Logique & Analyse **141-142** (1993), 25-38

MODALITIES IN SUBSTRUCTURAL LOGICS

Greg RESTALL

1. Logics

Logic is about valid deduction. One central result in logic is the *deduction theorem.*

$$A,\ \Sigma \vdash B \text{ if and only if } \Sigma \vdash \mathrm{A} \to \mathrm{B}$$

This result ties a fact from the metalanguage (that B follows from A, in the context of Σ) to a fact in the object language ($A \to B$, in the context of Σ). This connection is very important, because it shows how the properties of the conditional in a logic depend on the way premises are collected together — represented here by the comma. Standard brands of logics, like intuitionistic and classical logic, allow for all sorts of rearrangements in premises. They have *structural rules* like these:

$$\frac{\Gamma(A, A) \vdash B}{\Gamma(A) \vdash B} \text{ Contraction (WI)} \qquad \frac{\Gamma(A) \vdash C}{\Gamma(B, A) \vdash C} \text{ Weakening (K)}$$

$$\frac{\Gamma(A) \vdash B}{\Gamma(A, A) \vdash B} \text{ Mingle (M)} \qquad \frac{\Gamma(A, (B, C)) \vdash D}{\Gamma((A, C), B) \vdash D} \text{ Commuted Associativity (CB)}$$

$$\frac{\Gamma(A, (B, C)) \vdash D}{\Gamma((A, B), C) \vdash D} \text{ Associativity (B)} \qquad \frac{\Gamma(A, B) \vdash C}{\Gamma(B, A) \vdash C} \text{ Commutativity (CI)}$$

In a *substructural* logic, one or more of these rules are rejected. This means that our premise collections have *more* structure than would otherwise be the case. The number of premises, their order, their bracketing, or what *isn't* a premise, matters. Relevant logics, linear logics, and logics like BCK and fuzzy logics are all substructural logics, because these logics keep track of premises in ways that logics with all the structural rules don't.

However, even if we reject certain structural rules in *general*, by adopting a substructural logic, we may be interested in cases when substructural rules do apply. We may not think that all premises commute — but there may be a small class of premises which do commute, and this may be important. Similarly, we may not be able to contract all premises — but some might contract validly. So, it is interesting to see how we can keep track of structural rules with a limited applicability. That is the topic of this note. We will take our cue from Girard, and add *modalities* to our logics to regain structural rules [2].

It is easiest to give our account of modalities in substructural logic if we use an algebraic presentation of the formal systems. This will make the theorems easier to prove than would otherwise be the case. So it is to this that we will turn.

Substructural logics (like any) can be modelled by *propositional structures*. Think of the elements of these structures as propositions, and the operations as ways to form new propositions out of old.

Definition 1 A *basic propositional structure* is a 5-tuple $\langle P; \leq, \cdot, e, \Rightarrow \rangle$ where

- P is a nonempty set of propositions.
- $\leq$ is a partial order on P, representing entailment between propositions.
- The binary operation $\cdot$ on P, called fusion. This represents premise combination. Fusion preserves the entailment ordering. If $a \leq a'$ and. $b \leq b'$ then $a \cdot b \leq a' \cdot b'$.
- The element $e \in P$ is a right identity for fusion. This means that for all $a \in P$, $a \cdot e = a$. Because of this, e represents logical truth.
- The binary operation $\Rightarrow$ on P *residuates* fusion. For all a, b, $c \in P$, $a \cdot b \leq c$ if and only if $b \leq a \Rightarrow c$. This means that $\Rightarrow$ is our *conditional* operation on P. (Recall the deduction theorem, tying the conditional to valid deduction. This condition is the deduction theorem in algebraic form.) If $a \leq b$ then by residuation and identity, $e \leq a \Rightarrow b$. So, if a entails b, then the conditional $a \Rightarrow b$ is a logical truth. This is a desirable connection between entailment, logical truth, and the conditional.

If we wish to model conjunction and disjunction, we require that $\leq$ be a lattice order. In these structures, conjunction is the greatest lower bound, and disjunction, the least upper bound.

Definition 2 A *basic propositional structure with conjunction and disjunction* is a basic propositional structure where the underlying entailment ordering is a lattice.

Definition 3 A *basic propositional structure with a distributing conjunction and disjunction* is a basic propositional structure where the underlying entailment ordering is a distributive lattice.

If we wish to model negation, we can add a unary operator — on P, with suitable conditions.

Definition 4 A *basic propositional structure with an intuitionistic negation* is a basic propositional structure equipped with a unary operation — such that for each $a, b, c \in P, a \cdot b \leq -c$ if and only if $c \cdot b \leq -a$..

This condition provides us with contraposition in the form $a \Rightarrow -b = b \Rightarrow -a$, and double negation introduction, $a \leq --a$.

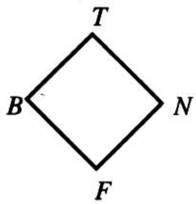
Definition 5 A *basic propositional structure with a de Morgan negation* is a basic propositional structure with an intuitionistic negation — such that, in addition, for each $a \in P, a = --a$.

We use propositional structures to model valid deduction in the usual way. Take a propositional language with connectives $\rightarrow$, $\circ$ and t and any of $\wedge$, $\vee$ and $\neg$ as desired. We define the interpretation $h(A)$ of a formula A in a propositional structure P recursively, from arbitrary assignments to atoms, using these clauses:

$$h(t) = e \quad h(A \circ B) = h(A) \cdot h(B) \quad h(A \rightarrow B) = h(A) \Rightarrow h(B)$$
$$h(A \wedge B) = h(A) \cap h(B) \quad h(A \vee B) = h(A) \cup h(B) \quad h(\neg A) = -h(A)$$

Take a class X of propositional structures. We say that a formula A is X-*valid* if for each structure $P \in X$ and for each interpretation h into P, we have $e \leq h(A)$. We write this '$\vdash_X A$.' For largely uninteresting historical reasons, the class of all propositional structures with a de Morgan negation, conjunction and disjunction is called *LDW*, and its cousin which contains only those structures in which disjunction and conjunction distribute is called *DW*.

One interesting propositional structure which illustrates our definition is BN4. Belnap [1], Meyer, Giambrone and Brady [3] and Slaney [4] all extol its virtues. The structure is simple.



$\cdot$	T	B	N	F
T	T	T	N	F
$e=B$	T	B	N	F
N	N	N	F	F
F	F	F	F	F

$\Rightarrow$	T	B	N	F
T	T	F	N	F
B	T	B	N	F
N	T	N	T	N
F	T	T	T	T

	$-$
T	F
B	B
N	N
F	T

Conjunction and disjunction are given by greatest lower and least upper bounds on the lattice ordering — which is indicated here by the Hasse diagram. This structure has a number of notable features: not least being the fact that negation has two fixed points, B and N. This shows that a de Morgan negation need not satisfy the usual classical conditions. Furthermore, premise combination is associative and commutative, but not idempotent. So *BN*4 satisfies some of the structural rules we've seen, but not all. It is helpful to catalogue structural rules in algebraic form, like this:

WI	$a \leq a \cdot a$
K	$b \cdot a \leq a$
M	$a \cdot a \leq a$
B	$(a \cdot b) \cdot c \leq a \cdot (b \cdot c)$
CB	$(a \cdot c) \cdot b \leq a \cdot (b \cdot c)$
CI	$a \cdot b \leq b \cdot a$

We can tack selections from these onto our basic logic to make all sorts of interesting formal systems. Given a class of propositional structures X, the class $X + Y$ is the subclass of X of all structures which satisfy condition Y. Girard's original linear logic (without exponentials) is then given by *LDW* + *B* + *CI*. So, in linear logic premise combination is associative and commutative. The contraction-free relevant logic C adds distribution to linear logic, so it is given by *DW* + *B* + *CI*. The relevant logic R is given by *DW* + *B* + *CI* + *WI*. We elide the distinction between classes of propositional structures and logics, so we will say things like $R = C + WI$ from time to time.

Now to round this section off we'll prove an important result which shows that we can restrict our attention to propositional structures of a particular kind, when using them to model a logic.

Definition 6 A propositional structure P is said to be *complete* if for every $X \subseteq P$, the join and meet $\bigvee X$ and $\bigwedge X$ exist.

Fact 1 *Any propositional structure can be embedded as a substructure in* *a complete propositional structure.*

Proof: Take a propositional structure P. We take the elements of the *completion* of P, P^* to be the *ideals* of P. A subset $X \subseteq P$ is an ideal if for each $x \in X$, if $y \leq x$ then $y \in X$ too. If P is equipped with disjunction, we require that $x, y \in X$ only if $x \in X$ too.

Conjunction in this structure is simply intersection of ideals (the intersection of any set of ideals is an ideal). So, arbitrary meets exist in this structure. This means that containment, $\subseteq$, is the ordering in the new structure.

If disjunction is present in P, then the disjunction $X \cup Y$ is the set $\{z\colon \exists x \exists y(x \in X,\ y \in Y \text{ and } z \leq x \cup u\}$. Otherwise, we can define disjunction as union of ideals. (If ideals are merely downwardly closed sets, then the union of two ideals is an ideal.) Arbitrary joins exist in this structure too. If disjunction is simply union, then ideals are closed under infinite unions. If disjunction is defined in terms of disjunction in P, the definition is a little more complex.

$$\bigvee \{X_i : i \in I\} = \{z\text{:for some } i_1, \ldots i_n \in I \text{ and } \mathrm{x}_{\mathrm{i}_1} \in X_{i_1}, \ldots x_{i_n} \in X_{i_n},\ z \leq x_{i_1} \cup \ldots \cup x_{i_n}\}$$

Fusion, the residual, e^* and negation (if present) are defined in similar ways

$$X \cdot Y = \{z\colon \exists x \exists y(x \in X,\ y \in Y \text{ and } z \leq x \circ y)\}$$
$$X \Rightarrow Y = \{z\colon \exists x \exists y(x \in X,\ y \in Y \text{ and } z \leq x \rightarrow y)\}$$
$$e^* = \{x\colon x \leq e\}$$
$$-X = \{z\colon \exists x(x \in X \text{ and } z \leq -x)\}$$

It is simple to check that the conditions on propositional structures are satisfied in the completion P^*, so P^* is a propositional structure. The original structure P lives in P^* in the guise of its principle ideals. The map $f\colon x \mapsto \{y\colon y \leq x\}$ is an injection of P into P^*.

If our original structure P satisfied a structural rule, then it is clear that P* also satisfies that rule (check the definition of fusion). ◁

This result shows us that complete propositional structures are enough, when it comes to modelling logics. The class of validities in all $\mathfrak{R}$ propositional structures is the same as the class of validities in all complete $\mathfrak{R}$ propositional structures, because any counterexample to validity in an incomplete structure survives as a counterexample in the completion of that structure. This fact is important, because we will need to assume that structures are complete in one result coming up.

Note that in complete propositional structures P, $\bigwedge P \leq x$ for any $x \in P$. So, complete propositional structures have a least element. In what follows, we will use '0' to denote the least element in a propositional structure.

Excursus This embedding result also motivates a short polemical point about conjunction and disjunction. The result has a simple corollary that any structure without conjunction and disjunction can be extended to include conjunction and disjunction in a natural, painless way. Furthermore, in this extended structure, conjunction distributes over disjunction. This is especially important for devotees of linear logic. In linear logic, additive conjunction does not distribute over additive disjunction. Our result shows that this is not a feature of the substructural nature of the logic — rather, it is a bare fact about the additive connectives. Since it is quite hard to interpret additive disjunction in linear logic, it is quite hard to see what a failure of distribution amounts to. Unless some interpretation for disjunction can be found, in such a way that motivates the failure of distribution, there is little reason to favour linear logic over its distribution-added cousin C. □

Leaving polemics about distribution aside, we will assume from now, that all logics come equipped with disjunction and conjunction (whether they distribute or not) because this will make the modal conditions simpler to state.

2. *Modalities*

There are a number of ways to introduce modalities to substructural logics. This note will focus on extremely popular modalities in the vicinity of $S4$ and $S5$. In our context, an $S4$ *necessity* satisfies the following conditions.

$T_\Box$	$\Box A \to A$	$\wedge_\Box$	$\Box A \wedge \Box B \to \Box(A \wedge B)$
$4_\Box$	$\Box A \to \Box\Box A$	$\to_\Box$	$\Box(A \to B) \to (\Box A \to \Box B)$
$N_\Box$	If $\vdash A$ then $\vdash \Box A$		

If a de Morgan negation is present, and if the $S4$ necessity also satisfies

$5_\Box$ $A \to \Box\neg\Box\neg A$

it is said to be an $S5$ *necessity*. These necessities have interesting properties in substructural logics.

Fact 2 All *S4 necessities satisfy* $\vdash \Box(\Box A \vee \Box B) \leftrightarrow \Box A \vee \Box B$

Proof: $\vdash \Box(\Box A \vee \Box B) \rightarrow \Box A \vee \Box B$ by $T_\Box$. For the other direction, we have $\vdash \Box A \rightarrow \Box A \vee \Box B$, so by $N_\Box$, $\vdash \Box(\Box A \rightarrow \Box A \vee \Box B)$, and $\rightarrow_\Box$ with $4_\Box$ gives, $\vdash \Box A \rightarrow \Box(\Box A \vee \Box B)$. Similarly we have $\vdash \Box B \rightarrow \Box(\Box A \vee \Box B)$, so we have our result, using the lattice properties of $\vee$. ◁

We'll call a proposition N a *necessitive* if and only if for some A, $\vdash N \leftrightarrow \Box A$. It is simple to show that if $\Box$ is an $S4$ necessity, that N is a necessitive if and only if $\vdash N \leftrightarrow \Box N$. This fact means that necessitives are closed under disjunction (where present). Necessitives are also closed under fusion.

Fact 3 In logics with fusion, all $S4$ necessities satisfy $\vdash \Box(\Box A \circ \Box B) \leftrightarrow \Box A \circ \Box B$.

Proof: Clearly, $\vdash \Box(\Box A \circ \Box B) \rightarrow \Box A \circ \Box B$. For the converse, we have $\vdash \Box B \rightarrow (\Box A \rightarrow \Box A \circ \Box B)$. So, $\vdash \Box\Box B \rightarrow \Box(\Box A \rightarrow \Box A \circ \Box B)$. This quickly gives $\vdash \Box B \rightarrow \Box A \rightarrow (\Box A \circ \Box B)$ and hence $\vdash \Box A \circ \Box B \rightarrow \Box(\Box A \circ \Box B)$ as desired. ◁

In addition, if the modality is $S5$ish, then necessitives are closed under negation.

Fact 4 All $S5$ necessities satisfy $\vdash \Box\neg\Box A \leftrightarrow \neg\Box A$.

Proof: Left to right is $T_\Box$, and right to left is given by $4_\Box$, $5_\Box$ and double negation elimination. ◁

In propositional structures we can model a necessity by adding a unary operator on the structure.

Definition 7 In a propositional structure P, a function $I: P \rightarrow P$ is an $S4$ *interior* operator if it satisfies these conditions.

$$
\begin{array}{llll}
T_I & I(a) \leq a & \cap_I & I(a \cap b) = I(a) \cap I(b) \\
4_I & I(a) = I(I(a)) & \cup_I & I(I(a) \cup I(b)) = I(a) \cup I(b) \\
e_I & I(e) = e & \cdot_I & I(I(a) \cdot I(b)) = I(a) \cdot I(b)
\end{array}
$$

If, in addition, it satisfies

$$5_I \quad a \leq I(-I(-a))$$

it is said to be an *S5-interior* operator. A propositional structure with an interior operator (whether *S*4 or *S*5) is said to be a *modal* propositional structure.

These conditions on *I* are forced by the axioms for *S*4 and *S*5 necessities. (For the fusion and disjunction clauses, see Facts 1 and 2). *I* is called an 'interior' operator on a set of subspaces of a space because of conditions like 4_I and T_I.. This has not gone unnoticed in the algebraic semantics of modalities, and the observation provokes an alternate way of presenting the semantics of modal operators. Instead of defining an 'interior' operator *I*, we can restrict ourselves to the class $\mathbb{O}$ of 'open' elements. The objects a such that $I(a)=a$.. That is, the necessitives of our earlier discussion.

Fact 5 In any complete modal propositional structure the class $\mathbb{O}$ of open elements contains e and 0, and is closed under fusion, conjunction and disjunction.

Proof: This is simple. T_I gives us $I(0)\leq 0$, so $I(0)=0$. We have $I(e)=e$ by e_I. The conditions $\cap_I$, $\cup_I$ and $\cdot_I$ show that the class $\mathbb{O}$ is closed under conjunction, disjunction and fusion. $\lhd$

Let's call a class $\mathbb{O}$ on a propositional structure a *potential open class* if it contains e and 0, and is closed under fusion, conjunction and disjunction. Then we have another fact.

Fact 6 A propositional structure *P* with a class potential open class $\mathbb{O}$ is a modal propositional structure with interior operator *I* if we define

$$I(a)=\bigvee\{x: x\leq a \text{ and } x\in\mathbb{O}\}$$

provided that the infinite joins exist.

This proviso is unimportant in practice, because we can assume that propositional structures are complete, because of Fact 1.

Proof: It is simple to show that $I(a)\in\mathbb{O}$ for each $a\in P$, and that $I(a)=a$ for each $a\in\mathbb{O}$. It follows that $I(I(a))= I(a)$ and $I(a)\leq$ a for each $a\in P$. We have $I(e)=e$ as $e\in\mathbb{O}$. Similarly, as $\mathbb{O}$ is closed under disjunction (if present) and fusion, we have $I(I(a)\cup I(b))=I(a)\cup I(b)$ and $I(I(a)\cdot I(b))=I(a)\cdot I(b)$.

Finally, if conjunction is present, we can note that $I(a\cap b)\leq I(a)$ and $I(a\cap b)\leq I(b)$ gives $I(a\cap b)\leq I(a)\cap I(b)$. Conversely, $I(a)\cap I(b)\leq$

$a \cap b$ and $I(a) \cap I(b) \in \mathcal{O}$ gives $I(a) \cap I(b) \leq I(a \cap b)$, as desired. ◁

This is a useful fact, because we can define a modal operator on a propositional structure *merely* by specifying its set of open elements. This way of specifying a modal operator in a propositional structure is much simpler than defining a function I from scratch. So, in what follows we will define our interpretations of modalities by specifying the set of open elements.

This set of open elements provides an interesting structure *inside* the larger propositional structure. We can pick out these elements in our language by using the modal operator $\Box$. Whatever the formula A is evaluated as, we knowe that $\Box A$ will be evaluated as an open element. This is where adding *modalised* structural rules becomes interesting. We can add a number of axioms or conditions to our logic, saying that the open elements satisfy structural conditions which are not shared by all other propositions. Here are some axioms and conditions.

$WI_\Box$	$\Box A \to \Box A \circ \Box A$	$a \leq a \cdot a$	for each $a \in \mathcal{O}$
$K_\Box$	$\Box B \circ \Box A \to \Box A$	$b \cdot a \leq a$	for each $a, b \in \mathcal{O}$
$M_\Box$	$\Box A \circ \Box A \to \Box A$	$a \cdot a \leq a$	for each $a \in \mathcal{O}$
$B_\Box$	$(\Box A \circ \Box B) \circ \Box C \to \Box A \circ (\Box B \circ \Box C)$	$(a \cdot b) \cdot c \leq a \cdot (b \cdot c)$	for each $a, b, c \in \mathcal{O}$
$CB_\Box$	$(\Box A \circ \Box C) \circ \Box B \to \Box A \circ (\Box B \circ \Box C)$	$(a \cdot c) \cdot b \leq a \cdot (b \cdot c)$	for each $a, b, c \in \mathcal{O}$
$CI_\Box$	$\Box A \circ \Box B \to \Box B \circ \Box A$	$a \cdot b \leq b \cdot a$	for each $a, b \in \mathcal{O}$

These axioms say that necessitives satisfy conditions which propositions in general, in our logics, fail. The first axiom, $WI_\Box$, ensures that necessitives contract. In logics like *CK*, not all propositions contract. We 'regain' a measure of contraction by adding a modality which contracts. Similarly, we may be interested in modalities which give us weakening, or mingling for necessitives.

Definition 8 Given a non-modal logic *X*, we'll call $XS4 + C_{1\Box} \ldots C_{n\Box}$ the logic you get when you add an *S*4 modality satisfying the conditions $C_{1\Box}$ to $C_{n\Box}$. Similarly for $XS5 + C_{1\Box} \ldots C_{n\Box}$.

Adding structural rules to the open elements of a propositional structure provokes two questions. Firstly, does it do anything to the non-modal structure of the logic? That is, is the extention *conservative*?

Definition 9 A logic *X* is a *conservative extension* of *Y* if it extends *Y*'s vocabulary, but nothing that is *X* valid in the old vocabulary of *Y* isn't also Y valid.

Secondly, does the extension do us any good? We will answer these two questions in the next sections.

3. *Conservativeness*

Fact 7 $XS4$ and $XS5$ are conservative extensions of X.

Proof: Suppose A is not X valid. It has a counterexample in the propositional structure P. P can be made into an $S4$ or $S5$ modal structure by defining the class $\mathbb{O}$ of opens on P to be the class of all propositions. $\lhd$

Fact 8 Any logic X is conservatively extended by $XS4 + C_{1\Box}\ldots C_{n\Box}$, for any structural rules $C_{1\Box}$ tot $C_{n\Box}$.

Proof: This time, on any propositional structure P, take the set of opens to be $\{0, e\}$. This is a sublattice, and it satisfies any added structural rule $C_{1\Box}$. $\lhd$

Fact 9 Any logic X satisfying K is conservatively extended by $XS4 + C_{1\Box}\ldots C_{n\Box}$, for any structural rules $C_{1\Box}$ to $C_{n\Box}$

Proof: In logics satisfying K, $\{0, e\}$ is closed under negation. Firstly, note that $-0 = \top$ and $-\top = 0$ by contraposition, where $\top$ is the top element of the lattice. Then it is sufficient to show that given K, $e = \top$. But this is easy. Given K, $b = b \cdot e \leq e$ for all b. So, we have our result, as $\{0, e\}$ is a set of opens satisfying all structural rules. $\lhd$

However, the conservative extension results end here. In logics where $-e \neq 0$, the class of opens must at least contain $e, -e, 0, -0$. And in many logics, $-e$ is not guaranteed to satisfy structural rules. Let $f = \neg t$ Then we can collate together a number of facts.

Fact 10 Any logic X in which $\nvdash f \rightarrow f \circ f$ is not conservatively extended by $XS5 + WI_{\Box}$.

Fact 11 Any logic X in which $\nvdash f \circ f \rightarrow f$ is not conservatively extended by $XS5 + M_{\Box}$ or $XS5 + K_{\Box}$.

Fact 12 Any logic X in which $\nvdash f \circ f \rightarrow f \circ f$ is not conservatively extended by $XS5 + CI_{\Box}$.

Fact 13 Any logic X in which $\nvdash f \circ (f \circ f) \rightarrow (f \circ f) \rightarrow f$ is not conservatively extended by $XS5+B_{\Box}$.

These facts have bite, because they apply to interesting logics. It is not difficult to show that in linear logic or C, $f \to f \circ f$. So, these logics are not conservatively extended by their cousins with a contracting $S5$ modality. Similarly, in R, $\nvdash f \circ f \to f$. So, R is not conservatively extended upon the addition of an $S5$ modality with weakening or mingling. What is the significance of these results? It is hard to tell, beyond the fact that it is another reason to prefer $S4$ over $S5$. $S4$ gives us a sublattice of 'opens', closed under fusion. We have sublattices like these at hand which satisfy *every* structural rule, in every propositional structure. $S5$ forces the sublattices to be closed under negation, and this seems to be a Bad Thing in general. Only the case of logics satisfying K stands out as an exception.

So much for tracking the effects of adding modalities on the nonmodal fragment of logics. Now we will see what these modalities can do for us.

4. *Embeddings*

The substructure of opens in a modal propositional structure has a number of interesting properties. It is a sublattice, and it is closed under fusion. If the modal operator is $S5$ish, it is also closed under negation. However, in general, it is not closed under implication, and if the modality is $S4$ish, it is not, in general, closed under negation. However, we have a number of interesting results in *approximating* implication and negation in the substructure of opens.

Fact 14 If $\mathbb{O}$ is a collection of opens in a propositional structure P, then for each $a, b, c \in \mathbb{O}$, $a \cdot b \le c$ if and only if $b \le I(a \Rightarrow c)$.

Proof: If $b \le I(a \Rightarrow c)$ then $b \le a \Rightarrow c$ and hence, $a \cdot b \le c$. Conversely, if $a \cdot b \le c$ and $I(b) \le I(a \Rightarrow c)$. However, $b = I(b)$ and we have our result.$\lhd$

From now, let $a \supset b$ stand for $I(a \Rightarrow b)$. This result means that in the context of $\mathbb{O}$, $\supset$ residuates fusion. This is an important fact.

Fact 15 If $\mathbb{O}$ is a collection of opens in a propositional structure P equipped with a de Morgan negation $-$, then for each $a, b, c \in \mathbb{O}$, $a \cdot b \le I(-c)$ if and only if $c \cdot b \le I(-a)$,

Proof: If $a \cdot b \le I(-c)$ then $-I(-c) \cdot b \le -a$, and hence $I(-I(-c) \cdot b) \le I(-a)$. However, $c - I(-c)$, so $c \cdot b \le -I(-c) \cdot b$. But $I(c \cdot b) = c \cdot b$ (as b and c are opens), so $c\cdot b = I(c\cdot b)\le I(-I(-c)\cdot b)\le I(-a)$ as desired. $\lhd$

From now, let ~ a stand for $I(-a)$. This result means that in the context of $\mathcal{O}$, ~ is an intuitionistic negation. This, also, is an important fact.

Now we get to the payoffs. Let's show how we can embed logics with structural rules in logics which don't have those structural rules (which, in case you've forgotten, was the motivating thought behind this work). We need to define the translation $A^{\square}$ of a formula A. It is defined recursively:

$$
\begin{aligned}
p^{\square} &= \square p\\
(A\wedge B)^{\square} &= A^{\square}\wedge B^{\square}\\
(A\vee B)^{\square} &= A^{\square}\vee B^{\square}\\
(A\circ B)^{\square} &= A^{\square}\circ B^{\square}\\
(A\to B)^{\square} &= \square\left(A^{\square}\to B^{\square}\right)\\
(\neg A)^{\square} &= \square\neg\left(A^{\square}\right)
\end{aligned}
$$

Fact 16 In any modal propositional structure *P*, a formula of the form $A^{\square}$ is always interpreted as an open element.

Proof: Observe the construction of $A^{\square}$. Its atomic parts $p^{\square}$ are interpreted as open elements, and any of the original connectives in *A* are modified in $A^{\square}$ to always map into open elements too. $\lhd$

Fact 17 $A^{\square}$ is $XS5+C_{1\square}\ldots C_{n\square}-$valid if and only if *A* is $X+C_1\ldots C_n-$valid..

Proof: If $A^{\square}$ is $XS5+C_{1\square}\ldots C_{n\square}-$valid then it is true in all $XS5+C_{1\square}\ldots C_{n\square}$ structures. Let *P* be an $X+C_1\ldots C_n$ structure. It is clearly an $XS5+C_{1\square}\ldots C_{n\square}$ structure if we take the set of opens to be the set *P* of all elements. So, $A^{\square}$ will come out true in this structure by our assumption that it is $XS5+C_{1\square}\ldots C_{n\square}-$valid. However, $A^{\square}$ and *A* have the same interpretation, because I is the identity operation (as $\mathcal{O}$ = *P*). This means that *A* is $X+C_1\ldots C_n-$valid as desired.

Now let *A* be $X+C_1\ldots C_n$ valid. We wish to show that $A^{\square}$ is $XS5+C_{1\square}\ldots C_{n\square}-$valid. But this is simple. Take a $XS5+C_{1\square}\ldots C_{n\square}$ structure in which $A^{\square}$ is invalid. Let *h* be an interpretation which sends $A^{\square}$ to an element $x\not\ge e$ in *P*. The class of open elements $\mathcal{O}$ in *P* is a $X+C_1\ldots C_n$ structure, as we have seen, where we take $\supset$ to be the residual of fusion. We can define an interpretation *h'* into $\mathcal{O}$ by setting $h'(p)=I(h(p))=h(\square p)$. In this interpretation, $h'(A)=h\left(A^{\square}\right)$ by construction, and so, we have a counterexample to A in this structure. But we assumed that A has no counterexamples in $X + C_1 ... C_n$ structures, so we couldn't have a counterexample to $A^\Box$ in P. This means that $A^\Box$ is $XS5 + C_{1\Box} ... C_{n\Box}$ – valid as desired.

Fact 18 A is $XS4 + C_{1\Box} ... C_{n\Box}$ – valid if and only if $A^\Box$ is $X + C_1 ... C_n$ – valid, provided X is negation free.

Proof: Just like the proof in the previous fact. We can get away with using an $S4$ style modality because we have no need to show that the structure $\mathbb{O}$ of opens is closed under negation. What about negation and $S4$ modalities? Is there anything else we can say? There is, but it is incomplete. For now, if X is a logic, let JX be the corresponding logic given by 'liberalising' the negation laws to allow an intuitionistic negation. Then, we have the following result.

Fact 19 A is $XS4 + C_{1\Box} ... C_{n\Box}$ – valid (where X has a de Morgan negation) if $A^\Box$ is $JX + C_1 ... C_n$ – valid.

Proof: Just as before, but we note that the negation operator ~ on $\mathbb{O}$ satisfies the conditions for an intuitionistic negation, by Fact 15. So, the resulting substructure $\mathbb{O}$ inside an original modal stucture P is a $JX + C_1 ... C_n$ structure. ◁

It is a lot harder to prove the converse, because we would have to show that every $JX + C_1 ... C_n$ structure can be embedded within a $XS4C_{1\Box} ... C_{n\Box}$ structure. This is difficult, because we need to construct a de Morgan negation and a modality from 'thin air' while ensuring that they interact in the desired way (satisfying $\sim a = I(-a)$ for each $a \in \mathbb{O}$). This is a difficult task, and one I must leave for another occasion, and for other techniques.

5. *Conclusion*

We have learned a number of things in these few pages. We have shown how to model modalities in a general substructural setting, and we have seen the way that adding structural rules to the necessitives or 'opens' may leave the underlying non-modal structure unscathed – or it may change it. Finally, we've seen how adding structural rules gives us a translation from systems with structural rules in general, into systems which contain those structural rules in a qualified form. This has opened a way to see Girard's results in modalities in linear logic in a much wider setting.*

6. Note

*Thanks to Pragati Jain, Allegra Bencivenni and an anonymous referee for comments helping me to clarify my presentation. After writing and submitting this paper, I came across Kosta Došen [2], which covers similar ground (but not exactly the same ground) to this paper. Anyone interested in these issues should carefully study that paper.

Australian National University

REFERENCES

[1] Nuel D. Belnap, Jr. "A Useful Four-Valued Logic," pages 8-37 in J.M. Dunn and G. Epstein, editors *Modern Uses of Multiple-Valued Logics*, D. Reidel, Dordrecht, 1977.

[2] Kosta Došen. "Modal Translations in Substructural Logics," *Journal of Philosophical Logic 21* (1992) 283-336.

[3] Jean-Yves Girard. "Linear Logic," *Theoretical Computer Science 50* (1987) 1-101.

[4] Robert K. Meyer, Steve Giambrone and Ross T. Brady. "Where Gamma Fails," *Studia Logica 43* (1984) 247-256.

[5] John K. Slaney. "The Implications of Paraconsistency," *Proceedings of the 12th International Joint Conference on Artificial Intelligence, Vol. 2* (1991) 1052-1057.