prove that each $\aleph_0$-categorical structure is Cameron-universal, but this proof is more difficult (one has to use König's Infinity Lemma for trees) {3}. In particular, if D is the class of all possible worlds and if the real world S is a symmetrical universal element of D, then in fact each possible world is realised as a substructure of the real world with its complete group of symmetries as "real" symmetries (or as symmetries of S).

6. *The amalgamation property.*

A class D of finite relational structures on a finite relational language L has the amalgamation property if, whenever we have structures A, B_1, B_2 of D and embeddings $f_i: A \to B_i (i=1,2)$ then there exists a structure C and embeddings $g_i: B_i \to C(i=1,2)$ so that $g_1 f_1 = g_2 f_2$. So, any two finite structures of D with a common substructure can be jointly embedded so that their intersection contains at least this common substructure. If S is homogeneous, then by property (A) $F\text{Age}S$ is an amalgamation class. In particular, the class of all finite L-structures where L is a given finite relational language satisfies (in a trivial way) the amalgamation property. This is also the case for linear graphs, m-coloured graphs, n-uniform hypergraphs, m-coloured n-uniform hypergraphs, systems of m-dimensional subspaces of projective spaces with countable dimension on a finite field and many other structures {4, 11}. For quasi-orderings (or reflexive and transitive relations) and for posets (or antisymmetrical quasi-orderings) the amalgamation property is no longer trivial but still easy to prove ({11}). If (X,d) is a metric space and $D = \mathrm{Im}\, d - \{0\}$, then we can associate with it a relational structure $((X, R_i) i \in D)$ where $R_i(x, y) \leftrightarrow d(x, y) = i$. If $D \subseteq \mathbb{R}^+$ is a given finite or countable set of distances, it depends on properties of D if the class of all finite metric spaces with $\mathrm{Im} d - \{0\} \subseteq D$ is an amalgamation class or not. Sufficient conditions are given in{11}. In particular, if D is the set of positive rational numbers, or positive real algebraic numbers, or positive Turing computable real numbers, or $\mathbb{N}$, or $\{1, 2, ..., n\}$ with $n \in \mathbb{N}$, then the amalgamation property is true.

In general, topological spaces cannot be described by finite or countable relational structures, but finite topological spaces can. If $Q(x)$ is in the intersection of all open sets containing a point x, then the binary relation $R(x, y) \leftrightarrow Q(x) \subseteq Q(y)$ is a quasi-ordering which describes completely the topology. This can be generalized for infinite topologies with the property that each transfinite intersection of open sets is open. These topologies are called Alexandrov topologies and it is well known{15} that there exists a covariant bijective functor between the category of the quasi-orderings with the relational morphisms and the category of the Alexandrov-topologies with the open continuous functions as morphisms. Since the finite quasi-orderings form an amalgamation class, this will also be the case for the class of finite topologies.

7. The theorem of Fraïssé.

We know that if S_1 and S_2 are two homogeneous countable L-structures with $F\mathrm{Age}S_1 = F\mathrm{Age}S_2 = D$ then S_1 and S_2 are isomorphic. So, if D is a class of finite L-structures, then $D = F\mathrm{Age}S$ for at most one homogeneous countable L-structure S. What are the conditions for D so that such a homogeneous countable L-structure exists? This is answered by the theorem of Fraïssé.

Theorem of Fraïssé. If D is a class of finite L-structures, where L is a finite relational first order language, then there exists up to an isomorphism exactly one homogeneous countable L-structure S with $F\mathrm{Age}S = D$ if and only if:

[1] D is closed under isomorphisms (or, if $A \in D$ and $f: A \rightarrow B$ is an isomorphism then $B \in D$)
[2] D is closed under taking substructures (or, if $A \in D$ and B is a substructure of A then $B \in D$)
[3] D contains only countable many non-isomorphic members
[4] D is an amalgamation class.

We call S the *Fraïssé-limit* of D and denote it by $F(D)$.

For the proof of this theorem we refer to {4} or {7}. It is a constructive proof, starting with D a complete construction of $F(D)$ is given. In general, the Fraïssé-conditions 1), 2) and 3) are trivial and only the amalgamation property had to be verified. Since L is a finite relational language, $F(D)$ is also $\aleph_0$-categorical and so Cameron-universal. Hence, if D is the class of all finite models of a first order theory T on a finite relational language L which is an amalgamation class, then the other Fraïssé conditions are satisfied and we call the Fraïssé limit of D the Fraïssé limit $F(T)$ of this theory T. Then $F(T)$ will be countable T-universal (since it is finite T-universal and Cameron universal). If T is the first order theory of linear graphs, m-coloured graphs and n-uniform hypergraphs respectively, I have constructed models of the Fraïssé limit $F(T)$ which show that $F(T)$ is also symmetrical countable universal {11}. So, if T is a first order theory on a finite relational language which satisfies the amalgamation property and if the countable models of T are the class of all possible worlds, then $F(T)$ is the unique possible world which is homogeneous and countably T-universal. Therefore, the Fraïssé-limit $F(T)$ is the unique model which has the highest form of symmetry or harmony since it is homogeneous, and which has also the richest possible variety of substructures since it is countably universal. So, $F(T)$ can be considered as the best of all possible worlds in the sense of Leibniz. The theorem of Fraïssé can also be considered as a model theoretic analogue of the principle of composability of Leibniz. Indeed, if the finite models of the theory T form an amalgamation class, this means that any two finite models of T can always be embedded together in a finite model of T even if it is required that they had to have an arbitrary substructure in common.

Therefore, if this is the case, the maximum number of finite configurations of finite substructures can be realised together. By the theorem of Fraïssé they will then be the class of all finite substructures of the best of all possible worlds, namely the Fraïssé limit $F(T)$.

However, it is not true that $\aleph_0$-categoricity and finite universality imply homogeneity. In the case of linear graphs, for instance, Macpherson, H.D. and Droste, M. have constructed $2^{\aleph_0}$ non-isomorphic countable linear graphs which are all $\aleph_0$-categorical and which have all finite linear graphs as finite substructures. But the Fraïssé limit of the class of finite linear graphs is the unique linear graph of this class which is also homogeneous {6}.

8. The tree of an age, the Cantor topology and residual properties.

Using topological arguments we prove that if T is a first order theory on a finite relational language whose finite models satisfy the amalgamation property, then almost any countable model of T is isomorphic with the Fraïssé limit $F(T)$. This is the model theoretic analogue of the metaphysical statement that almost any possible world is isomorphic with the best of all possible worlds.

In a complete metric space (X,d) a subset $A \subseteq X$ is called *residual* if its complement is meagre or is a set of the first category which means that this complement is a countable union of nowhere dense sets {15}. In a complete metric space, A will be residual if and only if A contains a countable intersection of open dense sets. By the Baire category theorem a residual set in a complete metric space is always dense (and so, in particular, never empty). Residual sets in complete metric spaces are regarded as "very large". If there is no measure theory, these residual sets replace in fact the subsets with measure 1.

If the set of all points of a complete metric space which have a certain property P form a residual set, then one says that almost any point of this metric has property P, or that the property P is "forced in category" in this metric.

For each model of the theory T with n points we take $[n] = \{0, 1, 2, ..., n\text{-}1\}$ as universe and we denote this set of models by F_n.

We make the class of finite models of T into a tree $t(T)$ in the natural way: the nodes at level n are the elements of F_n and a node $\alpha_n \in F_n$ is adjacent with a node $\beta_{n+1} \in F_{n+1}$ if and only if the restriction of the structure β_{n+1} to $[n]$ is α_n. So, we obtain a tree with the trivial model ω on $\{0\}$ as base point. Since L is a finite relational language, the valence of each node or the number of edges adjacent with each node is finite. For each countable model of the theory T we take $\mathbb{N}$ as universe. If M is a countable model of T, it defines an finite path in the tree $t(T)$ starting with the base point: $\{\omega, \alpha_1, \alpha_2, ..., \alpha_n, ...\}$ where α_n is the restriction of M to $[n]$.

Conversely, every infinite path of this tree $t(T)$ with ω as starting point defines a countable model of T on $\mathbb{N}$.

The infinite paths with ω as first node are also called the points at infinity of the tree $t(T)$ with ω as base point. So, there is a 1-1 correspondence between the points at infinity of $(t(T),\omega)$ and the countable models of T. If M and N are two countable models of T on the set $\mathbb{N}$, then we define the distance $d(M,N) = 2^{-n}$ if and only if the associated infinite paths with base point ω agree to level n but no further. Or if the countable models M and N correspond with the paths $(\omega, \alpha_1, \alpha_2, \ldots)$ and $(\omega, \beta_1, \beta_2, \ldots)$ of $(t(T), \omega)$ respectively, with $\alpha_i = \beta_i (1 \leq i \leq n)$ but $\alpha_{n+1} \neq \beta_{n+1}$.

This defines an ultra metric on the class of countable models of T for $1 \leq i \leq n$, since for any three models M_1, M_2 and M_3 $d(M_1, M_3) \leq \max\{d(M_1, M_2), d(M_2, M_3)\}$.

In this metric each open sphere is also closed and if two spheres are not disjoint, then one of them contains the other. This metric is called the Cantor metric, and it is complete since a Cauchy sequence of paths has the property that its members agree on longer and longer initial segments, so these initial segments define a path of $(t(T),\omega)$ which is then the limit path of this Cauchy sequence.

The associated topology is the Cantor topology which can be characterized as the unique topology which is metrisable, totally disconnected, perfect and compact {15}. In particular, the topology of any compact metric space is a continuous image of the Cantor topology.

Is s a $(\forall, \exists)$-sentence in the language L, then the set of points of infinity of $(t(T),\omega)$ or the set of countable models of T for which s is true is always a countable intersection of open dense sets in the associated Cantor topology {4, 11} and so this set is residual. The Fraïssé limit $F(T)$ of T is completely characterized by the property (A) which translated in the first order language L gives a countable recursive set of $(\forall, \exists)$-sentences, the so called Alice's Restaurant axioms. Since a countable intersection of residual sets is again residual, and since each countable model of T which satisfies the Alice's Restaurant axioms is isomorphic with $F(T)$, we have that the class of countable models of T on $\mathbb{N}$ which are isomorphic with the Fraïssé limit form a residual set in the natural Cantor topology on the set of countable models of T. Hence, almost any countable model of T is isomorphic with $F(T)$, or the property of being isomorphic with the Fraïssé limit of T is forced in category {3, 11}.

9. The theory of almost true sentences of a given theory, zero-one laws and countable random models.

Let T be a theory on a finite relational first order language L. The number of models of T on a universe of n points is then finite and we denote it by $A(n)$. If s is a sentence of L, we denote the number of models of T on n points for which the sentence s is true by $B(n,s)$ and we call

$$Prob_n(s,T) = \frac{B(n,s)}{A(n)}$$

the probability that the sentence s is true in a model of T on n points. If $\lim_{n\to\infty} Prob_n(s,T)$ exists, then we call this limit the *asymptotic probability* of the sentence s in the theory T and we denote it by $Prob(s,T)$. We say that a sentence s of L is *almost true* in the theory T if and only if $Prob(s,T) = 1$ (this idea comes from Carnap). We denote the set of all sentences in the language L which are almost true in the theory T by T^*. Of course, if $\beta \in T^*$ and $s \models t$ then $t \in T^*$ and so T^* is also a theory. We call T^* the theory of the almost true sentences of the theory T. If the theory T^* is complete (i.e. $s \notin T^* \Rightarrow \sim s \in T^*$), then we say that the theory T satisfies the zero-one law (the asymptotic probability in T of each sentence of L is 0 or 1). Each sentence of L is then almost true or almost false. Any model of T^* is called a random model of T. In the case that the theory T^* is not only complete but also $\aleph_0$-categorical, there is up to an isomorphism a unique countable model of T^* which we call the countable random model of T and we denote it by R(T). In the case that T is the theory of the L-structures (for an arbitrary finite relational first order language L), Faggin has proved that T* is $\aleph_0$-categorical and that $R(T) = F(T)$ {7}. In this case the unique countable random model of T is isomorphic with the Fraïssé-limit of the class of finite models of T. The same is true if T is the first order theory of linear graphs, of m-coloured graphs, and of n-uniform hypergraphs. These proofs are easy and use only elementary probabilistic arguments {7}. The result $R(T) = F(T)$ is the model theoretic analogue of the metaphysical statement that the best of all possible worlds is also the most probable of all worlds or that it is a random model.

When is $R(T) = F(T)$, in other words when is the Fraïssé-limit of a first order theory isomorphic with the countable random model of this theory?

So, we have to consider a theory T on a finite relational first order language L such that the finite models of T form an amalgamation class and such that the theory T of all almost true sentences of T is $\aleph_0$-categorical. If we denote the set of the Alice's Restaurant axioms by $A(T)$, then the theory generated by $T \cup A(T)$ is always $\aleph_0$-categorical since the axioms $A(T)$ (or property (A)) are exactly what is necessary to apply the back and forth argument. But, in general, $T \cup A(T)$ is not consistent, this will be the case if and only if the finite models of T form an amalgamation class and then $F(T)$ is the unique countable model. If $A(T) \subseteq T^*$or if the Alice's restaurant axioms are almost true, then the unique countable model of $T \cup A(T)$ or $F(T)$ had to be the unique model of T^* or $R(T)$ and conversely. So, $F(T) = R(T)$ if and only if Alice's restaurant axioms are all almost true or have asymptotic probability equal to 1. As an example take for T the theory of linear graphs or irreflexive symmetric binary relations. So, L is the first order language with binary relation symbol R and T is the theory generated by the axioms: $\forall x(\sim R(x,x))$ and $\forall x \forall y((R(x,y) \& R(y,x)) \vee (\sim R(x,y)$ $(\sim R(y,x)))$ Now for linear graphs property (A) is equivalent with the following property (B): If A_1 and A_2 are two disjoint finite sets of vertices (or points), then there exists a vertex z which is adjacent with all vertices of A_1 and with no vertex of A_2. But property (B) is also equivalent with the following countable set of axioms $C_n(n \in \mathbb{N})$.

C_n says that if A_1 and A_2 are two disjoint sets of vertices each of cardinality at most n, then there is a vertex z outside $A_1 \cup A_2$ which is adjacent to every vertex of A_1 but not to any vertex of A_2. Now C_n can be translated in the language L : $\forall x_1 \forall y_1 \forall x_2 \forall y_2 \ldots \forall x_n \forall y_n \exists z((R(x_1,z) \& R(x_2,z) \& \ldots \& R(x_n,z) \& \sim R(y_1,z) \& \sim R(y_2,z) \& \ldots \& \sim R(y_n,z)) \vee (x_1 = y_1 \vee x_1 = y_2 \vee \ldots \vee x_n = y_1 \vee \ldots \vee x_n = y_n))$ If $m > 2n$, what is $\mathrm{Prob}(C_n, T)$? We denote the set of vertices by X_m. Let $z \in X_m - (A_1 \cup A_2)$; then the probability that z is adjacent to all vertices of A_1 and to no vertex of A_2 is 2^{-2n} and so the probability that it is not the case is $(1-2^{-2n})$. So, the probability that for the two disjoint sets A_1 and A_2 C_n is not true, or that there exists no vertex $z \in X_m - (A_1 \cup A_2)$ with the required property is

$(1-2^{-2n})^{m-2n}$. Therefore, the expected number of bad pairs (A_1, A_2) is at

most $\binom{m}{2n}\binom{m-n}{n}(1-2^{-2n})^{m-2n}$ which approaches zero as $m \to \infty$. This

means that $Prob(C_n, T) = 1$(for all $n \in \mathbb{N}$).

So, the Fraïssé-limit of the first order theory of linear graphs is the unique countable random model of this theory and this graph is called the countable random graph.

For models and interesting properties of this graph we refer to {2, 4, 11}. For a model of the countable random graph which proves that it is also symmetrical T-universal see, for instance, {11}.

However, there exist counterexamples of theories on a finite relational first order language L for which $F(T)$ and $R(T)$ exist but are not isomorphic. For instance, if T is the first order theory of posets (or reflexive, anti-symmetric and transitive relations). In this case $T \cup A(T)$ and T^* are both $\aleph_0$-categorical theories, but they are different or $F(T) \neq R(T)$.

Here Alice's Restaurant axioms say, in fact, that for any three disjoint sets A,B and C each of size at most n, such that no element of B lies above any element of A or below any element of C and no element of A lies below an element of C, there is any element z which lies below every element of A, and above every element of C while being incomparable to everything in B. The unique countable poset satisfying these axioms (or $F(T)$) is of course, homogeneous and countable T-universal, but it is not the countable random poset. Indeed, these Alice's Restaurant axioms are in this case almost false (i.e. they have asymptotic probability 0)! {16, 11}. However, the theory T^*

of almost true sentences in *T* is also $\aleph_0$-categorical and *R*(*T*) is a certain graded poset of heigth 3 which is certainly not universal, since graded posets of heigth greater than 3 cannot be embedded, and which is certainly also not homogeneous since the symmetry group has exactly three orbits on the points. For a complete description of *R*(*T*) we refer to {9}. Of course, the class of countable posets which are isomorphic with the Fraïssé limit *F*(*T*) remains residual in the Cantor topology but residuality and probability or measure do not always agree. So, for the class of posets the best of all possible worlds is not the most probable of all possible worlds.

10. Random symmetry and universality.

For many first order theories *T* on a finite relational language there exists a unique countable random structure *R*(*T*) which is isomorphic with the Fraïssé limit *F*(*T*) and this random structure is very symmetric since it is homogeneous, and also each countable model of *T* can be embedded in it, therefore it is also very universal. In many cases *R*(*T*) is also symmetric T-universal. Since *R*(*T*) has a large group of symmetries whose order (as for each oligomorphic permutation group) is $2^{\aleph_0}$, each finite or countable model of *T* appears infinitely many times in it as a substructure including the whole structure itself. For instance, for the countable random graph not only each sphere but also each finite intersection of spheres satisfies property (*B*) and so it is isomorphic with the whole random graph again. Therefore, in these cases the best of all possible worlds contains all possible worlds including itself infinitely many times and so each possible world is in fact realised.

This idea is not new; for instance, Whitehead describes in his book "Process and Reality: an essay in cosmology" how in the fullness of time all possible worlds will exist (see {14}). Whitehead's cosmology is, broadly speaking, the same as the "chaotic inflationary" models developed recently by A. Linde in which the visible portion of the universe is just one of an infinite number of bubbles in an overall chaotic random universe. (see {1}).

On a sufficiently large scale, the universe is pictured as chaotic, for assuming global chaos obviates the problem of assuming certain initial conditions. Also the Many-Worlds interpretation of quantum theory gives a world model which allows evolution to occur on a global scale while simultaneously allowing all possible universes to exist (in the Hilbert space of realised possibilities) {1}.

As we have seen, such universal possible worlds which contain all possible worlds can be random structures.

But these best of all worlds are not only universal but also homogeneous and so highly symmetric. Not only as a permutation group Aut*F*(*T*) describes the highest possible symmetry for a countable model of T, also as an abstract group it has very rich properties. For instance, Truss {13} has proved that the symmetry groups of the countable random m-coloured graphs (and for $m = 2$ this is the countable random graph) are all simple groups not belonging to the class of simple groups of Lie type, so they are in fact sporadic simple infinite groups!

Symmetry plays an increasingly important role in modern physics (the standard model, Yang-Mills theory, supersymmetry theories, string theory and so on ...).

But already in the time of Newton it was a problem how high forms of symmetry have appeared in the world (see, for instance, the second half of the book "Natural Theology" of William Payley {12}).

From our examples in model theory it follows that it is possible that in an infinite world high forms of symmetry are simply the result of a random process.

So, for many classes of models, no God is necessary to choose out of the class of all possible worlds the best one. Almost certainly, any random choice will do this. However, this has nothing to do with the ontological argument of Leibniz about the existence of God.

On the problem "why not nothing?" one could answer that the empty model is not a random model. But, of course, the real problem is why some possible world is a real world.

University of Ghent

REFERENCES

[1] J. D. Barrow and F. J. Tipler. *The Anthropic Cosmological Principle.* Oxford University Press, Oxford", 1986".

[2] P. J. Cameron. Aspects of the random graph. In Bollobas, editor, *Graph Theory and Combinatorics*, London Academic Press, 1984, pp. 65-79.

[3] P. J. Cameron. *Oligomorphic Permutation Groups*. Volume 152 of *Lecture Notes in Math.* Cambridge University Press, London, 1991.

[4] P. J. Cameron. The age of a relational structure. *Discrete Math.* Volume 95, 1991, pp. 49-67

[5] G. Cantor. Beitrage zur Begrundung der transfiniten Mengenlehre. *Matematische Annalen*, volume 26, 1895, pp. 481-512.

[6] M. Droste and H. D. Macpherson, On k-Homogeneous Posets and Graphs. *Journal of Combinatorial Theory*, volume A 56, 1991, pp. 1-15.

[7] R. Faggin. Probabilities of finite models. *Journal of Symbolic Logic*, volume 41, 1, 1976, pp. 50-58.

[8] R. Fraïssé. Sur certains relations qui généralisent l'ordre des nombre rationels. C. R. *Acad. Sci. Paris*, volume 237, 1953,pp. 540-542.

[9] D. J. Kleitman and B. L. Rothshild. Asymptotic enumeration of partial orders on a finite set. *Trans. Amer. Math. Soc.*, volume 205, 1975, pp. 205-210.

[10] G. W. Leibniz. On the Radical Orgination of Things (1697). In L. E. Loemker, editor, *Philosophical Papers and Letters*, pp. 486-491, Dordrecht, D. Reidel, 1969.

[11] W. Mielants. $\aleph_0$-categorische structuren. *Simon Stevin*, volume 66 and 67, 1992-1993.

[12] W. Paley. Natural theology (1802). In H. Brougham and C. Bell, editors, *Paley's Natural Theology with Illustrative Notes*, 1836. 2 volumes.

[13] J. Truss. The group of the countable universal graph. *Math. Cambridge Philos. Soc.*, volume 98, 1985, pp. 213-245.

[14] A.N. Whitehead. *Process and Reality: an essay in cosmology* (1928). The Free Press, Cambridge, 1969.

[15] S. Willard. *General Topology*. Addison-Wesley series in Mathematics, 1970.

[16] P. Winkler. Randomstructures and Zero-one laws. In R. Woodrow, editor, *Finite and Infinite Combinatorics of Sets and Logic*, Kluwer Academic Publishers, 1992, pp. 1-22.