Logique & Analyse **139-140** (1992), 313-333

FINITE MODELS FOR SOME SUBSTRUCTURAL LOGICS

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This paper is a report of computer-driven research into the modelling of propositional systems related both to linear logic and to relevant logic. One result is an exponential lower bound on the number of models of given size validating such systems. Another theorem explores dualities within models for such systems in order to explain striking regularities in the distribution of models of the system C. In these researches the computer figures not as a prover of theorems, nor even as a proof assistant, but as a source of quasi-empirical data.

1. *The Logics C, CK and R*

As is explained at more length in [5], premises of arguments—items in databases—may be bunched together in several different ways, conclusions (queries) either following or failing to follow logically from these bunches. Here we consider two ways of putting statements together for the sake of argument: collection into sets and collection into multisets. The two operations may be nested inside each other to any finite depth, giving a technical notion of a "bunch" of formulae as follows.

1. Any formula is both an extensional and an intensional bunch.
2. Any set of intensional bunches is an extensional bunch.
3. Any multiset of extensional bunches is an intensional bunch.

Bunches (intensional and extensional) are just what are required to be so by this definition. For the logical purposes of this paper only finite bunches are really required, though the general theory of such complex objects is of interest and deserves a proper mathematical study. It is worth pointing out before entering into details of the logic that the null set $\varnothing$ and the null multiset $\Box$ are here being treated as different items with very different inferential properties. On the other hand, we shall be fairly loose about the difference between $\{A\}$, the set singleton of A, and its multiset singleton $[A]$, since both have just the same logical force as A itself.

It is convenient to have some notation for intentional and extentional combinations of bunches of arbitrary sorts. For this purpose it is easiest simply to overload the symbols for set union $\cup$ and multiset union $\sqcup$. To form the set union over a family of bunches, let each of them which is not already a set be replaced by its set singleton and then the union formed. Analogously, the multiset union over a family of bunches may be defined by substituting $[X]$ for each constituent X which is not already a multiset. Thus, for example, where M is a bunch which is a multiset, S a bunch which is a set and A a formula

$$M \sqcup S \sqcup A = M \sqcup [S] \sqcup [A]$$
$$M \cup S \cup A = \{M\} \cup S \cup \{A\}$$

Clearly each union operation is associative and commutative. In addition, set union is idempotent though multiset union is not.

The objects with which logic deals are *sequents*. A sequent is an ordered pair consisting of a bunch and a single formula.(1) We write

$$X \vdash A$$

to say that the sequent with premises X and conclusion A can be proved in a given logical system, subscripting the turnstile with the name of the system if the context does not make it plain. $\Gamma(X)$ is a bunch in which X occurs as a sub-bunch, and $\Gamma(Y)$ differs from it exactly in that the displayed occurrence of X is replaced by Y. The systems begin with one axiom scheme

$$A \vdash A$$

Any formula is derivable from itself. The rest of the rules transform this trivially valid argument form into less trivial ones. Apart from the unspoken principles flowing from combinatory properties of the operations $\cup$ and $\sqcup$, such as associativity and commutativity, there is one basic structural rule saying that inference from a *set* need only appeal to a subset

(1) This concept of a sequent is really a special case: it is sometimes more convenient to think of a sequent as a pair of bunches, giving multiple conclusions as well as multiple premises. The present paper uses only the attenuated version, however.

$$\frac{\Gamma(X) \vdash A}{\Gamma(X \cup Y) \vdash A} \quad \textbf{EK}$$

For the logic **C** with which we begin there is no corresponding principle for multisets.

To formulate the propositional logic **C**, add to the basis various logical rules governing connectives. The first connectives to consider are two types of conjunction corresponding to extensional and to intensional combination of premises.

$$\frac{X \vdash A \qquad Y \vdash B}{X \cup Y \vdash A \,\&\, B} \quad \textbf{\&I}$$

$$\frac{X \vdash A \,\&\, B \qquad \Gamma(A \cup B) \vdash C}{\Gamma(X) \vdash C} \quad \textbf{\&E}$$

$$\frac{X \vdash A \qquad Y \vdash B}{X \sqcup Y \vdash A \circ B} \quad \circ\textbf{I}$$

$$\frac{X \vdash A \circ B \qquad \Gamma(A \sqcup B) \vdash C}{\Gamma(X) \vdash C} \quad \circ\textbf{E}$$

It will readily be seen that the connective symbolised by the ampersand has the familiar semilattice character of classical conjunction. It exhibits the usual truth-functional relationship, a conjunction following from (the set of) its conjuncts and in turn entailing each of them. The intensional conjunction or "fusion", on the other hand is entailed by the *multiset* of its conjuncts and in general entails not those conjuncts individually but whatever can be got by using one of them as an inference ticket and the other as a departure point.

Negation in **C** is an involution: an operation of period 2 which reverses the sense of a proposition and reverses its inferential properties. Thus it is governed by the two-way double negation rule and by a version of *modus tollens*.

$$\frac{X \vdash A}{X \vdash \neg\neg A} \quad \textbf{DNI} \qquad\qquad \frac{X \vdash \neg\neg A}{X \vdash A} \quad \textbf{DNE}$$

$$\frac{X \sqcup A \vdash B \qquad Y \vdash \neg B}{X \sqcup Y \vdash \neg A} \quad \mathbf{MT}$$

In terms of negation and the two conjunctions we can define several other connectives. Each conjunction has a dual disjunction, and as might be expected in a logic there is an implication

$$\begin{aligned} A \vee B &=_{\text{df}} \neg(\neg A \,\&\, \neg B) \\ A + B &=_{\text{df}} \neg(\neg A \circ \neg B) \\ A \rightarrow B &=_{\text{df}} \neg(A \circ \neg B) \end{aligned}$$

Introduction and elimination rules for these are derivable but are omitted here for brevity.

It is technically useful to add sentential constants to the language. The true constant t has, as premise, the same inferential force as the null multiset □, while the trivial constant T corresponds similarly to the null set ∅. Each of these can be negated to produce the false and absurd constants f and F respectively. There are two further axioms **tI** and **TI**

$$\begin{aligned} \Box &\vdash \mathrm{t} \\ \varnothing &\vdash \mathrm{T} \end{aligned}$$

and one more rule

$$\frac{X \vdash \mathrm{t} \qquad \Gamma(\Box) \vdash A}{\Gamma(X) \vdash A} \quad \mathbf{tE}$$

Finally, there are two more definitions.

$$\begin{aligned} \mathrm{f} &=_{\text{df}} \neg\mathrm{t} \\ \mathrm{F} &=_{\text{df}} \neg\mathrm{T} \end{aligned}$$

Clearly, $\mathrm{t} \rightarrow A$, is equivalent to A, as are $\mathrm{t} \circ A$ and $\mathrm{f} + A$, while $A \rightarrow \mathrm{f}$ is equivalent to $\neg A$.

To convert **C** into the stronger logic **CK** it suffices to add the intensional version **IK** of the structural rule **EK**.

$$\frac{\Gamma(X) \vdash A}{\Gamma(X \sqcup Y) \vdash A} \; \mathbf{IK}$$

The effect of the two rules together is of course to allow arbitrary weakening of premise bunches, so they can be combined into a single rule

$$\frac{\Gamma(X) \vdash A}{\Gamma(\Delta(X)) \vdash A}\ \mathbf{K}$$

One way of viewing **IK** is as the result of identifying $\varnothing$ with $\square$ or equivalently T with t. This identification takes the system close to classical logic, but not all the way since multiset union remains non-idempotent.

The main logical use of the idempotence of a premise-combining operation is to secure the rule of contraction or absorption: that what follows from two applications of a premise can be regarded as resulting from just one. The intensional form of contraction can be added to **C** without assuming full idempotence, and the result of adding it is the relevant logic **R**. **R** is **C** plus

$$\frac{\Gamma(X \sqcup X) \vdash A}{\Gamma(X) \vdash A}\ \mathbf{W}$$

In the context of **C** the rules **IK** and **W** can be seen as opposites in that an equivalent of **IK** is the principle that intensional bunching is stronger than extensional

$$\frac{\Gamma(X \cup Y) \vdash A}{\Gamma(X \sqcup Y) \vdash A}\ \mathbf{K}^{\sqcup}$$

while **W** is equivalent to the converse

$$\frac{\Gamma(X \sqcup Y) \vdash A}{\Gamma(X \cup Y) \vdash A}\ \mathbf{W}^{\sqcup}$$

The result of imposing both extra conditions would be to identify the two forms of bunching, producing classical logic. To help place the various systems in the logical landscape, note that **C** is exactly the non-modal fragment of linear logic plus the distributivity of & and $\vee$. **CK** is similarly the distributive partner of the logic **BCK**.

2. *Modelling the Logics*

An algebraic model for **C** in the simplest sense is just a set on which are defined functions corresponding to the connectives and to the bunching operations. A "**C** structure", then, is a complete distributive lattice, extensional bunching corresponding to general lattice meet and the lattice order serving to model the relation of implication. It is also equipped with a function μ from multisets of its elements back into the algebra such that for any a

$$\mu[a] = a$$

and for any multisets of elements $M_1 \ldots M_n$

$$\mu(M_1 \sqcup \ldots \sqcup M_n) = \mu[\mu M_1 \ldots \mu M_n]$$

Moreover there should be strong lattice ordering in the sense that for any set X and for any element a

$$\mu[a, \bigvee X] = \bigvee\{\mu[a,x] \colon x \in X\}$$

To model negation there must be an involution $^-$ satisfying the postulates

$$\bar{\bar{a}} = a$$
$$\mu[a,b] \leq \bar{c} \Leftrightarrow \mu[a,c] \leq \bar{b}$$

Extensional conjunction and disjunction (& and $\vee$) are modelled straightforwardly by lattice meet and join, and negation by $^-$. Intensional conjunction, $\circ$, is modelled for interpretation $\Im$ by

$$\Im(A \circ B) = \mu[\Im(A), \Im(B)]$$

and the bunching operations $\cup$ and $\sqcup$ are treated exactly like & and $\circ$ respectively. $\Im$(t) is always $\mu\Box$, and $\Im$(T) is always lattice I, the top element. Modellings of the other connectives follow from their definitions.

An **R** structure is just a **C** structure in which for every element a

$$a \leq \mu[a,a]$$

and a **CK** structure is a **C** structure in which

$$\mu\Box = \mathrm{I}$$

Formula A is *true* for interpretation $\Im$ iff $\mu\Box \leq \Im(A)$ and *valid* in a particular algebra iff true for every interpretation in that algebra. Sequent $X \vdash A$ *holds* on interpretation $\Im$ iff $\Im(X) \leq \Im(a)$ and is *valid* in an algebra iff it holds on every interpretation therein. We say that X *implies* A where the sequent holds and that X *entails* A where the sequent is valid. A is valid simpliciter iff it is valid in every algebra for the appropriate logic, and X entails A, without qualification, likewise.

This paper concerns attempts to generate small models—the largest with 16 elements—using the program MaGIC (Matrix Generator for Implication Connectives) developed at the Australian National University by the author.(2) For the remainder of this paper, therefore, attention may be restricted to finite algebras of the kinds outlined above. Finite lattices are trivially complete, and in finite algebras there is no need to consider the function μ in general since it is fully reducible to its binary case. We therefore write

$$ab =\mathrm{df}\ \mu[a,b]$$
$$e =\mathrm{df}\ \mu\Box$$

It is clear that fusion, the binary operation symbolised by juxtaposition, is associative and commutative, that e is an identity for it, and that

$$ab \leq \overline{c} \Leftrightarrow ac \leq \overline{b}$$

Given that the partial order is that of a distributive lattice and that $^{-}$ is of period 2, these properties could be stipulated as postulates were fusion taken as primitive rather than defined. Naturally, the algebras model **R** or **CK** iff they also satisfy

$$a \,\&\, b \leq ab$$

or

(2) MaGIC is in the public domain and is available by anonymous ftp from `arp.anu.edu.au` where it is in directory `/pub/MaGIC`. Full documentation, data files, Makefile and online `man` pages are included with the sources. MaGIC is in 'C'.

$$ab \leq a \,\&\, b$$

respectively. Proofs of soundness (that all provable sequents are valid) and completeness (the converse) are routine and will not be rehearsed here.

3. *Numbers of Models*

To generate some comprehensible data concerning the behaviour of MaGIC applied to **C** it proved useful initially to concentrate on totally ordered structures. These provide an easily graspable series increasing in size in a regular way. Because of total order there can be no isomorphisms between distinct such structures provided the elements are presented in some standard sequence: we choose to make this simple by identifying the implication order with numerical order. Where the partial order of implication is a chain, the lattice operations become trivial: the meet of a and b either is a or is b, whichever is the lower, and their join is the other. Where the elements are numbered from 0 at the bottom to n at the top, $\overline{a}$ can only be n-a, so the De Morgan quasi-complement operation is also determined by the ordering. Moreover, provided for every a

$$0a = 0$$

residuation is guaranteed, for $0a \leq b$ whence $\{x : xa \leq b\}$ is nonempty; but in a finite chain the lattice join of any set is a member of that set, so the residual b/a of b by a is well defined as the greatest such x. For the purposes of logic it is customary to write b/a as $a \rightarrow b$ in order to emphasise the relationship with implication. The easy way to generate totally ordered models of **C** and like systems is thus to work as MaGIC does not with fusion directly but with the operation $\rightarrow$. The modelling condition for the arrow reads

$$\Im(A \rightarrow B) = \Im A \rightarrow \Im B$$

Now clearly $a \rightarrow b$ is a true element (in the positive cone) iff $a \leq b$, so each cell of the implication matrix can take either true values only or false ones only. This regularity may be written into the search space at an initialisation stage. It is easy to ensure that in every matrix tried

$$a \to b = \overline{b} \to \overline{a}$$

just by forcing the matrices tried to show symmetry about the diagonal. As a general fact about **C** models, $e \to a = a$ for all a. Given all of this, and given that for all a, b, c

$$a \leq b \Rightarrow c \to a \leq c \to b$$

it is only necessary to test for satisfaction of the one postulate

$$a \to b \leq (b \to c) \to (a \to c)$$

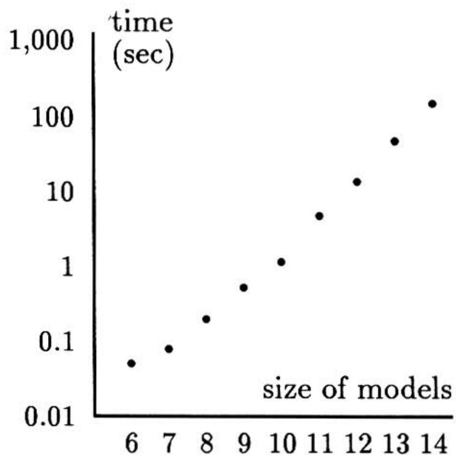

Figure 1: Runtimes

This MaGIC does reasonably quickly, though no doubt a program using a search function designed specifically for this system rather than a general-purpose one would out-perform it.

Here are the runtimes on a Sparc-2 for generating totally ordered models of **C** by MaGIC 2.0. The sizes are numbers of elements and the times are in seconds.

Size	6	7	8	9	10	11	12	13	14
Time	0.05	0.08	0.20	0.52	1.41	4.74	13.7	45.3	153

It seems clear that these times are increasing at least exponentially with the number of elements. This impression is strongly confirmed by graphing the same results on a logarithmic scale (Figure 1). This would seem to indicate that the problem of generating structures such as these ordered monoids is of exponential time complexity. However, a closer examination reveals another interesting function of the size of models: their number. Figure 2 is a similar graph of the number of totally ordered **C** models of sizes from

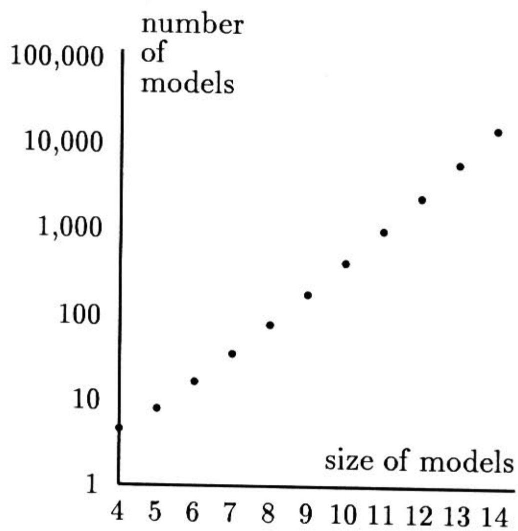

Figure 2: Numbers of Models

4 to 14. So apparently the number of models of a given size is also bounded below by an exponential function of the size. This places the earlier figures for runtimes in a new light, for it now appears quite possible that the model-generation problem *looks* hard only because the *density* of solutions in the

search space does not decrease much with size. The unit task for the problem is really the generation of a single model rather than all of them. Hence a fairer measure of its difficulty is the time taken per model found. If the number of solutions is increasing exponentially then *any* method of producing them—even reading them from a file—will have an exponential time bound, so in these circumstances such a bound is not necessarily devastating.

The time taken per model generated, by size again, is as follows. These times are in milliseconds.

Size	6	7	8	9	10	11	12	13	14
Time	2.9	2.2	2.5	2.8	3.2	4.5	5.3	7.0	9.2

This function is certainly increasing with the size of the models, but it is not obvious that the increase is exponential. Figure 3 is the graph, this time on an ordinary linear scale.

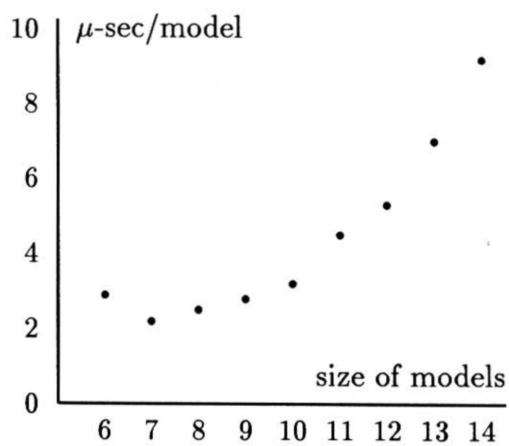

Figure 3: Milliseconds per Model

Some increase in the time taken per model found is to be expected, both because large models occur comparatively rarely in comparison with the size of their search space and because the time taken to test a matrix of size $n \times n$ for satisfaction of a postulate with k variables is likely to be around $O(n^k)$. In the present case the serious postulate to be tested has 3 variables, so certainly the time spent testing a good matrix of size n is proportional to n^3, and it would be unsurprising to find the time taken over bad ones to show broadly similar behaviour. Given this, the above results are quite encouraging, suggesting that the model-generation problem is not so severely intractable as at first appears. Indeed, up to size 10 the increase in time seems to be no worse than linear. One upshot is that even linear improvements in processing speed—for instance those due to parallelisation—may bring about significant advances in what can be computed.

The next need is for a firm theorem to the effect that the numbers of models do indeed increase exponentially with their size. It is easiest to establish this for the special case of totally ordered models for **CK** which are **C** models in which the identity e is the greatest element under the lattice ordering. Begin, then, with a finite chain

$$a_0 < a_1 < \ldots < a_{n-1} < a_n$$

The key is to pick what are to be the idempotent elements for the fusion operation. Let the idempotents be some subset Σ including a_0. Clearly there are 2^n different possible selections and what we want to show is that each of them leads to a model. The point of requiring a_0 to be idempotent is to ensure that for any a_i there is some idempotent a_j with $j \leq i$. Now define

$$a_i \circ a_j =\text{df} \bigvee\{a_k : k \leq i, k \leq j, a_k \in \Sigma\}$$

It is obvious that this operation is commutative, and a moment's thought shows it to be associative as well. Moreover, it clearly satisfies lattice ordering in that

$$a_i \circ (a_j \vee a_k) = (a_i \circ a_j) \vee (a_i \circ a_k)$$

So we have a totally ordered commutative semigroup. To make it into a monoid we add an extra element e with the property that $a_i < e$ for all i and stipulate that it be an identity for $\circ$:

$$a_i \circ e = a_i = e \circ a_i$$

This does not upset associativity, commutativity or lattice ordering. Residuation is also fine:

$$a_i \to a_j = \bigvee\{a_k\colon a_i \circ a_k \leq a_j\}$$

All that is now required is to carry these properties over into a structure with a De Morgan quasi-complement $^-$ and for this we use a construction deriving from one originated by Meyer for the purposes of showing conservative extension for negation in logics such as **R**. Let f and $b_0 \ldots b_n$ be objects distinct from each other and from e and $a_0 \ldots a_n$. Extend the total order to them thus

$$f < b_n < \ldots < b_0 < a_0 < \ldots < a_n < e$$

and define quasi-complementation in the obvious way, letting $\overline{e}$ be f, $\overline{f}$ be e and for any $i \leq n$ letting $\overline{a_i}$ be b_i and $\overline{b_i}$ be a_i. It remains to extend the operation $\circ$ to the enlarged structure, which is easily done. For any element x let

$$x \circ e = x = e \circ x$$

and let

$$x \circ f = f = f \circ x$$

Then for any i and j from 0 to n,

$$\begin{aligned} a_i \circ b_j &= \overline{a_i \to a_j} \\ b_i \circ a_j &= \overline{a_j \to a_i} \\ b_i \circ b_j &= f \end{aligned}$$

Checking that all of the postulates for a totally ordered **CK** model hold of the structure is routine and will not be detailed here. Evidently each of the 2^n original choices of idenpotents leads to a different model, and each of these models has exactly $2n+4$ elements. Hence where m is any even number greater than 2 there are at least $0.25(\sqrt{2})^m$ totally ordered **CK** models with m elements.

The analogous argument for **R** models again proceeds by picking an arbitrary set of idempotents, this time including a_n but not necessarily a_0 and defining

$$a_i \circ a_j =\text{df} \bigwedge\{a_k : i \leq k, j \leq k, a_k \in \Sigma\}$$

Again associativity and commutativity are secured unproblematically. The added identity e is ordered *below* the a_i rather than above them, and it is necessary to add a further element $\bot$ even below e. The definition of $a_i \rightarrow a_j$ is exactly as before. Now the b_i are placed up above the a_j and a new top element $\top$ is added to give the ordering

$$\bot < e < a_0 < \ldots < a_n < b_n < \ldots < b_0 < f < \top$$

The definition of fusion requires a slight amendment:

$$b_i \circ b_j = \top$$

This structure has more added elements than that for **CK**, so the lower bound this time is only $0.125(\sqrt{2})^m$. This is still exponential, however.

It will be noticed that the results of this section place lower bounds on the numbers of models of even-numbered sizes only. I fully expect that an extension to odd numbers is possible, but I also expect it to be uninteresting so it is not pursued here.

4. *Duality*

It must have struck the reader that the construction used above to show the number of totally ordered **R** models exponentially bounded was in a straightforward sense dual to the argument for **CK** models. The purpose of the present section is to make this duality precise and to explain it. Here are the numbers of totally orderd **C** models of all sizes up to 14 elements listed according to the choice of e. The "middle" element M of any finite chain is the one such that either $\bar{M} = \bar{M}$ (if the number of elements in the chain is odd) or $\bar{M} = M$-1 (if the number of elements is even).

Value of e \ Size	2	3	4	5	6	7	8	9	10	11	12	13	14
$M+6$												2067	6148
$M+5$										329	944	1212	2946
$M+4$								59	161	204	477	643	1390
$M+3$						12	31	38	85	109	229	367	748
$M+2$				3	7	8	17	20	41	60	121	218	438
$M+1$		1	2	2	4	4	8	10	20	32	64	126	252
M	1	1	1	1	1	1	1	1	1	1	1	1	1
M-1			1	2	2	4	4	8	10	20	32	64	126
M-2					3	7	8	17	20	41	60	121	218
M-3							12	31	38	85	109	229	367
M-4									59	161	204	477	643
M-5											329	944	1212
M-6													2067

The first striking regularity (Fact 1) is that when e is M there is only one **C** model of any given size. The second (Fact 2) is that the numbers below the middle in any column exactly mirror those above it in the previous column. To explain these facts we need some theorems.

Let us say that a **C** structure is *complete* iff for every element a either a is in the positive cone or $\overline{a}$ is, and that it is *pseudo-complete* iff for every a, $e \leq a \vee \overline{a}$. In totally ordered finite algebras, of course, these two amount to the same thing, though in a more general setting they can come apart. The first fact to note is that for any element a of any pseudo-complete **C** model with 0

$$a \mathbin{\&} (a \to 0) = 0$$

The proof of this is quite easy. First note that since $e \leq \overline{0}$ and $e \to 0 = 0$

$$\overline{0} \to 0 = 0$$

From this it is immediate that for any a

$$a \to 0 = a \to (\overline{0} \to 0)$$

whence by permutation in the right-hand side

$$a \to 0 = \overline{0} \to (a \to 0)$$

Applying the facts that in general $x \to y = \overline{y} \to \overline{x}$ and that $^{-}$ is of period 2 we get

$$a \to 0 = \overline{a \to 0} \to 0$$

whence

$$\overline{a \to 0} \leq a \to 0 \to 0$$

But in a pseudo-complete structure

$$e \leq (a \to 0) \vee \overline{a \to 0}$$

so by the monotonicity properties of lattice ordered structures

$$e \leq (a \to 0) \vee (a \to 0 \to 0)$$

Clearly both sides of the join are less than or equal to $a \,\&\, (a \to 0) \to 0$, so very quickly

$$e \leq (a \,\&\, (a \to 0)) \to 0$$

which is to say

$$a \,\&\, (a \to 0) = 0$$

What this amount to in the case of totally ordered complete structures is that for any nonzero element a, $a \to 0 = 0$. In the literature on algebraic treatment of relevant logics this property is called *rigorous compactness*.

There is a stronger property of *rigorous supercompactness*, defined as rigorous compactness plus that the result of deleting elements 0 and $\overline{0}$ be a subalgebra. The next observation is that any totally ordered complete **C** structure with 0 in which $\overline{e} \leq e$ is rigorously supercompact. That it is rigorously compact we have already established, and that the result of deleting the extreme elements is a De Morgan lattice is obvious. It remains therefore to show that if $ab = 0$ then either $a = 0$ or $b = 0$ and that if $ab = \overline{0}$ then either $a = \overline{0}$ or $b = \overline{0}$. The former is easy given total order and rigorous compactness, for if $ab = 0$ then by residuation $a \leq b \to 0$ whence by rigorous compactness $a \,\&\, b = 0$. For the latter, suppose $ab = \overline{0}$ and $a \neq \overline{0}$. As we have shown, $\overline{a}$ being nonzero, $\overline{a} \to 0 = 0$ which is to say $\overline{a}\overline{0} = \overline{0}$. On the supposition that $ab = \overline{0}$, therefore, $ab\overline{a} = \overline{0}$. Now in any **C** structure and for any a and b, $a\overline{a} \leq \overline{e}$ and $e \leq b \to b$, so given the remaining condition that $\overline{e} \leq e$ it is immediate that $a\overline{a} \leq b \to b$ or in other words $a\overline{a}b \leq b$. Putting everything together, $\overline{0} \leq a\overline{a}b \leq b$ whence $b = \overline{0}$.

It is now easy to show Fact 1: that for any finite(3) n there is only one complete totally ordered n-element **C** structure with $\overline{e} \leq e$. Proof is by induction on n. Actually there are two inductions: one for the case in which n is odd and one for the case in which it is even. Either way the base of the induction can be established by inspection, and the induction step only requires us to note that for an n-element structure of the relevant kind all fusions involving 0 or $\overline{0}$ are fixed by rigorous compactness, while all the rest are fixed by rigorous supercompactness and the induction hypothesis.

The key to Fact 2, the more general duality evident in the numbers of models is that between fusion and fission, $\circ$ and $+$. It is easy to exploit the fusion-fission duality to turn any totally ordered finite **C** structure in which $\overline{e} \leq e$ into a complete totally ordered **C** structure with one more element. The construction is to add a new top element T and to define fission on the extended structure thus:

$$a + b = \begin{cases} \mathrm{T} & \text{if } a = T \text{ or } b = T \\ a \circ b & \text{otherwise} \end{cases}$$

Obviously the operation $^-$ must be newly defined for the enlarged structure. There is only one way to do this: where the elements are numbered in their natural order from 0 up to $n = T$,

$$\overline{a} = n - a$$

(3) This regularity does not extend to the infinite case: for a completely different denumerable **C** structure consider the integers in their usual numerical order with addition as fusion and zero as the identity e.

Then fusion $\star$ is defined for the new structure

$$a \star b = \overline{\overline{a} + \overline{b}}$$

It is clear that the result of this construction is still totally ordered and has one more element than the original. To establish that it really is a **C** structure it is necessary to check out the postulates. Clearly + is commutative, and so $\star$ is as well. For associativity, note that the following are equal

$$a \star (b \star c)$$
$$\overline{\overline{a} + \overline{b \star c}}$$
$$\overline{\overline{a} + (\overline{b} + \overline{c})}$$
$$\overline{(\overline{a} + \overline{b}) + \overline{c}}$$
$$\overline{\overline{a \star b} + c}$$
$$(a \star b) \star c$$

Now where e is the identity of the old (small) structure and a is any element of the new one, $a \star \overline{e}$ is $\overline{\overline{a} \circ e}$ by definition and the fact that $^{-}$ is of period 2. $\overline{\overline{a}} \circ e$ reduces to $\overline{\overline{a}}$ which is just a. Hence there is an identity for the new structure. Lattice ordering is similarly inherited, and finally residuation holds because for any a and b

$$\overline{T} \star a \leq b$$

so the set $\{x : x \star a \leq b\}$ is nonempty whence $a \rightarrow b$ is well defined as the join over it. Since the identity of the new structure is the quasi-complement $\overline{e}$ of the old one, the old condition that $\overline{e} \leq e$ has gone over into the new condition that the structure be complete.

The above construction establishes a mapping from the totally ordered **C** structures of any finite size with $\overline{e} \leq e$ into the complete totally ordered **C** structures of the next size up. The numbers of models below the middle line are therefore at least as big as those in the top half of the previous column. To establish Fact 2, it remains to show that this mapping is in fact onto, for which it is necessary to show that it can be reversed. Consider, then, a complete totally ordered finite **C** structure. It has already been shown, as part of the rigorous supercompactness argument above, that if

$ab = 0$ then either $a = 0$ or $b = 0$ (the rest of the argument for rigorous supercompactness relies on the supposition $\overline{e} \leq e$ and fails for the general case of complete **C** structures, but we do not need it for present purposes). Dualising, $a + b = T$ if and only if either $a = T$ or $b = T$ so the result of deleting T leaves fission well defined, although it may destroy fusion. To complete the reversal of the earlier construction, $^{-}$ gets redefined in the natural way and $\circ$ is recovered:

$$a \circ b = a + b$$

Since + is fully associative and commutative these properties are inherited by $\circ$, as is the existence of an identity and respect for lattice order. The only remaining postulate is residuation, and again it suffices for this that for any a and b

$$0 \circ a = 0$$

That is, in the large structure, for every a except T

$$0 + a = 0$$

But this is exactly the property already established for complete totally ordered **C** structures. Fact 2 therefore stands proved.

5. *Conclusion*

What has been set out in this paper is the outcome of a new kind of logical research in which the main tool is a computer. The reasons for developing programs such as MaGIC had nothing to do with exploring the fission-fusion duality in **C**, and nor was it driven by any interest in bounds on the numbers of **C** models of given sizes. What did motivate the programming project was the desire for a source of models suitable for showing this or that formula to be a non-theorem of a chosen logic. Testing a formula for validity in all the **C** models with up to about 10 elements does not take very long on a computer and yields, as Meyer points out, a *de facto* decision method for **C**. The same goes for other logics in the vicinity such as the undecidable **R**. If a formula survives all attempts to refute it in some thousands of different **C** structures, the most likely explanation is that it is a theorem of **C**.

Since logics of the sort considered in this paper have no finite characteristic matrices, there will be *some* nontheorems which are nonetheless valid in all small models—indeed in the case of **R** there are some which are valid in all finite models—but empirical experience suggests that these are rare. Hence MaGIC is useful as a guide to provability. It was also realised early in the project that examination of many small models for a system of logic can give a logician some sort of "feel" for that system. An important part of research in any of the formal sciences is getting to know one's way around; one must acquire a sense of how the structures with which one deals tend to behave. For such a purpose, acquaintance with examples is invaluable.

It was only later, after efficient model-generating programs had been produced and had been run a good deal, that interest grew in the numbers of models being found and in their patterns of distribution. Regularities in the numbers of totally ordered **C** models were immediately striking and were noted as calling for explanation long before any explanatory theorem, or even conjecture, was available. So were the exponential growth rates of the numbers of models of most non-classical logics. Thus the hypotheses leading to theorems like those above were generated from the empirical data, having been totally unlooked-for at the time of production of the data.

It is quite evident that the researches so far undertaken using programs such as MaGIC have no more than skimmed the surface of a large subject. Theorems of significant generality are not proved directly by such programs, but experience thus far shows that they can be suggested by perusal of the mechanically produced low-level facts. When computers first became available for logical research in the 1970s one of our initial reactions was of joy that hard work had been banished from logic for ever. That thought turned out to have been premature at best, but at least now we can divide the labour better, passing off much of the drudgery onto machines and freeing creative logicians to create.

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