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ON SELF-MEMBERED SETS IN QUINE'S SET THEORY NF*

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In NF, there are many self-membered sets (for example, the universal set, the set of all infinite sets, ...). D. Scott [2] proved the relative consistency of "no singleton is self-membered" with NF. T.Forster [1] (chapter 5) asked for a relative consistency proof of "every self-membered set is infinite" with NF. In this short Note, we prove the relative consistency of "no self-membered set is strongly cantorian" with NF. This implies the relative consistency of "every self-membered set is infinite" with NF + Rosser's axiom of counting.

We will use the permutation method of Bernays-Rieger described in chapter 3 of [1]. The basic notions concerning NF are given in chapter 2 of [1].

In NF, let us define $A(x)$ as the biggest set A such that x is disjoint from $A \times V$ (where V denotes the universal set and the cartesian product is defined in terms of Quine's ordered pairs). Since x and $A(x)$ have the same type, we have defined a function $A: V \to V$.

Lemma 1. If x is strongly cantorian, then $V - A(x)$ (the complement of $A(x)$) is strongly cantorian.

Proof. If x is strongly cantorian, then Px (the power set of x) is strongly cantorian, thus $USC(V - A(x))$ (the set of singletons of elements of $V - A(x)$) is strongly cantorian, since the map $\{y\} \mapsto x \cap (\{y\} \times V)$ is an injection from $USC(V - A(x))$ into Px.

Lemma 2. There is a function $F: V \to V$ such that $F(x) \in A(x)$ for any strongly cantorian set x.

Proof. Let U be the operation on ordinal numbers corresponding to the operation $RUSC$ on well-orderings. Since U raises the types by 1, there is a function $F: V \to V$ defined by

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$$F(x)=\begin{cases} U \text{ (the smallest ordinal in } A(x)) & \text{if } A(x) \text{ contains ordinals,} \\ 0 & \text{otherwise.} \end{cases}$$

If x is strongly cantorian, then $V-A(x)$ is strongly cantorian, thus $A(x)$ contains ordinal numbers. Let α_x be the smallest ordinal in $A(x)$. Since $\{\alpha | \alpha < \alpha_x\}$ (which is included in $V-A(x)$) is strongly cantorian, we have $U(\alpha_x)=\alpha_x$, thus $F(x)\in A(x)$.

Theorem. If *NF* is consistent, then *NF*+"no self-membered set is strongly cantorian" is consistent.

Proof. Let F be as in Lemma 2 and let p be the involution of V exchanging x and $\langle F(x), \langle x, V\rangle\rangle$ for any x of power different from the power of V (the brackets denote Quine's ordered pair). Let V_p be the permutation model corresponding to p, i.e. the model with domain V in which the $\in$-relation is interpreted by the relation $\in_p$ defined by $x \in_p y \leftrightarrow x \in p(y)$. If x is strongly cantorian in V_p, then $p(x)$ is strongly cantorian, thus (by Lemma 2) $p(p(x)) \notin p(x)$, thus $x \notin p(x)$ (since p is an involution), i.e. $x \notin_p x$.

Since Rosser's axiom of counting is preserved by the permutation method, and since this axiom implies that any finite set is strongly cantorian, we have the

Corollary. If *NF*+Rosser's axiom of counting is consistent, then *NF*+Rosser's axiom of counting + "every self-membered set is infinite and non strongly cantorian" is consistent.

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[2] D. Scott, *Quine's individuals*, in Logic, Methodology and Philosophy of Science, Stanford University Press, 1962, p. 111-115.