Logique & Analyse **141-142** (1993), 3-24

PARACONSISTENT CLASSICAL LOGIC

Richard SYLVAN and Igor URBAS

Abstract.
The objective is to reformulate classical (propositional) logic, preserving all theses, so that the spread rule $A, \sim A / B$, is avoided. There are many ways of doing this - some less, some more satisfactory, all exact formulations no doubt an improvement on standard classical logic. So results a cluster of paraconsistent classical logics. These systems are surveyed and rudimentarily classified, several of them separated, and some exhibited and discussed in more detail. Among the *pure* systems of the cluster are Hiz's logic $\mathring{H}$, a Hilbert-style reformulation of Arruda-da Costa logic J_3, a cut-free formulation of Schütte's system K_1, and various normal-forming logics. Some of these pure systems fit Brasilian and Belgian criteria for paraconsistency very neatly, and significantly better than any Brasilian or Belgian systems.

There is a tension, of dialectical character, between classical logic and paraconsistent logic. Classical logic, Frege-Russell logic, as usually conceived is *explosive:* from a contradiction you can go any damn where (as the late Wittgenstein expressively put it). But a paraconsistent logic just is a *non*-explosive logic: from contradictions you can't go deductively any damn where. But an appealing synthesis can be achieved by exploiting a slackness in the notion of *classical* logic, between extensional and intensional forms. We can retain classical logic as a system of tautologies (just as the early Wittgenstein saw it), while adjusting the rules (and rule *following*) to meet paraconsistent standards.

The trouble with classical logic, paraconsistently speaking,[1] is the rule of Material Detachment,

MD. $A, A \supset B / B$

[1] It is not, of course, the only major trouble with classical logic. But other troubles, such as those with the substitution principles and with the orthodox theory of quantification, are separate from paraconsistent issues. For a fuller account of paraconsistency, see OP. For the other major problems, see JB.

also known, in degenerate regions where material-implication is taken as an implication, as Modus Ponens. For from this rule, as re-expressed – upon making the approved elimination of material-implication (i.e.$\supset$), thus: $A, \sim A \vee B / B$ – it follows just by Addition and received metalogical machinery (specifically, / Transitivity) that

EFQ. $A, \sim A / B$

i.e. contradictions trivialise.

Why pick on MD, when other principles are involved? Firstly, it is the *sole* rule of classical (sentential) logic as most commonly presented, and the classical axioms *are* tautologies. Secondly, other routes trivializing inconsistency - than that through Rule Addition, $A / A \vee B$ - put the spotlight on MD; for instance (in condensed notation) $A, \sim A / A, \sim (A \& \sim B) / A, A \supset B / B$. And while dropping Addition but not MD does lead to neat containment systems, such as Parry logics, these are far removed from classical logic and, as regards the main systems, not paraconsistent (see RLR, p.101). As a matter of in fact we shall soon sharpen the problem so that excision of MD becomes obligatory; Addition may also be ditched.

The problem that at once arises, of *how to formulate classical logic without* MD, has *many* solutions. Our problem was solved incidentally, more than half a century ago, by Gentzen, through his Cut-free systems. That solution involved extra apparatus beyond the rule and axiom (logical ruler and compass) methods of Hilbert. A recent deliberately paraconsistent variant on a "Gentzen solution", the system J_3 of Arruda-da Costa, also exceeds Hilbertian means. In fact however, a șolution conforming to Hilbert strictures was earlier discovered by Hiz̊ and a neater, even earlier, solution is there for the taking in Schütte's Hilbertian adaptation of Gentzen. As well, there are attractive solutions nearby in adaptations of familiar normal-forming systems.

All these systems form part of a cluster - the *classical* (propositional) *cluster* – of paraconsistent systems, having the same theorems (tautologies) as classical sentential logic *S*, but different rule structures. The usual (New World) systems of relevant logic also belong to this classical cluster, should they be conceived as adding to classical apparatus a rule connection $\rightarrow$ (like Gentzen's sequent sign) which perhaps admits of nesting. Certainly the system *FD* of first-degree relevant logic (with $\rightarrow$ not nested) belongs to the cluster (for information on *FD* see RLR).

Evidently then, three progressively more comprehensive classes of (minimally) paraconsistent classical logics can be distinguished;

I. *Pure* systems, which have just the morphology of classical logic *S*, in particular say, standard connective set $\{\supset, \&, \vee, \sim\}$;

II. *Rule* extensions, which add a "rule-forming functor", such as a sequent symbol;

III. *Implicational* extensions, which add a connective (or connectives) enabling the definition of a rule-sustaining implication.

These are not the only extensions of pure systems. Some other extensions, such as by functors and operators, can be removed for the present by the expedient of restricting considerations to propositional logics. Other extensions, by dangling connectives, could be excluded by confining extensions to rule enabling ones. Even so there is a broader cluster which admits nonclassical theories in classical connectives, and a narrow quite pure cluster which contains no such theories. All the systems envisaged exactly contain classical logic *S*, in the sense that all the theorems of classical logic in the standard set of connectives are theorems of the systems, which *conservatively* extend *S*. Plainly MD, although not a good or extendible rule, is an admissible rule for the (quite) pure *S* component of these systems, i.e. whenever *A* and $\sim A \vee B$ are theorems of *S* so also in fact is *B*.

The cluster of systems thereby generated is unexpectedly large. It includes not only an interesting who's-what of systems of paraconsistent logic - some of Arruda-da Costa *J* and *C* systems such as J_3 and C_1, Priest's *LP*, relevant logics - but also, unless desirable restrictions are imposed upon paraconsistency[2], some surprises - such as certain neighbours of minimal logic *J*. The reason in the latter case, is that such systems can (like intuitionist logic, which however guarantees EFQ) be reformulated as systems extending classical logic, in connective set $\{\&, \sim\}$, through connectives $\vee$ and $\rightarrow$; so they afford implicational extensions, which do *not* admit MD over the full extension (see further Sylvan 87).

[2] Although minimal logic is technically paraconsistent (e.g. in da Costa's sense), it is not properly paraconsistent since it guarantees the rule $A, \sim A / \sim B$. It is easy, and desirable, to define a tighter notion which excludes such systems. More controversial would be the upgrading of the notion to exclude any system which guarantees a rule of the form $A, \neg A / \delta B$, where connective $\neg$ is a negation definable in the system, and δB is some syntactical complication of B. Such an upgrading, to a strong paraconsistency, would (properly) exclude many of da Costa's systems, e.g. C_1.

A good deal is known about some systems among the *rich* variety of extensions. In the present exercise we will focus, in the main, on pure systems and questions concerning them. We set aside, at the outside, cheap solutions to the main problem. Very cheap solutions, easily afforded, are given by typical Hilbert-style axiomatics for classical logic, subject to the proviso that the premisses of the sole rule, MD, are theorems. Such solutions, through system-restricted rules, supply no mechanism for derivation from *arbitrary* premisses, which is at least as important as the mere generation of tautologies.[3] Thus, what we seek is a general and preferably "natural" rule structure which, with the help of some axiom schemes, yields all classical tautologies. An extendible unrestricted rule, enabling derivation of B from arbitrary well-formed premisses $A_1,\ldots,A_n$ is symbolised, as usual; $A_1,\ldots,A_n$ / B (sometimes alternatively written: $\frac{A_1,\ldots,A_n}{B}$).

Many familiar rules are not so extendible, and accordingly should be differently symbolised, e.g. substitution rules. A first obvious fact is that the sought extendible rule structure must not result in MD being derivable; for, if it were, then the tautology $A \supset (\neg A \supset B)$ would enable derivation of Spread, i.e. EFQ. Requisite solutions must eschew MD, and indeed adequate solutions will eschew modified forms of MD (such as those that require that the conclusion B have a certain implicational form) as well.

1. On earlier pure systems of paraconsistent classical logic.

There are no doubt *many* such pure systems, coinciding in their theorems, but differing in their formulations and essentially in their rule structures. There appears to have been little systematic investigation of such *rule* weakening, of theory change of this elementary sort. We begin with two such systems drawn from the literature: $\mathring{H}$ and J_3. The first, due to Hiż, was presented as, and has been taken as, a curiosity – not as a system having paraconsistent relevance. By contrast, the second, essentially due to Arruda and Da Costa, was introduced primarily for paraconsistent ends.

Hiż, remarks that his 'extendible calculus', hereafter called $\mathring{H}$, formulated using connectives $\supset$ and ~, is based on Church's text. So presumably connectives & and $\vee$ are defined as in Church; but Hiż does not

[3] Hunter, who offers a helpful account of Hiż's system (discussed below) curiously restricts the rules of the system to theorems, thereby missing a main part of the point of such systems (see his p.118).

say, though he assumes he is presenting a full propositional logic. For definiteness we adopt, to start with, the morphology of Church's system P_2, reformulated schematically (p.188), and his slightly idiosyncratic definitions (p.78);

$$A \vee B =_{Df} A \supset B \supset B;\ A \& B =_{Df} A \not\subset B \not\subset B,$$
$$\text{where } A \not\subset B =_{Df} \sim(\sim B \supset A)$$

(Definitions "equivalent" within mainstream classical logic are not, without further ado, equivalent in rule weakenings.) The postulate structure of $\mathring{H}$, is simply this:

Axiom schemes	*Rules*
$\sim(A \supset B) \supset A$	$A \supset B, B \supset C / A \supset C$
$\sim(A \supset B) \supset \sim B$	$A \supset (B \supset C), A \supset B / A \supset C$
	$\sim A \supset B, \sim A \supset \sim B / A$

The main result Hiż establishes for his extendible system is a completeness theorem (p.201), namely:

Theorem. Every classical (i.e. two valued) tautology is a theorem of $\mathring{H}$.

One corollary that emerges therefrom, a feature more conspicuously exhibited by the J systems, is that the standard rule structure (of classical logic which includes MD) is far stronger than needed for the recursive delivery of all classical theorems. The standard stronger rules do of course have a role, a less than desirable role, in *excluding extensions* (within the vocabulary); in establishing *strong* completeness of the standard calculus. That is, the addition of any nontautology as a thesis leads to triviality, to every wff being derivable. By contrast, $\mathring{H}$ is not strongly complete, but is extendible; indeed it admits infinitely many distinct non-trivial extensions (p.196). Among the nontrivial extensions not mentioned by Hiż are connexive ones, by principles such as Aristotle's thesis, $\sim(A \supset \sim A)$ in present notation. The system $\mathring{H} + \sim(A \supset \sim A)$ is a dialethic one, including contradictory pairs of theses.

A different paraconsistent classical logic is incidentally supplied by Arruda and da Costa in their study of J systems, systems tentatively designed to reinstate a full set-theoretic abstraction principle by qualifying ponibility.[4] But a result that comes out of further investigations of these systems is this: that, whilst shunning MD, the systems nonetheless guaranteed sweeping general cases of it, in particular every case where the consequent was of $\supset$ –implication form, i.e. effectively

MD$\supset$. $A, A \supset (B \supset C) / B \supset C.$

The derivability of this extendible rule (established by Bunder) in fact destroyed the J systems as far as intended motivation was concerned, and showed that the J systems were only of the same weak order of paraconsistency as minimal logic, i.e. satisfying the letter of a narrow law but not strictly paraconsistent. For any J theory containing explicitly contradictory propositions, such as those of naive set theory, would also include all implication wff.

The J systems are not pure systems; for as well as a "full complement" $\{\&, \vee, \sim, \supset\}$ of sentential connectives, they involve a Gentzen-style non-iterable connective $\rightarrow$. However they admit of pure truncation, in a sense to be explained, using MD$\supset$ or equivalent rules. We concentrate on the system J_3 which coincides theoremwise with classical logic.[5]

What we have so far obtained in the direction of Hilbert-style reformulations of J_3 are decidedly less compact than formulations of other systems we consider. The first formulation we offer, of a quasi-Hilbert -style system KJ_3 can be seen as building, extensively and heavily, on da Costa's basic C system C_ω, much of the construction being required by one variation, modification of MD. System C_ω is formulated as follows:-

1. Positive schemes (of C_ω and J_1):
   $A \supset . B \supset A; A \supset B \supset . A \supset (B \supset C) \supset . A \supset C;$

   $A \& B \supset A;$ $A \& B \supset B;$ $A \supset . B \supset A \& B$

   $A \supset A \vee B;$ $B \supset A \vee B;$ $A \supset C \supset . B \supset C \supset . (A \vee B) \supset C$

[4] Blocking logical paradoxes in such a way forms part of an older project going back at least to Fitch's "basic logic". Eschewing Curry-style paradoxes by modifying ponibility of course bears a close resemblance to avoiding such paradoxes by removing such Assertion themes as $A \& (A \rightarrow B) \rightarrow B$, the approach of depth relevant logics, recommended in JB, Appendix.

[5] But we intend elsewhere to make analogous studies of other J systems and of related (absolute based) logics; for many systems, including such systems as Hilbert's positive logic, permit recasting using modified ponibility rules.

2. Negative schemes (dual intuitionistic in character):
   $A \vee \sim A; \sim\sim A \supset A$.

3. Sole rule MD of (C_ω).

Our start towards obtaining an appropriate formulation of J_3 replaces MD by one or more of the following rule sets:

$$A, A \supset (B \supset C) / B \supset C; \quad A \supset B, B \supset C / A \supset C; B / A \supset B;$$
$$A \supset (A \supset B) / A \supset B; \quad B / A \supset B; A \supset B, A \supset (B \supset C) / A \supset C;$$

We shall work with the first rule. To make up the ground lost by the absence of MD - elementary inferences Arruda and da Costa want, such as $A \& B / A$ and $A / A \vee B$ - a good deal of supplementation is inevitable.

Axiom schemes of KJ_3 All schemes of C_ω together with the following negation schemes:-

i. the intuitionistic schemes (of Kleene's system):
   $A \supset \sim\sim A; \sim (A \& \sim A); \quad A \supset B \supset. A \supset \sim B \supset. \sim B$

ii. schemes matching rules;

$$\sim A \& \sim B \supset \sim (A \vee B); A \& \sim B \supset \sim (A \supset B); \sim A \supset \sim (A \& B); \sim B \supset \sim (A \& B).$$

Rules and rule schemes of KJ_3

$A, A \supset (B \supset C) / B \supset C;$
$A \& B / A;\ A \& B / B;\ A, B / A \& B$
$A / A \vee B;\ B / A \vee B;$ where $D, A / C$ and $D, B / C$ then $D, A \vee B / C$
(scheme $\vee I$)
$A / \sim\sim A;\quad \sim\sim A / A;\quad \sim A, \sim B / \sim (A \vee B);$
$A, \sim B / \sim (A \supset B);\ A, \sim B / \sim (A \& B);\quad \sim A, B / \sim (A \& B);\quad \sim A, \sim B / \sim (A \& B)$

While the (apparent) positive part of the system is not so inelegant, apart from one hitch, the negative part is undoubtedly something of a shambles, owing to rule weakening of contraposition and reduction principles. The difficulty is not so simply rectified; for straightforward addition of the rule $A \supset B / \sim B \supset \sim A$, or expected weakenings of it, voids paraconsistency, delivering EFQ (see Urbas p.24). The hitch to our present formulation lies not however with negation, which is simply messy, but with disjunction.

For, with the scheme $\vee$I, we have stepped outside strict Hilbertian bounds. Except by switching to a rule framework which permits multiplicity on the right (whereupon $\vee$I, is replaced by the elegant rule $D, A \vee B \,/\, D, A, B$), also slightly out of Hilbertian bounds, we do not presently see a way of avoiding the hitch (inductive replacement of $\vee$I so far fails). So we persist with KJ_i (system HJ_i has yet to be located, *if* it can be).

Theorem. The system KJ_i reformulating J_i is fully equivalent to the original system J_i of Arruda-da Costa (for i = 1, ..., 4) in the following sense: $A_1, \ldots, A_n \rightarrow B$ [in the null case, where $n = 0, \rightarrow B$] is derivable in J_i iff $A_1, \ldots A_n \,/\, B$ is a derivable rule [nullwise, B is a theorem] of KJ_i

Proof. We consider only the relevant case for classical paraconsistent logic where i = 3 (other J_i systems, which diverge from classicality, can be dealt with along similar lines). One half of the theorem, from J_i to KJ_i is a straightforward matter of verifying that every axiom scheme of KJ_i is a theorem of J_i and that every rule (in / form) of KJ_i is supplied (in $\rightarrow$form) in J_i. The converse, from KJ_i to J_i, is more arduous. Strictly, we should show that we can conservatively extend rule connection, /, into a structural connective $\rightarrow$ with the right features. Less rigorously, we shall show that we can capture the effects of all the postulates of J_i in KJ_i. The $\rightarrow$-postulates $\rightarrow_1$ - $\rightarrow_5$ of J_3 (and of all J_i systems) merely correspond to essentially unformalized structural properties of proofs in axiomatic systems. To capture $\supset_1$) namely $G, A \rightarrow B \,/\, G \rightarrow A \supset B$ of J_3 , we need an implicational deduction theorem *and* we need to ensure that for every primitive rule of KJ_3 there is a matching theorem (whence considerable initial redundancy). But both have been ensured, in previous work (see Urbas). To cover $\supset_2$), $(A \supset B) \& (A \supset B \supset C) \rightarrow A \supset C$ of J_3, a scheme of C_ω and the rule $A,\ A \supset (B \supset C \,/\, A \supset C)$ suffice. And so on.

Corollary. KJ_3 has the same theorems as classical logic.

Proof. Apply theorem 6 of Arruda-da Costa, which establishes the open part of the result for J_3.

Now the *mere* overlap of $\mathring{H}$ and J_3 can be established.

Theorem. Although $\mathring{H}$ and J_3 have the same theorems (those of S), neither includes the other.

Proof:

1. $\mathring{H}$ is not included in J_3.

The rule $\sim A \supset B, \sim A \supset \sim B / A$ (and similarly $A \supset B, A \supset \sim B / \sim A$), representing the third rule of $\mathring{H}$, is not derivable in J_3. Proof is by the matrices supplied for theorem 7 of Arruda-da Costa[6], or better by the following finite matrices, U_3 , analogous to those:

$\rightarrow$	0	1	2
*0	0	0	2
*1	0	0	2
2	0	0	0

$\supset$	0	1	2	$\sim$
0	1	1	1	2
1	1	1	1	1
2	1	1	1	0

&	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\vee$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

These matrices, based on the chain, $2 \rightarrow 1 \rightarrow 0$, validate the postulates of J_3 but invalidate the rule $\rightarrow \sim A \supset B, \rightarrow \sim A \supset \sim B / \rightarrow A$ when A is assigned the value 2 and B is assigned any value at all.

2. J_3 is not included in $\mathring{H}$.
The rule $A, A \supset (B \supset C) / B \supset C$ is not derivable in $\mathring{H}$. The matrix system M_3 supplied by Hiż to establish features of his system shows that the rule is not derivable. $\mathring{H}$ is modelled by M_3:

$\supset$	1	2	3	$\sim$
*1	1	3	3	2
*2	1	1	1	1
3	1	1	1	1

Now set A = 2, B = 1, C = 2 or 3. Then $B \supset C$ has value 3, which is undesignated.

Alternatively, observe that all rules of $\mathring{H}$ have implicational premisses; it is thus to be expected that weakening, $B / A \supset B$, is not a derived rule of $\mathring{H}$. Matrix M_3 serves to verify this (take B = 2, e.g. a negated theorem, and A = 1; then while B is designated, $A \supset B$ is not).

$\mathring{H}$ and $K J_3$ are thus different - in *rule* structure, though not in theorems. There are no doubt other different systems in the vicinity of these systems,

[6] Note that this theorem 7 of Arruda-da Costa is astray in claiming that $A \rightarrow \sim A \supset B$ (our negation symbol) and $\sim A \rightarrow A \supset B$ are not derivable in J_3. They are derivable, and are listed as derivable in the weaker J_2 in theorem 4 of the same paper. The lapse is one of many errors that have crept into the paper. More generally, readers should beware of claims made in the literature concerning these J systems; quite a number are unreliable.

variations upon $\overset{\circ}{H}$ and KJ_3 and certain combinations of them. An obvious question (presupposing however that 'different systems' becomes well-defined) is :- *How many* pure classical paraconsistent (*pcp*) systems are there? A low-risk conjecture is the following:

Theorem ? There are infinitely many pcp systems.
Argument proposal: A pcp system can be represented through a many valued logic with a further connective >, beyond the classical set, satisfying a Detachment rule but not EFQ (in form $A \& \sim A > B$). Matrix theory will serve to show that there are infinitely many such many valued systems which validate just the classical tautologies in classical connectives.

An open problem suggested is making some suitable survey of pcp systems, with a view to determining their variety and significant kinds.

Assuming such variety, the literature so far encountered has not done us proud. For neither KJ_3 nor $\overset{\circ}{H}$ is particularly satisfactory. KJ_3 is not even strictly paraconsistent; given any pair A and $\sim A$ it will deliver all $\supset$-implicational wff, of form $B \supset C$. KJ_3 is in fact the first of a chain of unsatisfactory pcp systems which are similarly paraconsistent but not strictly so. A chain of systems can be designed with each system S^n having the rule

$$\text{MD}^n. \qquad A, A \supset (A_2 \supset \ldots (A_n \supset B)\ldots) / A_2 \supset \ldots (A_n \supset B)\ldots),$$

with each successive S^n presumably requiring supplementation from additional rules (such as Permutation). But MD^n will in turn permit the derivation of EFQ^n, i.e. $A, \sim A / B_1 \supset (B_2 \ldots (B \supset C)\ldots)$; so S^n will not be strictly paraconsistent either. For it tolerates the spread of theoremhood to vast and quite irrelevant domains (merely syntactically delimited).

By contrast, our objections to $\overset{\circ}{H}$ are evaluative rather than technical. It has the audacity to start out from the counterintuitive, from blatant paradoxes of implication as *axioms*, as self-evident truths; and even if these are inoperative or really mean something different in the given logical setting, the practice offends uncorrupted relevant sentiments. However it is not difficult to adjust $\overset{\circ}{H}$ to remove the superficial offence. Consider a reformulation $H\upsilon$ of $\overset{\circ}{H}$ with connectives $\vee$ and $\sim$. Then simply re-expressing $\overset{\circ}{H}$, system $H\upsilon$ takes the following shape:-

Axiom schemes	*Rules*
$\sim\sim(\sim A \vee B) \vee A$	$D \vee B, \sim B \vee C / D \vee C$
$\sim\sim(D \vee B) \vee \sim B$	$D \vee (\sim B \vee C), D \vee B / D \vee C$
	$\sim\sim A \vee B, \sim\sim A \vee \sim B / A$

With $\supset$ defined in familiar way, $A \supset B =_{Df} \sim A \vee B$, this delivers $\mathring{H}$ forthwith. Now plainly $H\upsilon$ is a trifle extravagant in two *different* respects (even when $\sim A$ is replaced by D):

• eliminable double negation forms in the axiom schemes. The schemes look as if they should be reducible to the pair

$$\left.\begin{array}{l}(\sim A \vee B) \vee A \\ (D \vee B) \vee \sim B\end{array}\right\} \text{ and the third rule to the form } A \vee B,\ A \vee \sim B \,/\, A$$

• dangling disjunctions, most obviously in the axiom schemes. The schemes look as if they should be reducible by Associativity (and Commutation) to forms like $D \vee (B \vee \sim B)$, whence the single scheme, $B \vee \sim B$, together with an Addition rule would deliver both axiom schemes. Classical logic would reduce to a certain shuffling of *ors* and *nots* in expansions of LEM, $A \vee \sim A$.

There are two catches to the second "simplification" of $H\upsilon$. Firstly, such an Addition rule[7], say inflating A to $A \vee B$ is not derivable in Hiż systems. This is evident from the complex premisses of all the rules of Hiż systems; and it can be confirmed using the matrix system M_3 (for take $A \vee B$ as $\sim A \supset B$, set $B = 3$ and $A = 2$; then from a designated value 2, a nondesignated value 3 would result). Secondly, worse, if Rule Addition were freely available, EFQ would result, as follows applying the third rule; $B, \sim B / \sim A \vee B, \sim A \vee \sim B / A$. So the neat "simplification" is not available, without relinquishing some other part of Hiż systems.

Fortunately we know, from relevant experience, *one* way to modify $H\upsilon$ so as to obtain the sought style of simplification without instating EFQ. The condensed (multiple conclusion) argument: $A, \sim A / A, \sim A \vee B / B$, reveals that two rules are intricated in reaching EFQ, namely Addition and γ (i.e. MD_v), and accordingly two types of pcps: nonadditive, like $H\upsilon$, and those just relinquishing γ.[8] While this suggests relevant logics, in fact

[7] Addition was deployed as a primitive rule as long ago as 1937 by Stone: see Church p.144.

[8] There is a further option, that of tampering with the rule structure, which is of course part of the presupposed derivational machinery. With transitivity of / repudiated, both Addition and γ can be retained, but EFQ resisted. While such repudiation is a desperate and, as far as *derivation* is concerned, ill-motivated expedient, it undoubtedly does open a fresh, and largely uninvestigated, range of logics. (Marketed nontransitive logics are typically presented in terms of nontransitivity of some implication, not of the inferential connection of Hilbert-style systems.)

the additive class is considerably wider, including J_3 for instance (for $A\&(\sim A \vee B) \rightarrow B$ is falsified under the following U_3 matrix assignments: $A = 1, B = 2$).

2. *The matter of relevantly satisfactory formulations of classical logic.*

Neither $\mathring{H}$ nor $H\upsilon$, nor systems sufficiently like them in rule structure such as J_3, can really be considered satisfactory for relevant paraconsistent purposes. Consider the first rule of $\mathring{H}$, a derived rule of J_3,

Barb. $A \supset B, B \supset C / A \supset C$

Put into disjunction form, as the replacements of H_υ require, this becomes the "resolution" form, $\sim A \vee B, \sim B \vee C / \sim A \vee C$. The equivalent form already reveals to the wary that some funny business is going on concerning B and $\sim B$, that they are "cancelling" one another. What is happening can be brought out by assuming & and $\vee$ normality. For then the rule becomes:

$(\sim A \vee B) \ \&(\sim B \vee C) / \sim A \vee C$, i.e. distributing
$(\sim A \& \sim B) \vee (\sim A \ \& C) \vee (B \ \& C) \vee (B \ \& \sim B) / \sim A \vee C$.

Plainly each disjunct, except the last, delivers, as a part either $\sim A$ or C, whence the result could be put together using Praeclarum principles. *Except* for the last clause, which requires a licence to proceed from $B \& \sim B$ to $\sim A \vee C$, i.e. using legitimate replacements, to proceed from $B, \sim B$ to D. But that is precisely the illicit inference paraconsistent enterprise seeks, above all else, to avoid.

A satisfactory formulation will accordingly not include Material Rule Transitivity, Barb, insofar as that involves eliminating (cutting out) B by $\sim B$. Does eschewing Barb make a satisfactory *pure* formulation of classical logic impossible to achieve? An important preliminary question is, evidently: when is a formulation *satisfactory*? Reflective investigation of relevant logic has supplied an answer to this question. To make that answer less tendentious, *relevant* satisfactoriness will be defined. A finite schematisation *RS* of classical logic *S* is relevantly satisfactory iff each axiom scheme A_i is relevantly valid and each rule R_j preserves relevant validity (i,j finite). More specifically, a finite schematisation consists of axioms or schemes A_i, $1 \leq i \leq n$, each of which will of course be classical tautologies, and rule schemes R_j, $1 \leq j \leq m$, each of the form $A_\alpha, \ldots, A_\lambda \ / \ A_\mu$ (or similarly multiple on the right, in a more generous formulation). Such a rule preserves relevant validity iff $A_\alpha \& \ldots \& A_\lambda \rightarrow A_\mu$ is relevantly valid, i.e. valid in the system *FD* of first degree relevant logic.

It is immediate that the orthodox formulation of classical logic with MD is unsatisfactory. All that is satisfactory really is the alternative rule, $A, A \supset B \,/\, B, A \& \sim A$, in multiple formulation. Similarly, it follows that Barb is unsatisfactory: all that is is its displacement, $A \supset B,.\ B \supset C \,/\, A \supset C,\ B \& \sim B$. To be sure, it may be argued that the problem results from converting $A \supset B$, now seen as representing some sort of genuine implication, to $\sim A \vee B$ (as in *Principia Mathematica*), and can easily be resolved by cutting the connection. No doubt it does, and can to some extent. But that connection is essential to *material*-implication; to sever it is already to move beyond an orthodox *classical* system.

By proceeding beyond pure formulations of classical logic, satisfactory formulations can be obtained - in many ways. Most obviously, extend *S* by connective $\rightarrow$ of *FD* or by an equivalent rule, and use *FD* or an equivalent. (A rather different extension adds suitable assertion and rejection functors, $\vdash$ and $\dashv$, in terms of which eliminations of contradictions like $B \& \sim B$ can be made from rules.) But for pure formulations, is there any such relief in sight? What makes the problem *look* intractable is that a relevantly satisfactory pcp system must be generated from such shortest schemes by expansion; there can be no cutting of the usual sort, that MD and Barb sanction. But the very statement of the difficulty suggests a line of resolution, namely cut-free methods.

Conveniently there is relief, both right nearby and buried within standard logic exercises. In "simplifying" H_υ we almost arrived at such a system, effectively set down by Schütte, as a variant on Gentzen. It amounts to Gentzen's rules on the right of the sequent sign, for the $\{\sim, \vee\}$ classical (sub)system.[9] In Dunn's reformulation, the system, labelled K_υ here, is as follows (with association in multiple disjunctions to the right):

[9] The relation of the system to Gentzen's formulation is nicely explained in Dunn and Meyer, where the cut-free system is investigated in some detail. In the text we in fact consider only the propositional part of the systems considered in Schütte and in Dunn and Meyer; evidently our investigation could be extended to full classical quantification logic, simply building again on the shoulders of others.

Axiom scheme: $A \vee \sim A$ *LEM*

Rules [I. Structural] $\dfrac{M \vee A \vee B \vee N}{M \vee B \vee A \vee N}$ $\dfrac{N \vee A \vee A}{N \vee A}$

[II. Operational] $\dfrac{N}{N \vee B}$ $\dfrac{N \vee \sim A \quad N \vee \sim B}{N \vee \sim (A \vee B)}$ $\dfrac{N \vee A}{N \vee \sim\sim A}$

In every case but Rule Addition (i.e., in this context, Weakening), the *side* wff M and N may be omitted.

K_υ is a very pleasant system, in that the whole of classical logic is extracted (or spewed out) from *LEM* by *shuffling* disjuncts around and *ad-libing in* further disjuncts and double negations as required. It all becomes a straightforward shunting task from *LEM*. To illustrate, consider condensed derivations of axiom schemes of H_υ, with Associativity derived first:-

ad Associativity: $A \vee (B \vee C) // A \vee (C \vee B) // C \vee (A \vee B) // (A \vee B) \vee C$

These double lines indicate two way (extendible) inferences.

ad $\sim\sim(\sim A \vee B) \vee A$: $A \vee \sim A / B \vee (A \vee \sim A) / A \vee (\sim A \vee B) / A \vee \sim\sim(\sim A \vee B) /$
$\sim\sim(A \vee B) \vee A$
ad $(D \vee B) \vee \sim B$: similar.

By contrast with these axiom schemes, the rules of H_υ all involve applications of the Cut rule, which eliminates segments of wff; i.e.

Cut $\dfrac{M \vee A \quad \sim A \vee N}{M \vee N}$

which is no more acceptable than its special case: $\gamma \ \dfrac{A \quad \sim A \vee B}{B}$

Theorem. System K_υ is a relevantly satisfactory pure formulation of classical logic.

Proof: K_υ is pure, as matter of inspection, and it is relevantly satisfactory. For $A \vee \sim A$ is a theorem of *FD*, and the $\rightarrow$ reformulations of the rules of K_υ are all theorems of *FD*. Consider, for instance, the operational rule $N \vee \sim A, N \vee \sim B / N \vee \sim (A \vee B)$, which becomes, in $\rightarrow$ form, $(N \vee \sim A) \& (N \vee \sim B) \rightarrow . N \vee \sim (A \vee B)$, a theorem of *FD*. The $\rightarrow$ analogue of Cut, $(M \vee A) \& (\sim A \vee N) \rightarrow . M \vee N$, is of course *not* a theorem of *FD*.

The harder part consists in showing that K_υ is a formulation of classical logic. Happily the requisite work is accomplished, though rather circuitously, in Schütte and in Dunn and Meyer; for their interest lies in showing that Cut or γ is an admissible rule. But, plainly, several less laborious routes, which aim directly at tautology-completeness, are feasible. Some of these run through normal-forming procedures, to which we shortly turn. One such makes completeness elementary.

Another variant upon Schütte's system from the relevance camp, this time from Anderson and Belnap, serves as a bridging system. On the one side, this intermediate system makes demonstration of tautology-completeness rather simple, *trivial* high-powered types would say.[10] But on the other side, the postulates of this system, $AB_\vee$, are readily established in K_υ. In schemata form, they amount to no more than disjunctive contextualisation of two operational rules of K_υ – to $D({\sim} A), D({\sim} B) / D({\sim}(A \vee B))$ and $D(A) / D({\sim\sim} A)$– and compression of the rest of the apparatus into a more complex, but immediately assessible axiom scheme: $DJ(A, {\sim} A)$, which consists of any disjunction containing A and ~A and perhaps other disjuncts as well (thereby taking care of Addition) in any order (thereby accommodating structure). The bridging system takes us back towards the schemes of $H_\vee$.

While, for a relevantly satisfactory amendment to $H_\vee$, the axiom schemes can be retained, the rules, which all involve cutting, have to be modified. Anderson and Belnap's variation reveals a simple way to do this. Both the axiom schemes of $H_\vee$ are of the form: $\ldots A \ldots {\sim} A \ldots$, of disjuncts of $A, {\sim} A$ and other wff. Let any wff, $DJ(A, {\sim} A)_\circ$, of this form serve as an axiom. (In fact Anderson and Belnap like Hiż use sentential variables in their axioms.) To evolve a new set of rules for the $\sim \vee$ logic, it pays to inquire as to what sort of $\sim \vee$ theorems of sentential logic the axioms schemes do not supply. Two evident cases are those of iterated negations, where ${\sim\sim} C$ replaces C in some disjunctive sentence context, and of negated disjunction, where ${\sim}(C \vee D)$ replaces ${\sim} C \vee {\sim} D$ in some context. These two negation-shuffling rules can be stated as follows:

$$D(A)/D({\sim\sim} A) \qquad D({\sim} A), D({\sim} B)/D({\sim}(A \vee B)),$$

[10] Completeness and consistency are said to be 'almost trivial' in Anderson and Belnap, who outline an appropriate argument in a mere four lines. A more detailed argument is given in Hunter, pp 131-2.

where $D(C)$ is any wff of which C is a disjunctive part. What is slightly less evident, but quickly proves to be the case, is that those two contextual rules suffice for derivation of all classical tautologies. Such is, in fact, a chief result of Anderson and Belnap's "simple treatment of truth functions".

Once again, in this system $AB_{\vee}$, stricter Hilbertian requirements upon formal systems have been exceeded. On this occasion, an important part of intersubstitutivity of provable coimplications has been incorporated into the rules, without the need for a separate intersubstitutivity rule.[11] A way around contextual rules is, in this case, easily found, for instance by reflection on Gentzen proof procedures. Indeed a way round was already provided by Schütte. A main component part of the Anderson-Belnap simplification feat consisted in replacing his explicit rules by a significantly smaller set of contextual rules.

While it is pleasant indeed to generate all of classical logic (under consideration) from one of the traditional law of thought, the principle of Excluded Middle, it strikes us as unfortunate that the start should be from the least reliable and most controversial of the three traditional principles. It would be better surely not to use as foundation rock what intuitionists and astute vagueness-proponents account the Big Illusion, but to build upon something more solid, in particular the principle of Non-Contradiction, which the Philosopher pronounced 'the most certain of all principles'. If the Big Illusion should come out in the classical wash, well that's not so bad as assuming it at the bottom. There is another reason too for starting just from Non-Contradiction (rather than Identity, which is the third traditional "alternative"): that is, that it offers some backing for the traditional theme that all logical truths flow from Non-Contradiction.

Technically, the trouble with Non-Contradiction is that it has a negation covering the conjunction of A with $\sim A$. It is accordingly easier to work with rejection, and thereby just mirror the disjunction-negation system. The strategy is this:- Begin with

$\dashv A \mathbin{\&} \sim A$ as sole rejection schema. Add a set of rejection rules reflecting the assertion rules for the $\vee$ $\sim$ system. Then simply characterise anti-theoremhood thus: A is a anti-thesis iff $\dashv A$. Naturally $\sim A$ is a thesis if A is an anti-thesis.

With a little notational cleverness in introducing $\dashv$, we can obtain systems $AB_{\&}$ and $K_{\&}$ as straightforward *rewrites* of corresponding systems $AB_{\vee}$ and $K_{\vee}$ with $\dashv$ suppressed. Thus, for instance, $K_{\&}$:

[11] While the rules are still extendible, this does have the effect of narrowing the class of theories closed under them.

Axiom scheme: $A \& \sim A$ (i.e. in fuller form $\rightarrow \sim (A \& \sim A)$)

Rules: $M \& A \& B \& N : M \& B \& A \& N$ $\quad N \& A \& A : N \& A$

$N : N \& A \quad N \& \sim A,\ N \& \sim B : N \& \sim (A \& B) \quad N \& A : N \& \sim\sim A$

Soundness is readily confirmed; completeness is not quite as evident. But since the system, written thus, is a strong translation of K_v (with '$\vee$' simply translated as '&'), translations of all results for K_v hold for $K_\&$. So, in particular, $K_\&$ is classically adequate in that it rejects all classical contradictions, i.e. asserts all classical tautologies.

Deployment of connective & in this dual role soon suggests another group of relevantly satisfactory pcp systems, namely *normal-forming systems*. More exactly, what we do is to reexploit what finally paid off in computing, all those dreary normal-forming techniques (enforced as exercises on generations of reluctant students) to supply systems. Essentially we *reverse* the techniques, by starting from the "obvious by inspection" wff to which wff are usually reduced, and using the reduction rules (or rather parts of them) as rule schemes to *expand* upon the obvious. Normal-form reductions techniques are invariably extravagant in the principles given for reduction, so (for those who like them) economies can be effected.

Consider first a pretty full and no doubt redundant *CNF* system, CNF_1, which reflects the normal-forming procedures of several standard textbooks; indeed it was extracted from them. CNF_1 is an axiom economic but rule profuse system:-

Axiom Scheme: $A \vee \sim A$

Rules:

$A / A \vee B$ $\qquad A, B / A \& B$

$K(A \vee B) / K(B \vee A)$ $\qquad K(A \& B) / K(B \& A)$

$K(A \vee (B \vee C)) / K((A \vee B) \vee C)$ $\qquad K(A \& (B \& C)) / K((A \& B) \& C)$

$K((A \vee B) \& (A \vee C)) // K(A \vee (B \& C))$ $\qquad K((A \& B) \vee (A \& C)) // K(A \& (B \vee C))$

$K(A) // K(\sim\sim A)$

$K(\sim (A \& B)) // K(\sim A \vee B)$ $\qquad K(\sim (A \vee B)) // K(\sim A \& \sim B)$

Dually disjunctive normal form systems may be devised. Plainly this system is classically sound. Its adequacy for classical logic will follow from orthodox textbooks (e.g. Hughes and Londey, Appendix 1), or (less laboriously) as follows:

Classical Completeness: If A is a classical tautology, then A is provable in classical system CNF_1.

Proof outline:-

(1) Where B is in CNF normal form, B is a classical tautology iff each conjunct of the form contains a disjunctive component and its negation in it has a valid normal form (by inductive proof).

(2) All valid normal forms are derivable.

(3) Suppose A is a classical tautology, then it has a valid normal form B. But B is provable. So A is provable.

System CNF_1 can be made into a fully reversible system CNF_2 by absorbing the first two rules into an expanded axiom schemata: $CNF\left(A_1, \sim A_1; A_2, \sim A_2 \ldots; A_n, \sim A_n\right)$. Reversibility facilitates a decision procedure. An intermediate system $AB_{\vee\&}$ results by adding conjunction, &, to $AB_{\vee}$. System $AB_{\vee\&}$ with connective set $\{\sim, \vee, \&\}$ apparently has the following postulates:

Axiom scheme: $DJ(A, \sim A)$, i.e. as for $AB_{\vee}$

Rules:

$A, B / A \& B$ (whence the conjunctive forms)

$K(A) / K(\sim\sim A)$

$K(\sim A \& \sim B) / K(\sim (A \vee B))$ } De Morgan Rules

$K(\sim A \vee \sim B) / K(\sim (A \& B))$ }

$K((A \vee B) \& (A \vee C)) / K(A \vee (B \& C))$ Distribution

It is worth observing that within the framework that is developing, part of metalogic is becoming converted into syntactical or proof theory, establishing the equivalence or difference of systems. Semantics in particular, normally introduced in regard to completeness, disappears. Should it reappear, as it may, it too will have to change.

We conclude this section with some preliminary *observations on* the further business of *overhauling semantical theory.* Not only both the logical syntax and proof-theory of classical logic need to be overhauled; so also does the semantics. From a paraconsistent viewpoint, the standard semantics, which reflect truth tables, are not adequate. For they validate EFQ. (Finding suitably strong semantics for systems like H_V, semantics which duly validate the systems while invalidating EFQ, is a far from trivial open question.) The paraconsistent approach discloses however a more general shortcoming of standard semantics, which is this. While adequate semantics appropriately characterise the theorems of a system, distinguishing theorematic from non-theorematic wff, they do not similarly characterise rules, distinguishing types of satisfactory rules (of one sort or another) from less satisfactory or unsatisfactory rules.

Some elements of an improved semantical theory can be gleaned from the literature (e.g. the distinctions between weak and strong models). But to proceed further, it seems desirable to begin to take rule symbolisation seriously, and also to begin to distinguish more systematically types of rules. What is sought, in the first place, is an additional semantical rule for connexion /, which can serve in validating the rules of a given system while invalidating restricted rules. Thus in addition to normal rules like those for $\vee$ and &

$$I(A \vee B)=1 \text{ iff } I(A)=1 \text{ or } I(B)=1 \text{ and } I(A \& B)=1 \text{ iff } I(A)=1 \text{ and } I(B)=1$$

perhaps duly relativised, a rule evaluating $I(A_1, \ldots, A_n / B)$ is required. It needs little inspiration to see that a rule *like* that for implication, or conversely finite *consequence*, is what is sought. While there are several options, any adequate analysis along deductive lines must be duly truth-preserving over a sufficiently comprehensive class of situations.

3. Paraconsistent adequacy and classical logic.

Investigation of the classical cluster enables us to come substantially closer to formulating systems of genuinely paraconsistent merit meeting Brasilian conditions, than any of the systems of paraconsistent logic for which Brasil is justly acclaimed, such as the C and J systems (or than Batens' more classical adaptations of these). To be sure, these claims involve risky assertions. For it appears that paraconsistent systems, as generally (*not essentially*) subsystems of classical logic, involve deletion, removal of some principles or others. But *deletion* is notoriously nonunique in result. Moreover, a decent measure of *closeness* to the original is notoriously difficult to obtain. The alternative idea here is simple: that we cut through such difficulties by using *re*formulations of classical logic, paraconsistently acceptable reformulations (even if not always relevantly satisfactory reformulations).

Da Costa and Alves assert that, in general, systems of paraconsistent logic must satisfy the following conditions:

(I) From two contradictory statements A and $\sim A$, it must not be possible in general to deduce an arbitrary statement B; and

(II) Such systems should contain most of the schemata and deduction rules of classical logic that do not interfere with (I).

While other conditions have frequently been added, these two conditions have consistently served as the primary guiding principles for da Costa and his Brasilian collaborators in the design of their paraconsistent logics. (The other conditions and what they presuppose, e.g. prior selection of a negation $\sim$, are discussed in OP, e.g. p.54).

Conditions (I) and (II) can be maximally satisfied by a "classical logic" (extensional classical logic) without the Spread rule: $A, \sim A / B$. Such a logic is however *far* from uniquely specified. Indeed the situation is significantly worse than we have so far indicated. For tearing one principle, and inevitably *some* or other of its connections, from the classical web, makes room for the instatement of nonclassical principles. Hiz's system $\mathring{H}$ alone, admits of infinitely many extensions by negated wff, of the form $\sim_n A$, $n > 2$, where $\sim_n A$ is $\sim \ldots \sim A$ with n negations.

Theorem. There are infinitely many distinct nontrivial extensions (in the given classical vocabulary) of $\mathring{H}$ (see Hiz p.196).

The intended range of variation is considerably restricted by further conditions, not explicitly stated but regulary presupposed, on intended Brasilian systems of *propositional* paraconsistent logic:

(III) The systems should be subsystems of classical logic, i.e. should not contain nonclassical theses (such as $\sim\sim\sim\sim A, \sim(A \supset A), \sim(A \supset \sim A)$, etc.).

(IV) The systems should conform to the requirements (syntactical and semantical) of positive (intuitionistic) logic, i.e. should contain schemata of Hilbert system H that do not interfere with (I).

Da Costa and his collaborators assume these further conditions in *all* their practice involving mathematical applications. The conditions have the effect of confining the systems *within* the class of quite pure paraconsistent classical logics so far examined, where conjunction and disjunction behave rather normally. By virtue of (IV), the failure of $A, \sim A / B$ must be manifested in failure of MD (in disjunction): $A, \sim A \vee B / B$. For suppose the latter were to hold. Then $A, \sim A / A \& \sim A$, by conjunction normality. But by disjunction normality, $\sim A / \sim A \vee B$, so by conjunction again $A \& \sim A / A \& (\sim A \vee B)$. By transitive composition of the inferential procedure then: $A, \sim A / B$.

In fact the conditions imposed suggest a more precise location of Brasilian propositional paraconsistent logics, in so far *uninvestigated* systems: *BS systems* to name them. Any *BS* system will have, like intuitionism, a prima facie independent set of connectives $\{\supset, \&, \vee, \sim\}$, the first three of them positive, the last negative. To maximally satisfy condition (IV), BS will include *all* of positive logic *H*. This can be done (if again too cheaply) without offending (I) by restricting detachment (in $\supset$) to *H*, to positive wff, i.e. wff not containing connective $\sim$. Indeed to exclude Spread it suffices to add to MD the proviso: provided A is positive. But we should go further in this direction, with a view to satisfying condition (II), and include all of positive classical logic S^+, i.e. $H + (A \supset B) \supset A \supset A$. Then a *BS* system will result by addition of any set of postulates adequate for all negative classical logic which does not deliver Spread.[12]

While paraconsistent classical logic no doubt removes the worst deficiencies of classical logic as regards inconsistency, it represents a quite insufficient improvement upon classical logic. For even relevantly satisfactory systems are extremely impoverished. Most important, they fail to include satisfactory implication or conditional connectives. In fact for this reason, paraconsistent classical logic does not begin to combat other paradoxes and puzzles, in the way that a more radical change of logical base can succeed in doing. It does not get to grip with Curry-style paradoxes, with relevance paradoxes, with explicatory puzzles, and so on. Paraconsistent classical logic is but a first way-station on a route from classical logic to the not-so-remote promised land.

Australian National University
Canberra

REFERENCES

A.R. Anderson and N.D. Belnap, Jr. 'A simple treatment of truth functions', *Journal of Symbolic Logic* 24 (1959) 301-2.

A.I. Arruda and N.C.A. da Costa, 'Sur le schéma de la séparation', *Nagoya Mathematical Journal* 38 (1970) 71-84.

D. Batens, 'Paraconsistent extensional propositional logic', *Logique et Analyse* 23 (1980) 195-254.

M.W. Bunder, 'Arruda and da Costa's logics J_1 to J_5', *Journal of NonClassical Logic* 2 (1983) 43-48.

[12] Thus the approach taken is significantly different from that of Batens who builds on classical positive logic (as does da Costa with C_1), but weakens negation axioms rather than restricting detachment.

A. Church, *Introduction to Mathematical Logic,* Princeton University Press, Princeton, 1956.
N.C.A. da Costa, 'On the theory of inconsistent formal systems', *Notre Dame Journal of Formal Logic* 15 (1974) 497-510.
N.C.A. da Costa and E.H.Alves, 'Relations between paraconsistent and many-valued logics', *Bulletin of the Section of Logic, Polish Academy of Sciences* 10 (1981) 185-191.
J.M. Dunn and R.K. Meyer, 'Gentzen's Cut and Ackermann's γ', in *Directions in Relevant Logic* (ed. J. Norman and R. Sylvan), Nijhoff, Dordrecht, 1989.
F.B. Fitch, *Symbolic Logic*, Ronald Press, New York, 1952.
H. Hiż, 'Extendible sentential calculus', *Journal of Symbolic Logic* 24 (1959) 193-202
G.E. Hughes and D.G. Londey, *The Elements of Formal Logic,* Methuen London, 1965.
G. Hunter, *Metalogic*, Macmillan, London, 1971.
G. Priest and R. Routley, *On Paraconsistency,* Research Series in Logic and Metaphysics, RSSS, Australian National University 1984; referred to as OP.
R. Routley, *Exploring Meinong's Jungle and Beyond,* Research School of Social Science, Australian National University, 1979; referred to as JB.
R. Routley and others, *Relevant Logics and Their Rivals,* Ridgeview, California, 1982; referred to as RLR.
R. Sylvan, 'Intuitionistic logic - subsystem of, extension of, or rival to, classical logic', *Philosophical Studies* 53 (1988) 147-151.
I. Urbas, *On Brazilian Paraconsistent Logics,* Doctoral dissertation, Australian National University, 1987.