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FLEXIBLY STUCTURED PREDICATION (*)

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1. *Motivation*

Frege required that each predicate of his formal language should be of a given fixed degree or adicity; i.e., that for each such predicate F there should be some fixed number n such that exactly n terms are needed to combine with F to form a formula. But, in 'The Logical Form of Action Sentences' ([1966]), Davidson confronted the fact (previously emphasized by Kenny), that natural-language action-predicates apparently flout the Fregean requirement in being what Leonard and Goodman [1940] called 'multigrade'—i.e., in apparently possessing the capacity to combine with an *arbitrary* number of terms to generate a formula. Of course, Davidson went on to propose an elegant analysis which, if correct, showed the multigrade character of these predicates to be *merely* apparent, inasmuch as the underlying truth-relevant structure of action-sentences was to be displayed in a language all of whose predicates remained steadfastly Fregean. Still, though compelling, his analysis did not find universal acceptance; and anyway there remain apparently multigrade predicates of ordinary language (e.g. ⌜... are collinear⌝, ⌜... live together⌝) which fall outside its scope. In later work, Richard Grandy ([1976]) and Adam Morton ([1975]) urged and inaugurated a serious study of non-Fregean formal languages with multigrade predicates, both in order to illuminate these and similar features of natural languages and also because they think there is scope for deployment of multigrade resources to technical ends, such as the forging of truth-theories for languages containing Quinean predicate-functors (Grandy), or the investigation of Goodman's calculus of individuals (Morton).

Postponing for the time questions of the soundness of these motivations,

(*) A predecessor of this paper, 'Articulated Predication and Truth-theory', was written by Taylor for a *Festschrift* for Donald Davidson which never appeared. The present paper is a considerably rewritten version of that original. Hazen is primarily responsible for the sections on proof-theory and on Russell (the latter being part of further ongoing work), and for improvements to other formal details. Each author blames the other entirely for all faults which remain.

let us begin by considering in some detail the formal properties of languages containing multigrade predicates. We shall not, however, form these languages simply by adding multigrade predicates to classical languages already containing Fregean predicates, thereby treating multigrade predicates as a sort of generalization of their Fregean brethren, as most of the previously-mentioned writers have done (though we should acknowledge the beginnings of some finer discriminations in the original paper by Leonard and Goodman [1940], p. 53). Instead, our languages will contain what, for want of a better name, we call *flexibly structured predicates*, or just *flexible predicates* for short —generalized Fregean predicates of which both classical Fregean predicates and multigrade predicates are special cases.

The basic idea of a flexible predicate is that such a predicate has, like a Fregean one, a fixed number of argument-places; but that each such argument-place, rather than being occupied by a single term, can be occupied by a number of terms varying between specified limits. More precisely, let a *place-limitation* ℓ be a pair $[i;\alpha]$, where i is a natural number and α is an ordinal such that $i \leq \alpha \leq \omega$; i is called the *lower limit* set by ℓ, and α the *upper limit* it sets. Then a *type* for a flexible predicate F of degree n is a sequence $\langle \ell_1 \ldots \ell_n \rangle$ of place-limitations; and formulae come from F by putting in its i-th argument-place any finitely-long sequence of terms whose length lies improperly between the lower and upper limits set by ℓ_i. Multigrade predicates as standardly conceived are thus 1-place flexible predicates of type $\langle [0;\omega] \rangle$; a Fregean predicate of degree 3 is a 3-place flexible predicate of type $\langle [1;1][1;1][1;1] \rangle$. (Alternatively, we might consider regarding it as a 1-place flexible predicate of type $\langle [3;3] \rangle$; but a reason against this construal will emerge below.)

Clearly, there are flexible predicates which are neither multigrade nor Fregean; so our framework employs a more powerful generalization of the traditional conception than the minimum necessary to allow predication to be multigrade. This in itself would count for little, were it not for the fact that the gain in power is not idle, but can be used to mark some intuitive distinctions. Thus, Grandy in his [1976] found evidence of multigrade constructions in ordinary language in verbs which, like ⌜cooks⌝, can be used both transitively and intransitively, and hence which can apparently be combined optionally with either *one or two* terms to form a sentence; whilst Morton in the paper already cited took as his favoured vernacular candidates for multigrade treatment predicates like our earlier ⌜are collinear⌝ and ⌜live together⌝, which can combine with (almost) *any number* of terms in sentence-formation. Now whatever the ultimate merits of these examples, there are clear intuitive differences between them which are obscured by lumping cases of both kinds together into a single class of multigrade predicates. In fact, neither seems to be best treated as a genuine multigrade predicate —a 1-place flexible predicate of type $\langle[0;\omega]\rangle$— at all. Morton's ⌜live together⌝ comes close, but is better treated as of type $\langle[2;\omega]\rangle$ to make the point that it needs to be combined with two terms at least (a point Morton perforce glosses over, since his apparatus will not conveniently handle it); whilst Grandy's ⌜cooks⌝ is best viewed as a 2-place flexible predicate of type $\langle[1;1][0;1]\rangle$. So the framework of flexible predicates, allowing as it does for predicates which are neither Fregean nor in the strict sense multigrade, seems better able to handle the intuitions which the friends of the multigrade prize than can an account which simply welds multigrade predicates onto a Fregean base.

Moreover, in many applications of multigrade predication, it turns out to be useful to employ non-Fregean analogues of Fregean predicates; thus e.g., where F is a 1-place Fregean predicate, it is useful to have on hand a multigrade predicate F^* whose meaning is systematically related to that of F in such a way as to guarantee the equivalences

$$F^*(a) \leftrightarrow F(a);\ F^*(ab) \leftrightarrow F(a)\&F(b);\ F^*(abc) \leftrightarrow F(a)\&F(b)\&F(c);\ \ldots$$

But, when F is a *many*-place Fregean predicate, it is difficult to state a chain of equivalences which intuitively generalizes the above, unless it can be assumed that F^*, though non-Fregean, is nevertheless a predicate to which the concept of degree applies, and which applies moreover in such a way that it has the same degree as F. So Morton, whose only non-Fregean predicates are multigrade ones and who accordingly lacks degree-possessing predicates of the appropriate sort, is led into a quite counterintuitive characterization of F^* for many-place F, one for which no natural analogue of the given chain of equivalences holds. But within a framework of flexible predication the problem is easily solved. For then we can say that where F is a 2-place Fregean predicate, F^* will be of type $\langle[1;\omega][1;\omega]\rangle$, and the chain of equivalences it sustains can be given thus:

$$F^*(a_1;a_2) \leftrightarrow F(a_1a_2);\quad F^*(a_1b_1;a_2b_2) \leftrightarrow F(a_1a_2)\&F(b_1b_2);$$
$$F^*(a_1b_1c_1;a_2b_2c_2) \leftrightarrow F(a_1a_2)\&F(b_1b_2)\&F(c_1c_2);\ \ldots$$

(Semi-colons are used in these formulae to divide the argument-places of non-Fregean flexible predicates.) Similarly, if F is a 3-place Fregean predicate, F^* will be a flexible predicate of type $\langle[1;\omega][1;\omega][1;\omega]\rangle$ with a meaning guaranteeing the equivalences

$$F^*(a_1;a_2;a_3)\leftrightarrow F(a_1a_2a_3);\quad F^*(a_1b_1;a_2b_2;a_3b_3)\leftrightarrow F(a_1a_2a_3)\&F(b_1b_2b_3);$$
$$F^*(a_1b_1c_1;a_2b_2c_2;a_3b_3c_3)\leftrightarrow F(a_1a_2a_3)\&F(b_1b_2b_3)\&F(c_1c_2c_3);\ \ldots$$

In thus permitting a natural generalization of Fregean predicates F to their analogues F^*, the framework of flexible predicates scores once more over a rival treatment acknowledging only Fregean and multigrade predicates. But note that the generalization is a natural one only when e.g. a 3-place Fregean predicate is taken as being of type $\langle[1;1][1;1][1;1]\rangle$ rather than of type $\langle[3;3]\rangle$, the latter account suggesting no obvious analogous method of generalization; this is the ground previously promised for preferring the former construal.

Consideration of the predicates F^* does however suggest the need for one refinement of our framework. Take, for example, the generalized identity predicate $\ulcorner{=}^*\urcorner$. Our treatment so far would assign this to type $\langle[1;\omega][1;\omega]\rangle$, thus allowing it to be flanked on either side by any non-empty finitely long sequence of terms. But it seems natural to require further that these sequences should be of the same length, so that whereas e.g. $\ulcorner a_1b_1 =^* a_2b_2\urcorner$ and $\ulcorner a_1b_1c_1 =^* a_2b_2c_2\urcorner$ will be well-formed (and equivalent respectively to $\ulcorner a_1 = b_1 \& a_2 = b_2\urcorner$ and $\ulcorner a_1 = b_1 \& a_2 = b_2 \& c_1 = c_2\urcorner$), a formula like $\ulcorner a_1b_1 =^* a_2b_2c_2\urcorner$ will be ruled out as ungrammatical. This can be achieved by redefining a predicate-type for a flexible predicate of degree n as consisting not merely of a sequence of n place-limitations, but also of a symmetrical and transitive relation of *covariance* defined over the place-limitations in the sequence, with the proviso that place-limitations $[i;\alpha]$ and $[j;\beta]$ can be covariant only if $i=j$, $\alpha=\beta$, and $i\neq\alpha$. Formation rules can then be adjusted to require that covariant argument-places should always contain equal numbers of terms. We might informally represent the covariance relation graphically by linking specifications of covariant argument-places with a double-headed arrow. Then the predicate $\ulcorner{=}^*\urcorner$ will be not merely of type $\langle[1;\omega][1;\omega]\rangle$, but more precisely of type $\langle[1;\overset{\downarrow}{\omega}][1;\overset{\downarrow}{\omega}]\rangle$. And in general, when F is of degree n, F^* will be of type $\underbrace{\langle[1;\overset{\downarrow}{\omega}][1;\overset{\downarrow}{\omega}]\ \ldots\ [1;\overset{\downarrow}{\omega}]\rangle}_{n \text{ times}}$.

And, of course, if in the sequel we specify a type without mention of a covariance relation, it should be taken that the covariance relation intended is the trivial null relation $\langle\ \rangle$, which imposes no restriction on the way in which argument-places may covary in their number of terms.

An additional refinement suggests itself. This is to adapt a suggestion made by Morton, and to allow that the language of flexible predicates should also contain special variables, capable all by themselves of filling any argument-place of a flexible predicate, and subject to binding by quantifiers in the standard fashion. Dearly though we would wish to call these *plural* variables, in honour of their semantic role, we must yield place to George Boolos ([1984]), who beat us to the name. So we will christen them after the syntactic shape we choose for them, call them *vector variables*, and write them as $\ulcorner\vec{x}\urcorner$, $\ulcorner\vec{y}\urcorner$, etc. They will take as their substitution instances finitely-long sequences of ordinary variables and singular-terms, subject to the restriction that the length of the substituted sequence should be appropriate to ensure the well-formedness of the substitution instance. Once flexible predicates are introduced, the accompanying extension of quantificational apparatus to enable the binding of their argument-places is entirely natural, and as Morton points out it is apparently required to express the plural quantification involved in such sentences as ⌜All the Mortons live together⌝, which becomes expressible in terms of flexible predicates and extended quantification as

$$\forall\vec{x}\,(\text{Morton}^*(\vec{x})\rightarrow\text{Live together}(\vec{x})).$$

A further natural extension of this same device would allow vector variables to stand alongside other variables (even other vector variables), or other singular-terms, *within* a single argument-place. Then we could e.g. render ⌜Adam, Bernard and all the Mortons live together⌝ as

$$\forall\vec{x}\,(\text{Morton}^*(\vec{x})\rightarrow\text{Live together}(ab\vec{x}));$$

and ⌜All the Mortons and all the Goodmans live together⌝ would become

$$\forall\vec{x}\forall\vec{y}\,(\text{Morton}^*(\vec{x})\&\text{Goodman}^*(\vec{y})\rightarrow\text{Live together}(\vec{x}\,\vec{y})).$$

Accordingly, this is an extension we shall adopt. Since it will turn out (see below) that taking this further step —we shall call it the *adoption of concatenation terms*— has significant consequences for the proof-theory of the symbolism we are developing, it is a step worth motivating further, by pointing out how it leads to an increase in desirable expressive power of the projected formal language. For with its aid, we obtain the pleasing ability to characterize the properties of the quasi-logical notion of starred identity, the postulates for which will include at least these:

[=*I] $\forall\vec{x}\ \vec{x} =^{*} \vec{x}$

[=*II] $\forall\vec{x}\forall\vec{y}\forall x\forall y(\vec{x}x\vec{y} =^{*} \vec{x}y\vec{y} \rightarrow x = y)$.

The second of these would be unstatable without the adoption of concatenation terms. Note that the adequacy of this statement depends upon vector variables being interpreted, as we do, as capable of taking a null value.

So much for motivating intuitions and preliminaries. Now for a more precise description of the formal properties of a language with flexible predicates.

2. *Formalization*

A. *Syntax*

A *type* for a flexible predicate of *degree n* is a pair $\langle\langle \ell_1 \ldots \ell_n\rangle, \mathbf{C}\rangle$, where each ℓ_i is a place-limitation, and $\mathbf{C}$ is a covariance relation, as these notions were defined in the preceding section. The *primitive symbols* of a language $\mathcal{L}$ with flexible predicates are the following:

(1) a finite number of *flexible predicates*, each with a specified type. (We use ⌜*F*⌝, ⌜*G*⌝, etc. as metalinguistic variables over such predicates.)

(2) denumerably many *ordinary variables* ⌜x⌝, ⌜y⌝, ⌜z⌝, ⌜x_1⌝, ⌜y_1⌝, ⌜z_1⌝, ... ; the order which has just been indicated being their *alphabetic ordering*. (We use ⌜v⌝, as a metalinguistic variable over these ordinary variables, and generate further metavariables as required by using primes and subscripts.)

(3) denumerably many *vector variables* ⌜$\vec{x}$⌝, ⌜$\vec{y}$⌝, ⌜$\vec{z}$⌝, ⌜$\vec{x}_1$⌝, ⌜$\vec{y}_1$⌝, ⌜$\vec{z}_1$⌝, ... ; metalinguistic variable ⌜$\vec{v}$⌝, possibly decorated with primes or subscripts .

(4) *logical constants* ⌜~⌝, ⌜&⌝

(5) *universal quantifier* ⌜∀⌝

(6) *punctuation devices* ⌜(⌝, ⌜)⌝, ⌜;⌝.

(We exclude individual constants, function symbols etc. from $\mathscr{L}$ purely to avoid extraneous complications.)

The *wffs* of $\mathscr{L}$ we specify as belonging to two *levels*, the first comprising just wffs without vector variables and the second those in which vector variables occur. Let a *term-sequence* be any finitely-long sequence of ordinary variables of $\mathscr{L}$. Then

(1) If F is a predicate of $\mathscr{L}$ of type $\langle\langle \ell_1 \ldots \ell_n\rangle, \mathbf{C}\rangle$, and $\xi_1 \ldots \xi_n$ are term-sequences, then ⌜$F(\xi_1;\ldots;\xi_n)$⌝ is an *atomic wff* of $\mathscr{L}$ *of level 1*, provided that for each i and j where $1 \leq i \leq j \leq n$,

(a) $\mathrm{ll}(\ell_i) \leq \mathrm{lng}(\xi_i) \leq \mathrm{ul}(\ell_i)$ (where $\mathrm{lng}(\xi_i)$ is the length of ξ_i, and $\mathrm{ll}(\ell_i)$ and $\mathrm{ul}(\ell_i)$ are, respectively, the lower and upper limits which are set by ℓ_i);

and

(b) if $\xi_i \mathbf{C} \xi_j$, then $\mathrm{lng}(\xi_i) = \mathrm{lng}(\xi_j)$.

(2) If A and B are wffs of $\mathscr{L}$ of either level, then these are wffs of the same level: ⌜$\sim A$⌝, ⌜$(A \& B)$⌝, ⌜$\forall v A$⌝.

These two provisions between them specify all the wffs of level 1. Define the notion of free occurrence of ordinary variables in a level 1 wff in the usual way, and say that an occurrence of a term-sequence in a level 1 wff is free iff every variable-occurrence within the sequence-occurrence is a free one. Then the second-level wffs can be introduced:

(3) If A^1 is a level 1 wff, and A^2 comes from A^1 by replacing some free occurrences of term-sequences $\xi_1 \ldots \xi_n$ by vector variables $\vec{v}_1 \ldots \vec{v}_n$, then A^2 is a wff of level 2. (Note that we allow the term-sequences ξ_i to be subsequences of longer term-sequences; and, as a limiting case, to be empty.)

(4) If A is a level 2 wff, so is $\ulcorner \forall \vec{v} A \urcorner$.

Let a *concatenation-term* of $\mathscr{L}$ be a sequence of its ordinary or vector variables. Then it follows from these formation rules that every wff of $\mathscr{L}$ is either a *simple* wff, having the form $\ulcorner F(\tau_1;\ldots;\tau_n)\urcorner$ for some predicate F of degree n and concatenation-terms $\tau_1;\ldots;\tau_n$; or else can be constructed out of simple wffs using logical connectives and quantifiers. This fact is exploited in the definitions to come.

B. Semantics

A *model* for $\mathscr{L}$ is a pair $\langle D,\phi\rangle$, where D is a nonempty set and ϕ is a function assigning, to each predicate F of $\mathscr{L}$ of type $\langle\langle \ell_1 \ldots \ell_n\rangle,\mathbf{C}\rangle$, a set of n-tuples $\langle\sigma_1 \ldots \sigma_n\rangle$ meeting these conditions: for each i and j where $1 \leq i \leq j \leq n$,

(a) each σ_i is a finitely-long sequence of elements of D;

(b) $\text{ll}(\ell_i) \leq \text{lng}(\sigma_i) \leq \text{ul}(\ell_i)$;

(c) if $\ell_i \mathbf{C} \ell_j$, then $\text{lng}(\sigma_i) = \text{lng}(\sigma_j)$.

(And in particular, to explicate an informal notion of the last section, ϕ should assign, to the analogue F^* of a Fregean predicate F of degree n, the set of all those sequences $\langle\sigma_1 \ldots \sigma_n\rangle$ meeting these conditions:

(1) for each i and j where $1 \leq i \leq j \leq n$, $\text{lng}(\sigma_i) = \text{lng}(\sigma_j)$;

(2) for each $k \leq \text{lng}(\sigma_1)$, $\langle\sigma_1(k) \ldots \sigma_n(k)\rangle \in \phi(F)$ —where $\sigma_i(k)$ is the k-th member of σ_i.)

A *dividing function* is a partial function f whose domain is an initial sequence of the positive integers, and whose value for any such integer i is a pair of positive integers $\langle m,n\rangle$ such that $m \leq n$; we call m the *f-lower bound* of i ($f{\downarrow}(i)$), and n its *f-upper bound* ($f{\uparrow}(i)$). We count the null sequence $\langle\rangle$ as a trivial case of an ordered pair of integers, and so allow for the case $f(i) = \langle\rangle$; and further require of dividing functions f that they should be *strictly monotonic*, in the following sense: if i and j are both in the domain of f, $f(i) \neq \langle\rangle$, and j is the least integer greater than i for which $f(j) \neq \langle\rangle$, then $f{\uparrow}(i) + 1 = f{\downarrow}(j)$. Further, where f is a dividing function, and σ is a finite sequence of length lng(σ), we say that f *dissects* σ (Diss(f,σ) iff $f\downarrow(i)=1$ and $f\uparrow(j)=\text{lng}(\sigma)$, where i and j are respectively the least and greatest integers for which f yields non-null values. (Intuitively, then, if f dissects σ and f is defined up to n, then f divides the integers up to lng(σ) into n adjacent divisions $\langle f\downarrow(1)=1, \ldots, f\uparrow(1)\rangle$, $\langle f\downarrow(2)=f\uparrow(1)+1, \ldots, f\uparrow(2)\rangle$, $\ldots$, $\langle f\downarrow(n)=f\uparrow(n\text{-}1)+1, \ldots f\uparrow(n)=\text{lng}(\sigma)\rangle$; with the complication that some of the divisions may be empty.)

We now define some model-theoretic semantic notions relative to two finite sequences σ_1 and σ_2 of elements of D and to a dividing function f. Though stated for dividing functions generally, the primary and intended case is that in which f is a dividing function which dissects σ_2. The restriction to finite sequences involves a few tiresome complications which resort to infinite sequences would avoid, but will have a point in the discussion of applications in the next section.

Let τ be a concatenation-term of $\mathcal{L}$. Then either τ is null, or else τ is $\ulcorner\zeta_1 \ldots \zeta_n\urcorner$, where each ζ_i is either an ordinary variable or a vector variable. We define the *value* $\text{Val}_M(\tau,\sigma_1,\sigma_2, f)$ of τ on M relative to σ_1, σ_2, and f. If τ is null, $\text{Val}_M(\tau,\sigma_1,\sigma_2, f)$ is $\langle\rangle$. In the nontrivial case it is to be obtained by concatenating certain sequences, to be defined as the *denotations* of the variables ζ_i on M relative to σ_1, σ_2, and f: $\text{Val}_M(\tau,\sigma_1,\sigma_2, f) = \text{Den}_M(\zeta_1,\sigma_1,\sigma_2, f)* \ldots * \text{Den}_M(\zeta_n,\sigma_1,\sigma_2, f)$. As for denotation: if v is the ordinary variable occupying the i-th place in the alphabetic ordering of such variables, then the *denotation* $\text{Den}_M(v,\sigma_1,\sigma_2, f)$ is the unit sequence of i-th element of σ_1, if such i-th element there be (and is undefined otherwise). And if $\overrightarrow{v}$ is the vector variable occupying the i-th place in the alphabetic ordering of such variables, then $\text{Den}_M(\tau,\sigma_1,\sigma_2, f)$ is that subsequence of σ which lies between σ's $f\downarrow(i)$-th and $f\uparrow(i)$-th elements (if such subsequence there be; otherwise $\text{Den}_M(\tau,\sigma_1,\sigma_2, f)$ is undefined). So, for example, if $\sigma_2=\langle \mathbf{a}_1 \ldots \mathbf{a}_6\rangle$, $f\downarrow(2)=2$, and $f\uparrow(2)=4$, then $\text{Den}_M(\ulcorner\overrightarrow{y}\urcorner,\sigma_1,\sigma_2, f)=\langle \mathbf{a}_2 \ldots \mathbf{a}_4\rangle$.

We say that sequences σ_1 and σ_2 and dividing function f are *suited to* a wff A of $\mathcal{L}$ on M ($\text{Suit}_M(A,\sigma_1,\sigma_2, f)$) iff the following conditions hold:

(1) if the i-th ordinary variable occurs in A, then $\text{lng}(\sigma_1)\geq i$;

(2) if the i-th vector variable occurs in A, then f is defined at least up to i;

(3) f dissects σ_2;

(4) for every simple subformula B of A, if $B = \ulcorner F(\tau_1;\ldots;\tau_n)\urcorner$ and F is of type $\langle\langle \ell_1 \ldots \ell_n\rangle, \mathbf{C}\rangle$, then for each i and j where $1 \leq i \leq j \leq n$,

(a) $\mathrm{ll}(\ell_i) \leq \mathrm{lng}(\mathrm{Val}_M(\tau_i,\sigma_1,\sigma_2, f)) \leq \mathrm{ul}(\ell_i)$

(b) if $\ell_i \mathbf{C} \ell_j$, then $\mathrm{lng}(\mathrm{Val}_M(\tau_i,\sigma_1,\sigma_2, f)) = \mathrm{lng}(\mathrm{Val}_M(\tau_j,\sigma_1,\sigma_2, f))$.

Next comes the crucial notion of what it is for a wff A to be *satisfied* on M relative to σ_1, σ_2, and f ($\mathrm{Sat}_M(A,\sigma_1,\sigma_2, f)$); again, we define the notion generally, but in the primary case of interest σ_1, σ_2, and f are suited to a A. The clauses governing the nonquantificational cases are straightforward enough:

(1) If A is simple $= \ulcorner F(\tau_1;\ldots;\tau_n)\urcorner$, then $\mathrm{Sat}_M(A,\sigma_1,\sigma_2, f)$ iff $\langle \mathrm{Val}_M(\tau_1,\sigma_1,\sigma_2, f) \ldots \mathrm{Val}_M(\tau_n,\sigma_1,\sigma_2, f)\rangle \in \phi(F)$;

(2) If A is $\ulcorner {\sim} A\urcorner$, then $\mathrm{Sat}_M(A,\sigma_1,\sigma_2, f)$ iff not $\mathrm{Sat}_M(B,\sigma_1,\sigma_2, f)$;

(3) If A is $\ulcorner(B\&C)\urcorner$, then $\mathrm{Sat}_M(A,\sigma_1,\sigma_2, f)$ iff $\mathrm{Sat}_M(B,\sigma_1,\sigma_2, f)$ and $\mathrm{Sat}_M(C,\sigma_1,\sigma_2, f)$.

For the quantifier clause for ordinary variables, we need the notion of two sequences σ and σ' being *similar save perhaps at i* ($\mathrm{Sim}(i,\sigma,\sigma')$, meaning that σ and σ' have the same length and the same elements in all places save possibly the i-th; and require

(4) If A is $\ulcorner \forall v B\urcorner$ where v is the i-th ordinary variable, then $\mathrm{Sat}_M(A,\sigma_1,\sigma_2, f)$ iff for each σ_1' such that $\mathrm{Sim}(i,\sigma_1,\sigma_1')$, $\mathrm{Sat}_M(B,\sigma_1', f)$.

To state the final quantifier clause for vector variables, we need a more complex notion of sequence similarity: that of sequence σ as dissected by f being similar to sequence σ' as dissected by g save possibly at point i of dissection ($\mathrm{Sim\#}(i,\sigma, f,\sigma',g)$). Intuitively, this relation holds if σ and σ' are alike save at the subsequence of σ which f identifies as the value of the i-th vector variable (i.e., save perhaps over that portion between σ's $f{\downarrow}(i)$-th and $f{\uparrow}(i)$-th elements), where σ' may contain some other subsequence, perhaps of different length instead; and where f and g are so related as to extract from σ and σ' the same subsequences as values for all vector variables save perhaps the i-th. We can capture the relevant notion by requiring that $Sim\#(i,\sigma,f,\sigma',g)$ shall hold just when (a) f and g are defined over the same initial segment of the integers; (b) $f(k)=g(k)$ for $k<i$; (c) for some possibly negative constant integer c, $f(k)=g(k)+c$ for $k>i$;(d) g dissects σ'; and (e) sequences σ and σ' are identical up to their $f\uparrow(n)$-th element (where n is the greatest integer up to i, if such there be, for which f has a nonnull value), and from their $f\downarrow(m)+c$-th elements on (where m is the least integer after i for which f is nonnull, again if such there be). Then we can state the final quantifier clause:

(5) If A is $\ulcorner\forall\vec{v}\,B\urcorner$ where $\vec{v}$ is the i-th vector variable, then $Sat_M(A,\sigma_1,\sigma_2,f)$ iff for each σ' and g such that $Suit_M(B,\sigma_1,\sigma_2',g)$ and $Sim\#(i,\sigma_2,f,\sigma_2',g)$, $Sat_M(B,\sigma_1,\sigma_2',g)$.

A formula A is *true* on a model M ($Tr_M(A)$) iff $Sat_M(A,\sigma_1,\sigma_2,f)$ for each σ_1, σ_2 and f such that $Suit_M(A,\sigma_1,\sigma_2,f)$. Finally, A is *valid* ($\vdash A$) iff $Tr_M(A)$ for each model M.

C. Proof-theory

It is an old result that first-order logic supplemented with merely multigrade predicates is trivially axiomatizable, in the sense that any complete proof-procedure for first-order logic is complete also for first-order logic with multigrade predicates. For different-grade occurrences of a single predicate can be assimilated, in an extensional semantics, to occurrences of different single-grade predicates; and the standard completeness theorems are independent of the number of predicates in the language. (We know of no published source for this result, but it amounts to a corollary of Terence Parsons's completeness theorem for first-order logic with predicate modifiers, in an unpublished manuscript circulated in the early 1970's with the title *The Semantics of English*.) The problem would appear to be equally trivial for languages which contain flexible predicates like ours, and combine them with vector variables, but which do not permit vector variables to share an argument-place with other variables, vector or ordinary—that is, which do not permit concatenation terms. For now differently-structured occurrences of a single flexible predicate can be assimilated to occurrences of different Fregean predicates, so that by the previous argument the logic of the language can be treated as conventionally first-order—though this time as a *many-sorted* first-order logic, to allow for the different sorts of variable. (Note that occurrences of vector variables whose contexts impose different length requirements for well-formedness must, for this purpose, be treated as variables of different sorts.)

But the introduction of concatentation terms complicates matters to the extent of rendering the underlying logic (with identity) of our languages unaxiomatizable, a disadvantage offset by the way it adds sufficient expressive power to allow illuminating axiomatic characterization of some interesting concepts relative to the unaxiomatizable underlay. (Compare the existence of categorical axiomatizations of, e.g., the natural number system, in unaxiomatizable second-order logic.)

To see how this situation arises, consider first a language $\mathscr{L}_1$ with flexible predicates including ⌜=*⌝ and the Fregean identity predicate, and also containing an individual constant ⌜0⌝, (Fregean) arithmetical function symbols for the operations of addition and multiplication and for the successor function (respectively ⌜+⌝, ⌜.⌝, and ⌜succ⌝). Then we can encapsulate the basic properties of these fundamental arithemetical notions in the finitely-many axioms of Robinson's **Q**. Suppose further that $\mathscr{L}_1$ contains a flexible predicate ⌜Seg⌝ of type $\langle[1,\omega]\rangle$ with the intuitive meaning ⌜... form an initial segment of the natural number sequence⌝. Then we can axiomatize ⌜Seg⌝ with 'positive' axioms to guarantee that all genuine initial segments of the natural numbers fall within its extension

$$\mathrm{Seg}(0)$$

$$\forall\vec{x}\forall x\forall y(\mathrm{Seg}(\vec{x}x)\&y=\mathrm{succ}(x)\rightarrow\mathrm{Seg}(\vec{x}xy))$$

together with 'restricting' axioms to exclude all nongenuine segments

$$\forall x\forall\vec{x}(\mathrm{Seg}(x\vec{x})\rightarrow x=0)$$

$$\forall\vec{x}\forall x\forall y\forall\vec{y}(\mathrm{Seg}(\vec{x}xy\vec{y}\rightarrow y=\mathrm{succ}(x)).$$

Finally, we can express the principle —definitive, in the context of the axioms of **Q**, of the standard model for arithmetic— that every natural number has only finitely many predecessors, by adding the axiom

$$\forall x\exists\vec{x}\,\mathrm{Seg}(\vec{x}x).$$

Thus $\mathscr{L}_1$ allows a finite categorical characterization of arithmetic. By standard arguments, therefore, its underlying logic is not axiomatizable, even (because the characterization is finite) in the weak sense that all its valid formulae are provable. (True, $\mathcal{L}_1$ with its function symbols and individual constants is not quite a language meeting the formal specifications of the preceding sections. But by routine techniques we can recast the axiomatization into a language $\mathcal{L}_2$ using Fregean predicates (including identity) in place of function symbols and individual constants, so the result carries over to the logic of languages which satisfy our specifications to the letter.)

It might be objected that this result is simply an artefact of a decision to take the semantic values of vector variables to be finite sequences, which might be avoided by adoption of some alternative construal. As we shall see in the next section, the extent to which the semantics we have adopted really does commit us to the view of vector variables as taking finite sequences as 'values' is tendentious. Accepting the terms in which the objection is cast, however, it can be met by pointing to the implausibility, upon examination, of the alternatives to which the objector appeals. For what alternatives are there?

A first idea might be to take ω-sequences rather than finite sequences as the values of the vector variables. That will, however, immediately plunge us into the problem of explaining how only an initial segment of the variable-values is relevant to determining the truth-value of predications, and of finding some artificial way of coding the lengths of the initial segments relevant in various contexts. Supposing that problem can be overcome, there will be the further difficulty that a concatentation term containing a vector variable as other than its last component will not take as value a sequence of the same type as do vector variables themselves.

A second thought might be to regard vector variables as taking finite-or-ω sequences as values. But then we can, within a language like $\mathcal{L}_1$ (with vector-variables reconstrued) distinguish the finite values from the ω-long values by the definition

$$\mathrm{Fin}(\vec{x}) \leftrightarrow \exists \vec{y} \exists z (x = {}^{*}\vec{y} z).$$

By restricting the vector variables of the old axiomatization to finite values we can thus rewrite a finite categorical axiomatization of arithmetic within the new $\mathcal{L}_1$.

A final try might be to take the values of the vector variables as sequences which are finite in a nonstandard sense —as 'internally finite', in the jargon of the non-standard analysis, i.e. as sequences of individuals of the order-type of the natural numbers up to any given number *in some nonstandard model of arithmetic*. But a sequence of one of these nonstandard order types may be identical to one of its own proper initial segments. As a result, it appears that this proposal will be inconsistent with our earlier attempt to axiomatize $\ulcorner = {*} \urcorner$, since on any model with more than one individual in the domain the formula

$$\exists\vec{x}\exists x\exists y\exists\vec{y}(\sim x=y \,\&\, \vec{x}x\vec{y} =^{*} \vec{x}y\vec{y})$$

will be true, flouting [=*II] of the second last paragraph of §1.

3. *Application*

As we indicated in the first section, proponents of multigrade apparatus have advocated its use not just to illuminate the structure of ordinary language, but also as a tool for the forging of philosophical theory. We conclude with two examples of the use of flexible apparatus, both for the sake of further illustration, and to set up a final objection we need to confront.

Our *first example* comes from the history of philosophy. A number of themes that keep recurring in the various incarnations of Bertrand Russell's philosophy in the first two decades of this century —notably, the desire to 'Quine' propositions, the substitutional account of propositions, and the extremely fine-grained ramified theory of logical types found in *Principia Mathematica*— can be fitted into a moderately harmonious whole if we permit ourselves the use of flexible predication in a rational reconstruction of his position. The 'multiple relation' theory of judgement of [1913] will serve as a case in point.

As Russell conceived of them, propositions, the (purported) objects of the propositional attitudes, have constituents; he regularly spoke of them (and, confusingly, of many other things) as 'complexes', and insisted against Frege that even Mt Blanc, with all its snowfields, can be a part of a proposition. In addition to such concrete particulars as Mt Blanc (or, to take a later example, the white sensum that lasts for the second or so that I stare at a chalk mark), propositions also have universals as constituents —properties in the case of propositions which are monadic predications (if there are any such), relations in other cases. On the nature of these, later commentators (amongst them Quine) can be misleading. For Russell did *not* identify universals and propositional functions (the values of higher-order variables).

The domain of propositional functions (of a given type) must be closed under at least some logical operations if higher-order logic is to be of any mathematical utility, yet for Russell it was (often) an open, indeed empirical question whether, for example, a given universal has a complement. Again, universals being genuine entities, we include them all when we quantify over *absolutely all* things; but propositional functions must, on pain of paradox, be stratified into types. On a correct account, Russell's ontology of universals is a sparse one. There are universals; with some of these (those which can figure as constituents of propositions we can genuinely conceive) we are acquainted; others we can come to know only by description; but just which ones there are is a matter for empirical, not logical, investigation. To any genuine universal a propositional function corresponds, but beyond these the domains of propositional functions are padded out by including others with the dubious status of logical constructions.

Thus far, our exposition has accepted uncritically that there must *be* such things as propositions, the objects of the attitudes. But Russell did not. Some of the objects of the attitudes are false: people judge wrongly, hope in vain, and so on. On an ontology that accepted propositions as genuine entities, these commonplace facts would require that the furniture of the universe include *real falsehoods*, and this offended Russell's famously robust sense of reality. Others are less sensitive in this respect, but our present task is not to evaluate Russell's scruples, but to expound the multiple relation theory of judgement to which they led.

This arises entirely naturally: if propositions are to be rejected, we can use their constituents —or, more precisely, the things which would be their constituents, were they to exist— in their place. After all, even if there were propositions, a subject standing in an attitude toward one of them would, by simple composition of relations, also stand in *some* relation to the undoubted entities out of which the proposition was composed. When propositions are rejected, then, the attitudes can be reconstrued: to say a subject S judges (or whatever) that p is to predicate, not a dyadic relation of S and the pseudoentity p, but some polyadic ('multiple') relation of S and the however-many-there-are constituents of p. In a certain sense, this reduces propositions to their constituents, but the standard objections to such a reduction —that the existence of the constituents does not suffice for the truth of the proposition, and that the same constituents can compose quite different propositions— are avoided, because the specifically propositional connexions between the constituents can be taken to be implied in the polyadic relations corresponding to the attitudes: S may be acquainted severally with R, a, and b without its being the case that Judge(S,R,a,b), and, since a polyadic relation is not required by logic to be commutative with respect to any of its argument places, S can judge that Rab without judging that Rba (i.e., Judge(S,R,a,b) is logically independent of Judge(S,R,b,a)).

For our purposes, the important thing to note is that the polyadic relations which supplant the attitudes must be flexible. For we may judge that a particular has a property, or that two particulars stand in a relation, or that three particulars stand in a triadic relation, ...; or, for that matter, that some subject judges that we judge that ... . So it appears that the judgement relation (and other attitudinal relations) must be of some such type as $\langle[1,1][2,\omega]\rangle$, so we can write

Judge($S;F,a$)

Judge($S;R,a,b$)

Judge($S;R',a,b,c$)

Judge(S;Judge,S',F,a).

Of course, the theory needs extension, to cover nonatomic propositions. Russell himself provided no such further development in detail, though there are hints in his writings which reveal one possible shape the extension might take. In any case, even in the context of the partial theory to hand, it is clear that Russell will need not just flexible predicates, but also the other mainstay of our apparatus, vectorial quantification. For to say that there is some proposition which Ottoline accepts but Colette rejects is to say

$\exists F \exists x$(Judge(Ottoline;F,x)&Deny(Colette;F,x))

or, perhaps,

$\exists R \exists x \exists y$(Judge(Ottoline;$R,x,y$)&Deny(Colette;$R,x,y$))

or, perhaps, ...

The infinity (recall iteration!) of possible disjuncts means that we can express the idea in finite terms only with the aid of vector variables:

$\exists\vec{x}$(Judge(Ottoline;$\vec{x}$)&Deny(Colette;$\vec{x}$)).

Our *second example* of the use of flexible apparatus for philosophical purposes comes from the philosophy of logic. Consider a modal language *LN*, formed by adding a modal operator $\ulcorner\Box\urcorner$ to a first-order language *L*. To the standard Kripkean model-theory for *LN* there corresponds a truth-theory which generates statements of truth-conditions for the sentences of *LN* phrased in terms of an apparatus of worlds, an accessibility relation between them, and world-relativized correlates of *LN*'s predicates. But the (philosophical) interpretation of these truth-conditions is, of course, a familiar bugbear, and the whole approach poses a problem: are modal locutions *autonomously* intelligible, or rather, as the theory seems to suggest, ultimately accessible only through world-relativized reconstrual?

Concerned in part with the latter problem, Christopher Peacocke ([1978]) investigated the problem of constructing a new, homophonic truth-theory for *LN*. His initial ploy is the obvious one: to deal with modal sentences by adding to the Tarskian theory of satisfaction for unadorned *L* the clause

$$\mathrm{Sat}_{LN}(\ulcorner\Box A\urcorner,\sigma)\leftrightarrow\Box\,\mathrm{Sat}_{LN}(A,\sigma).$$

But—after turning a well-known objection to this approach by Wallace ([1972])—Peacocke finds himself faced by a nasty obstacle posed in conversation by Kit Fine.

Suppose *LN* to contain an existence-predicate $\ulcorner\mathrm{E}\urcorner$, and consider the *Fine formula*

$$\exists x\Diamond\exists y\sim\Diamond(\mathrm{E}(x)\&\mathrm{E}(y)).$$

This formula asserts the actual existence of an element *x* and the possible existence of an element *y* which do not possibly *co*exist. As such, it is intuitively true. But on Peacocke's theory, that truth requires the possible existence of a sequence containing both *x* and its incompossible companion *y* as elements; and if sequences are sets, and granted the plausible postulate that no set can exist in a world unless its elements do (Fine [1977] p. 126), the very truth of the Fine formula precludes the possible existence of any such sequence. (The Fine formula poses a problem for the Kripkean truth-theory too, but one which that theory can perhaps avoid, by making use of its accessibility relation. For the truth of the Fine formula by Kripkean lights bars the coexistence of *x* and *y* only in worlds *accessible from the* *actual world*, and so can allow their coexistence, and the existence of a sequence containing them both, in worlds not so accessible.)

Omitting frills, Peacocke's response, rendered into the present framework, is to reconstrue the satisfaction predicate of his truth theory as a flexible one, of type $\langle[1,1][0,\omega]\rangle$; and to use vector variables in place of variables over sequences. Thus the critical clause in the account of satisfaction becomes

$$\mathrm{Sat}_{LN}(\ulcorner\Box A\urcorner,\vec{x})\leftrightarrow\Box\mathrm{Sat}_{LN}(A,\vec{x})$$

and, freed from its commitment to existence of sequences as satisfiers, the truth-theory slides past its difficulty with Fine's formula.

Clearly, the success of Peacocke's manoeuvre stands or falls with an utter contrast between flexible predicates, and Fregean predicates of sequences; and correlatively, between vector variables and variables over sequences. The same point holds for our Russellian application; construing ⌜Judge⌝ as a Fregean predicate with a sequence as second argument-place would reintroduce the propositional objects which it is the whole point of the multiple relation theory to avoid. But, it may be objected, flexible predicates are not ultimately intelligible save as predicates of sequences, nor vector variables save as variables over sequences—a fact born out by the semantics offered in the last section, which manifestly treated flexible predicates and vector variables in precisely such terms.

The objection is manifestly a crucial one. But the apparent support which it gains from the semantics of the last section is illusory; for, by taking a leaf out of Peacocke's book and transporting flexible apparatus into our set-theoretic metalanguage, we can recast the whole of our semantics in a way which avoids the explicit invocation of sequences which beset our first attempt. Thus we will replace the sequence variables of the original with vector variables, and its Fregean predicates of sequences by flexible predicates of appropriate type. (It will be handy too, though not essential, to extend our flexible apparatus a bit, so Fregean function symbols can be replaced by flexible ones, without first having to be reconstrued in terms of predicates.) So, for example, the satisfaction predicate will be reconstrued as of type $\langle[1,1][1,\sigma],[1,\sigma][1,1]\rangle$, and in place of $\ulcorner\mathrm{Sat}_M(A,\sigma_1,\sigma_2,f)\urcorner$ we will write $\ulcorner\mathrm{Sat}_M(A,\vec{x}_1,\vec{x}_2,f)\urcorner$.

It was to facilitate this rewriting exercise that the semantics of the last section was cast in the way it was, e.g. in its otherwise clumsy use of finite sequences as satisfiers. It is not an exercise which will vastly impress our objector, who will rightly maintain that it merely transfers to the metalanguage the question of the right way to construe flexible apparatus. What it does do, however, is deprive the objector of any *formal* support. The substantive issue of the ultimate intelligibility of flexible apparatus as genuinely autonomous is not to be settled on formal grounds, but turns on the soundness or otherwise of intuitive motivation of the sort our first section sought to provide, and on the fruitfulness of the way it can be deployed to philosophical ends.

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