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DIALETHEISM AND PARADOXES OF THE BERRY FAMILY(¹)

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1. *Introduction.*

Dialetheism is the view that there are true contradictions, viz. dialetheias. This is a position that has been coined and, most notably, proposed by Richard Sylvan (Routley) and Graham Priest to provide a better approach towards the logical paradoxes, *i.e.* those paradoxes of self-reference that have beset (higher order) logic and semantics since the turn of the century. Priest has also suggested that dialectics requires dialetheism.

For about twenty five years now I have been absorbed in my own one-man-research-programme aiming at a *logical foundation of dialectic and speculative philosophy* with its main focus on Cantor's diagonal method. After an initial phase in which I played with something that would be called a *connexivist* logic today, I convinced myself that allowing for the provability of contradictions of the form $A \wedge \neg A$ is by no means required by the logical paradoxes. This suited my background in Hegelian philosophy where "kontradiktorische Widersprüche" and "dialektische Widersprüche" are commonly distinguished.(²) Moreover, it is essential for a theory of *positive* dialectic, *i.e.* speculative philosophy, to be able to distinguish also amongst dialectical contradictions. So I arrived at the following distinction:
"oppositional contradictions":(³) $A \wedge \neg A$.
"dialectical contradictions": $A \leftrightarrow \neg A$, $A \leftrightarrow (\neg A \rightarrow A)$, $A \leftrightarrow (\neg A \rightarrow (\neg A \rightarrow$

(¹) I struggled with myself over whether or not to make a stupid comment regarding its relation to the Adams family, but in the end I decided not to; or so I thought, at least.

(²) The point is not so much whether or not you have two kinds of contradictions, but where you draw the line. It is a matter of linguistic completeness to have a notion available on the formal level to distinguish contradictions of the kind that must be avoided and others that are allowed. In paraconsistent logic with an entailment connective $\rightarrow$ this would be something like $A \wedge (A \rightarrow \bot)$.

(³) This is a modest attempt to catch the meaning of "kontradiktorische Widersprüche" in English.

$A)), A \leftrightarrow ((\neg A \rightarrow A) \rightarrow (A \rightarrow \neg A))$, etc.(4)
In terms of this distinction my view is that all logical paradoxes fall into the second category.

On various occasions Graham Priest has claimed that there are *genuine* dialetheias amongst the logical paradoxes, *i.e.* oppositional contradictions which are provable without a detour via certain classical logical principles such as *reductio* or the law of excluded middle. The particular passages I have in mind are the following three.

> [S]et theoretic paradoxes can be produced which do not use the law of excluded middle or *reductio*. In Burali-Forti's paradox, a direct argument is given that the set of all (von Neumann) ordinals is not an ordinal, and a different argument that it is. An example with fewer technical presuppositions is Mirimanoff's paradox concerning the collection of all well founded sets.(5)

> [A]lthough no set theoretic paradox may be provable without the law of excluded middle, the case is different with the semantic paradoxes. Although some of these, such as the heterological paradox, go *via* an assertion of the form $\varphi \leftrightarrow \neg\varphi$, and hence use the law of excluded middle, the definability paradoxes, such as Berry's, Richard's and König's do not.(6)

> [T]he denial of the law of excluded middle would still not avoid dialetheism. This is for the very simple reason that there are proofs of contradictions which do not use it. Take Berry's paradox, for example[.](7)

For many years Graham and I have been discussing this issue. In the present paper I want to highlight some aspects of this discussion from my perspective. It should be clear, however, that although it is about my views it owes a lot to Graham Priest —not least because of the elimination of

(4) Given the appropriate care is taken with regard to the 'meaning' of the logical connectives.

(5) Priest [1987], pp. 36-37.

(6) Priest [1983], p. 161.

(7) Priest [1987], p. 20.

various errors and mistakes.(8)

2. *'Law of the excluded middle'*.

I want to begin with a concern regarding the phrase "law of excluded middle". I find it unclear in the following sense. On the one hand, Priest regards a detour "via an assertion of the form $\varphi \leftrightarrow \neg\varphi$" as using the 'law of excluded middle', on the other hand, the inference

$$\frac{A \leftrightarrow \neg A}{A \wedge \neg A}$$

is intuitionistically derivable; so if 'excluded middle' is needed for this inference, then intuitionistic logic does not exclude 'excluded middle'. The classic formulation of intuitionistic logic, however, is in terms of 'excluded middle' which is perfectly reflected in the Gentzen formalization *NJ*.

This seems to fit in with Bernays in Hilbert and Bernays [1939], p. 264, remarking that the principle of the excluded middle ("Satz vom ausgeschlossenen Dritten") is not necessary to derive a contradiction from the Liar sentence; what is needed, is a *reductio ad absurdum*. These two, however, are seen together in the first of the above quotations and even more so in the introductory remarks to Priest [1983]. In spite of this, the axiom system *T-W* that Priest suggests as an appropriate candidate for carrying out the proof of his formulation of Berry's paradox has $(A \rightarrow \neg A) \rightarrow \neg A$ available.(9)

This should be enough to illustrate the point of my dislike: I regard the standard of conceptual analysis as unsatisfactory; I don't think it is enough for the purpose of substantiating a claim about some 'logical principle' to just grab an axiom system and dump one of the axioms.

This is linked to questions regarding the nature of logic in general, and the point of non classical logic in particular. Although I do have a decisive view on these matters, I do not want to expound them here for fear of cluttering up the paper. Instead, I feel justified by what I regard as a general

(8) This raises the question, of course, about the responsibility for possible remaining ones and I hasten to add that Graham Priest is *not* to be blamed for them. I leave it to the reader, however, to work out whether this *not* is a paraconsistent one. Cf. also note 26 in this paper.

(9) Axiom A12 in Anderson and Belnap [1975], p. 340.

mess, to do in the context of my present paper what everybody else seems to be doing and just grab an axiom system which differs from one for classical sentential logic in that it lacks an axiom which has the form $A \vee \neg A$,([10]) to be precise, the following:

$$A \to (B \to A)$$
$$(A \to B) \to ((B \to C) \to (A \to C))$$
$$(A \to (B \to C)) \to (B \to (A \to C))$$
$$A \wedge B \to A$$
$$A \wedge B \to B$$
$$(C \to A) \to ((C \to B) \to (C \to A \wedge B))$$
$$A \to A \vee B$$
$$B \to A \vee B$$
$$(A \to C) \to ((B \to C) \to (A \vee B \to C))$$
$$(A \to B) \to (\neg B \to \neg A)$$
$$A \to \neg\neg A$$
$$\neg\neg A \to A$$

together with *modus ponens* as the only rule of inference. On the basis of this particular axiom system one obtains classical sentential logic by adding one of the following formulas as an axiom:

$$(A \to (A \to B)) \to (A \to B)$$
$$\neg(A \wedge \neg A)$$
$$A \vee \neg A$$
$$(A \to \neg A) \to \neg A$$
$$\neg(A \leftrightarrow \neg A)$$
$$A \wedge (A \to B) \to B$$

Now if you love to give names to formulas, you can call the above axiom system one that abolishes

excluded contradiction
tertium non datur (excluded middle)
reductio ad absurdum;

and if you have names for the other formulas as well you have an even

([10]) Hodges [1983], p. 32, talks of "tampering with the axioms" and seems to think that this is wonderful as an approach to non classical logic.

bigger choice. If you feel particularly witty, talk of the last formula as 'the principle of excluded middle' and point out to anyone who thinks this inappropriate that it comes down to the same thing.

Having expressed my distaste for this kind of approach to non classical logic and having just repeated it myself, I want to try, at least, to formulate my misgivings; one reason being that it will indicate in which sense the use of the above axiom system escapes my own criticism.

The formulations of all those 'logical principles' above involve theoretical constants, more specifically logical connectives. These, in turn, receive their exact specification from an axiom system through which they are 'implicitly defined' —if one doesn't want to assume their existence as independent entities in some Platonic Shrine.(11) So "tampering with the axioms" on the basis of 'logical principles' is a somewhat thoughtlessly circular business.(12) In this sense I agree with Quine's remark about non classical logicians and their opponents that "neither party knows what he is talking about"; when the deviant logician "tries to deny the doctrine he only changes the subject".(13) Such is indeed the predicament of non classical logicians who choose tampering with axioms as their approach to non classical logic, but not necessarily for those who employ Gentzen's symmetric sequential calculus. In the latter, rules governing theoretical constants are neatly separated from rules in which no theoretical constants occur. In other words, it is possible to design non classical logics in which none of the rules for the logical connectives need be touched. All that does need to be changed is the way we deal with assumptions; this is something that is incorporated in the side formulas of the operational rules and in the structural rules. No logical connective is involved in the latter; just a *geometry* of assumptions.(14)

I stop here with my tirade, not just because I have said enough to risk the argument of the dropped jaw from those who can't see the point of proof theory as "visualizing proofs", but because the catchword has been mentioned: *structural rules*.

As it happens, the above axiom system is equivalent to Gentzen's system

(11) Probably making Platonic Love and thereby creating new logical entities.

(12) To my mind, only the addition of new *truth values* can match this lack of systematics.

(13) Quine [1970], p. 81.

(14) With my apologies to Girard, *Towards a geometry of interaction*.

LK without contraction which, for reasons worked out elsewhere,([15]) I call *LD*.

This allows me to specify the question of my paper as follows: are there dialetheias in formalized theories of logic and/or semantics based on *LD*?

3. *A fixed point property.*

There is one further concern. Just as I feel uncomfortable with the way non classical logic tampers with axioms, I feel uncomfortable with the way discussions about paradoxes focus on a few hand picked examples, predominantly liars. Sure, the Liar is a paradigmatic example, but still it is only one particular instantiation of a more general schema and thus confines us to a small range of possibilities. So we have seen the advent of the Strengthened Liar, and I am waiting for the Super Liar and the Hyper Liar to make their appearances. To get over this agonizingly piecemeal approach to paradoxes, I propose to deal with a simple fixed point property for terms which comes as an almost immediate consequence of the following well known proposition.

3.1. *Proposition. Let Σ be a formalized theory which has a substitution function sub (substituting the i-th free variable by a numeral) available. To every one place nominal form $\mathfrak{Z}$ there exists a closed term f such that*

$$\Sigma \vdash f = \ulcorner \mathfrak{Z}[f] \urcorner$$

Proof. f is the term $sub(\ulcorner \mathfrak{Z}[sub(a_i, a_i)] \urcorner, \ulcorner \mathfrak{Z}[sub(a_i, a_i)] \urcorner)$. ∎

3.2. *Corollary. Let Σ be as before. To every one place nominal form $\mathfrak{Z}$ there exists a closed term ϕ such that*

$$\Sigma \vdash \phi = \mathfrak{Z}[\ulcorner \phi \urcorner]$$

Proof. Take ϕ to be the term $\mathfrak{Z}[f]$ with f according to 3.1, *i.e.* $f = \ulcorner \mathfrak{Z}[f] \urcorner$, hence $f = \ulcorner \phi \urcorner$. By substituting $\ulcorner \phi \urcorner$ for f in $\mathfrak{Z}[f]$ we obtain the required result. ∎

([15]) My [1992] typescript, and my [1980].

Thus the question I formulated at the end of the preceding section may now be reformulated: Is the fixed point property $\phi = \mathfrak{Z}[\ulcorner\phi\urcorner]$ compatible with contraction free logic *LD* enhanced by semantic predicates and/or functions?

Before I finally engage with it, I want to say a few words about the role of contraction.

4. *Logic without contraction.*

What's the point of abolishing contraction in relation to the paradoxes? The point is in fact a very simple one, but I don't know if it is at all well known: in the absence of contraction, the need for an induction on the length of the cut formula in the proof of cut elimination is confined to those cases in which inferences are involved that have more than one upper sequent, or more than one auxiliary formula in the upper sequent.(16) Gentzen's original cut elimination proof for classical first order logic(17) can be rewritten in such a way that contraction constitutes a separate case, the only one in which the induction on the length of the cut formula is required without exception. In all other cases we can escape to the criterion of a certain *height*, *i.e.* a maximum of lengths of branches, when the length of the cut formula threatens to get out of control, such as in quantification and abstraction.(18) Given a certain symmetry of rules is preserved, this result extends to a wide range of formalized theories, in particular, the ideal calculus without contraction. The latter seems to have been first realized by Grišin:

> [D]erivations of contradictions from comprehension axioms must use contraction rules or their equivalent, since it was proved in [Grišin [1974]] that the class of all comprehension axioms is consistent in logic

(16) Now I just hope that the reader will not identify contraction with the formula $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$ or a corresponding rule of inference, because my point regarding cut elimination would be wasted. The result might well be continuing attempts to derive in a logic without this kind of "contraction" a paradox involving semantic notions without paying attention to their reduction steps in a cut elimination proof.

(17) Cf., for instance, Takeuti [1975/87], §2; also with regard to the terminology I employ.

(18) As regards the situation of classical sentential logic, this is worked out in more, but still insufficient, detail in sections 21a and 21b of my [1992]. The proof of lemma 21.14 is incorrect as it stands.

without contractions.([19])

A nice example to illustrate the case is the following one in which a second order quantification inference precedes a contraction in a system of second order arithmetic with set formation (over natural numbers)

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta,\ \mathfrak{C}[P]}{\Gamma \Rightarrow \Delta,\ \forall X\, \mathfrak{C}[X]} \qquad \dfrac{\dfrac{\mathfrak{C}[\{x : \mathfrak{I}[x]\}],\ \forall X\, \mathfrak{C}[X],\ \Pi \Rightarrow \Lambda}{\forall X\, \mathfrak{C}[X],\ \forall X\, \mathfrak{C}[X],\ \Pi \Rightarrow \Lambda}}{\forall X\, \mathfrak{C}[X],\ \Pi \Rightarrow \Lambda}}{\Gamma,\ \Pi \Rightarrow \Delta,\ \Lambda}$$

The usual treatment of this case amounts to:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta,\ \mathfrak{C}[\{x : \mathfrak{I}[x]\}] \qquad \dfrac{\Gamma \Rightarrow \Delta,\ \forall X\, \mathfrak{C}[X] \qquad \mathfrak{C}[\{x : \mathfrak{I}[x]\}],\ \forall X\, \mathfrak{C}[X],\ \Pi \Rightarrow \Lambda}{\mathfrak{C},\ \mathfrak{C}[\{x : \mathfrak{I}[x]\}],\ \Pi \Rightarrow \Delta,\ \Lambda}}{\Gamma,\ \Gamma,\ \Pi \Rightarrow \Delta,\ \Delta,\ \Lambda}}{\dfrac{\text{exchanges and contractions}}{\Gamma,\ \Pi \Rightarrow \Delta,\ \Lambda}}$$

The first cut has a lower rank; the eliminability of the second depends on whether we have some measure to ensure that the cut formula has a lower grade of some kind, in order to apply an induction hypothesis.

If there is no contraction, then there is no such case like the above one, and one doesn't have to worry about the grade of a cut formula; a certain symmetry is sufficient, regardless of whatever order quantification:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \mathfrak{C}[P]}{\Gamma \Rightarrow \Delta,\ \forall X\, \mathfrak{C}[X]} \qquad \dfrac{\mathfrak{C}[\mathrm{p}], \Pi \Rightarrow \Lambda}{\forall X\, \mathfrak{C}[X], \Pi \Rightarrow \Lambda}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

This can be replaced by a cut of lower height as follows:

$$\dfrac{\Gamma \Rightarrow \Delta, \mathfrak{C}[\mathrm{p}] \qquad \mathfrak{C}[\mathrm{p}], \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

given that we can replace a free predicate variable by an arbitrary second

([19]) Grišin [1982], p. 42.

order term throughout a deduction.

These are well-known features of Gentzen style proof theory and I only mention them here to provide a background for why I consider abandoning contraction so safe a strategy against all kinds of paradoxes stemming from self reference. For the point of these paradoxes is the fixed point property, *i.e.* that a longer term may be contained, so to speak, in a shorter one. More philosophically: something is contained in a part of itself. Technically: some inferences have shorter conclusions than assumptions. But this is a feature that only causes trouble for cut elimination in the presence of contraction; or, of course, theoretical constants which somehow incorporate contraction, like 4-modality, for instance.([20]) With the eliminability of the cut rule we obtain a consistency proof in the usual way.

Another more or less well known feature of contraction free logic which deserves mention in this context is that its first order part is decidable.

I am now ready to engage with my question as formulated at the end of section 3.

5. *Dialetheias.*

As far as so-called "naive set theory" (ideal calculus) is concerned, the question is settled quickly: there are no genuine dialetheias. This is a consequence of the simple fact that the cut rule is eliminable in the ideal calculus without contraction.

There are still the so-called semantic paradoxes. Priest [1983] presents a reformulation of Berry's paradox which he claims makes no use of the law of excluded middle. In terms of my delineation above, however, it does: it employs contraction in the form of distributivity. It is, therefore, not a dialetheia in the sense delineated above. But the basic point is clear and can be formulated more shortly and, above all, without the use of a contraction. What we need is first of all the ε-operator obeying the following rule of deduction([21])

([20]) Cf. section 27d, p. 21 f, in my [1992].

([21]) I distinguish a rule of *deduction* from one of *inference* in that it must not depend on assumptions. The reason for this restriction on the usual ε-axiom will become a subject in the next section.

$$\frac{\exists x\ \mathfrak{C}[x]}{\mathfrak{C}[\varepsilon x\ \mathfrak{C}[x]]}$$

and secondly, a *den*-predicate obeying the rules (of inference)

$$\frac{s \neq t}{\neg den(\ulcorner s \urcorner, t)} \qquad \frac{s = t}{den(\ulcorner s \urcorner, t)}$$

where s must be closed.

In its simplest form we employ a fixed point $\phi = \varepsilon x \neg den(\ulcorner \phi \urcorner, x)$. We only have to find some term t such that $t \neq \phi$. What suggests itself is, of course, ϕ'. This enables us to carry out the following deduction:

$$\cfrac{\phi = \varepsilon x \neg den(\ulcorner \phi \urcorner, x) \qquad \cfrac{\cfrac{\cfrac{\phi \neq \phi'}{\neg den(\ulcorner \phi \urcorner, \phi')}}{\exists x\ \neg den(\ulcorner \phi \urcorner, x)}}{\neg den(\ulcorner \phi \urcorner, \varepsilon x \neg den(\ulcorner \phi \urcorner, x))}}{\neg den(\ulcorner \phi \urcorner, \phi)}$$

On the other hand we have:

$$\frac{\phi = \phi}{den(\ulcorner \phi \urcorner, \phi)}$$

This is it; all fat trimmed off; Berry in a nutshell.

So have we arrived at a perfect little dialetheia?

Not quite. True, nothing is involved that could by any stretch of imagination (mine, at least) be called an application of the 'law of the excluded middle'. But the above deduction has a flaw when employed to support the claim of dialetheias amongst *logical* paradoxes: it is not within *logic*; $\phi \neq \phi'$ is not a *logical* axiom.

Prima facie $\phi \neq \phi'$ is a venerable arithmetical truth. But, apart from the simple fact that I have come to distrust anything "*prima facie*" and even more anything "venerable", it is not a *logical* truth, and it is my understanding that we are concerned with logical paradoxes.([22])

([22]) The fixed point theorem looks like an arithmetical theorem too, but it can also be established by means of abstraction.

On the other hand, we are dealing with the ideal calculus without contraction, and one point of having unrestricted abstraction available is that this enables us to provide some substantial part of arithmetic; in particular we can prove $s \in N \rightarrow s \neq s'$ where N is defined in the usual way, namely as $\{x : \forall y\ (0 \in y \wedge \forall z\ (z \in z \rightarrow z' \in y) \rightarrow x \in y\}$. But this is not enough to establish a dialetheia; at best we may conclude that $\phi \notin N$.

We can try to rid ourselves of this constraint by taking to the following fixed point:

$$\varepsilon x\ (\ \forall y\ (y \neq y') \rightarrow \neg den(\ulcorner\phi\urcorner, x)) = \phi$$

This enables us to proceed as follows:

$$\cfrac{\cfrac{\cfrac{\forall y\ (y\neq y') \rightarrow \phi \neq \phi' \qquad \phi \neq \phi' \rightarrow \neg den(\ulcorner\phi\urcorner, \phi')}{\forall y\ (y\neq y') \rightarrow \neg den(\ulcorner\phi\urcorner, \phi')}}{\exists x\ (\ \forall y\ (y\neq y') \rightarrow \neg den(\ulcorner\phi\urcorner, x))}}{\cfrac{\forall y\ (y\neq y') \rightarrow \neg den(\ulcorner\phi\urcorner, \varepsilon x\ (\ \forall y\ (y\neq y') \rightarrow \neg den(\ulcorner\phi\urcorner, x))}{\forall y\ (y\neq y') \rightarrow \neg den(\ulcorner\phi\urcorner, \phi)}}$$

Now the paraconsistent logician who has entailment available —of the kind advocated in section 6.4 of Priest [1987]— gets $\exists y\ (y=y')$ with the help of $den(\ulcorner\phi\urcorner, \phi)$. In fact, we see that the reasoning does not in any way rely on the successor symbol, *i.e.* we can extend this result to any arithmetical function $\mathfrak{Z}$: $\exists y\ (y=\mathfrak{Z}[y])$.

Now this is a curious result, indeed; but, it is still not a dialetheia.

6. *Discussion.*

What we are dealing with here are paradoxes regarding the relation between *denotandum* and *denotatum* and they are not at all new. In essence, the above paradox can be found in Hilbert and Bernays [1939], pp. 271-277. I don't think it is translated, and it doesn't seem to be well known. So let me choose a couple of quotations, in my translation.

> The method of arithmetization of metamathematics was developed by Gödel for the purpose of establishing two general theorems which state the deductive incompleteness of every well defined, however not too narrow, logical-mathematical formalism.

> The idea of the proof by means of which Gödel obtained these theorems provides at the same time a method of mathematical tightening of those logical and set theoretical paradoxes in which the relation of designation and designated object plays a significant role[.]([23])
>
> A remarkable case of this kind is the impossibility of representing the notion "value of a number-determining expression" which, again, can be established given that a deductive formalism satisfies some very general conditions. [...]
>
> [...]
>
> [...] assumption: d_1) There exists a term $e(a)$, in which the number variable a is the only variable which occurs free, and which is such that, if n is the number of a term t which contains no free variable, then the equation
>
> $$e(n)=t$$
>
> is derivable[...].([24])

Now this gives rise to virtually the antinomy presented above and it can be found in Hilbert and Bernays, p. 276 f. Observe that e is *not* a number theoretic function.

By virtue of the fixed point property there exists a closed term f such that $f=\ulcorner\mathfrak{Z}[e(f)]\urcorner$. This gives us:

$$\frac{f=\ulcorner\mathfrak{Z}[e(f)]\urcorner \qquad \dfrac{\mathfrak{Z}[e(f)]=\mathfrak{Z}[e(f)]}{e(\ulcorner\mathfrak{Z}[e(f)]\urcorner)=\mathfrak{Z}[e(f)]}}{e(f)=\mathfrak{Z}[e(f)]}$$

This is a cute fixed point theorem which it takes quite some deviant style to enjoy. Needless to say, classical logicians won't like it. Take the successor function $'$, for instance, write ϕ for $e(f)$ and you get $\phi=\phi'$. Observe that it doesn't use ε-terms, nor cut, nor contraction; no negation, no implication, no 'excluded middle' is involved either.

I regard this fixed point result as some kind of a bottom line; something that has to be accepted by anyone who is committed to semantical closure, an absolute limit for theoretical constants, like the velocity of light for transmitting information.

([23]) Hilbert and Bernays [1939], p. 263.

([24]) Hilbert and Bernays [1939], p. 271 f.

Let me sketch its relevance with two examples.

Firstly, take the successor function. If you want a system in which not every numerical equation is provable you can't have $\sim(s=s')$ for some *strict* negation $\sim$,(25) *and* keep the usual arithmetical laws. For, the above fixed point property provides a term ϕ such that $\phi=\phi'$ and if you also have $\phi=y' \to 0=0'$ you get every numerical equation with the help of the recursive equations for addition and multiplication and the properties of $=$.

Dialetheists will want to retain $\neg(s=s')$ for some other form of negation $\neg$, one which is compatible with $s=s'$.(26) A negation of this kind may be convenient in certain situations,(27) but can it take the place of a strict negation? Not quite, it seems, for in chapter 6 of Priest [1987], two notions of entailment are introduced each of which, in turn, provides a notion of *strict* negation for which inconsistency cannot be admitted on pain of triviality.

Secondly, take the ε-operator. An ε-axiom in the form

$$\exists x\, \mathfrak{C}[x] \to \mathfrak{C}[\varepsilon x\, \mathfrak{C}[x]]\ ,$$

where $\to$ is again an entailment connective of the kind in section 6.4 in Priest [1987], cannot be retained. The reason will become clear from the following variation on our theme:

$$\dfrac{\dfrac{\dfrac{\dfrac{t\neq\phi \to \neg den(\ulcorner\phi\urcorner, t)}{\exists x\,(t\neq\phi \to \neg den(\ulcorner\phi\urcorner, x))}}{t\neq\phi \to \exists x\, \neg den(\ulcorner\phi\urcorner, x)} \quad \exists x\, \neg den(\ulcorner\phi\urcorner, x) \to \neg den(\ulcorner\phi\urcorner, \varepsilon x\, \neg den(\ulcorner\phi\urcorner, x))}{t\neq\phi \to \neg den(\ulcorner\phi\urcorner, \varepsilon x\, \neg den(\ulcorner\phi\urcorner, x))}}{t\neq\phi \to \neg den(\ulcorner\phi\urcorner, \phi)}$$

With $den(\ulcorner\phi\urcorner, \phi)$ we obtain $t=\phi$, *i.e.* in particular for 0 and 1, hence $0=1$

(25) For instance, $\sim A$ being defined as $A \to 0=1$ where $\to$ is something like Priest's entailment connective.

(26) Of course, the eminent philosophical question here is, which is the negation and which is the *other* one. (I am not sure if it is necessary to point out that this is meant sarcastically; but I am pretty sure that quite a few people will have a jolly good time discussing this question.)

(27) At this point I have to bite my tongue not to make a comment on alleged situations "When *No* doesn't mean *No*".

by transitivity.

The dialetheist who adheres to a notion of entailment like that in section 6.4 in Priest [1987] mentioned above will reject this formulation of the ε-axiom on the following grounds: it may be the case that $1 \in j_G(\exists x\, \mathfrak{C}[x])$, $0 \notin j_G(\exists x\, \mathfrak{C}[x])$, $1 \in j_G(\mathfrak{C}[\varepsilon x\, \mathfrak{C}[x]])$, and $0 \in j_G(\mathfrak{C}[\varepsilon x\, \mathfrak{C}[x]])$, for instance, when there are objects s and t such that $1 \in j_G(\mathfrak{C}[s])$, $0 \notin j_G(\mathfrak{C}[s])$, $1 \in j_G(\mathfrak{C}[t])$, and $0 \in j_G(\mathfrak{C}[t])$, with t being the choice of the ε-operator, a situation which renders the ε-axiom false.(28) One strategy is to restrict the ε-axiom to a rule of deduction as done in the preceding section or to escape to a non-contraposible notion of entailment as the one discussed in section 6.5 of Priest [1987]. In either case the possibility of substituting quantification, Hilbert's original motivation for introducing the ε-operator, is jeopardized.

Another way of looking at this counter example suggests that you can't have a choice operator which is restricted to objects for which the characterizing property is true only.

What I am trying to say is, it all depends on what you're after.

Dialetheists are after dialetheias, and with an appropriate mixture of tools like contraction, the restricted ε-operator and the e-function, there are quite amazing things to prove. Consider the following example:

$$\dfrac{\dfrac{\Rightarrow \varepsilon x\,(e(\ulcorner\phi\urcorner)\neq x)=\phi}{\Rightarrow e(\ulcorner\phi\urcorner)=\phi} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{0=0' \Rightarrow}{e(\ulcorner\phi\urcorner)=0,\ e(\ulcorner\phi\urcorner)=0' \Rightarrow}}{\Rightarrow e(\ulcorner\phi\urcorner)\neq 0,\ e(\ulcorner\phi\urcorner)\neq 0'}}{\Rightarrow \exists x\,(e(\ulcorner\phi\urcorner)\neq x),\ \exists x\,(e(\ulcorner\phi\urcorner)\neq x)}}{\Rightarrow \exists x\,(e(\ulcorner\phi\urcorner)\neq x)}}{\Rightarrow \varepsilon x\,(e(\ulcorner\phi\urcorner)\neq x)\neq e(\ulcorner\phi\urcorner)}}{\Rightarrow e(\ulcorner\phi\urcorner)\neq\phi}}{\Rightarrow \phi\neq\phi}$$

This is only something for the paraconsistent logician to enjoy since it involves contraction.

On the other hand, there is still the question whether dialetheists who indulge in this kind of extravagance are safe from results such as $0=1$ above.

There is reason to be cautious. The denotation predicate is unique; it goes

(28) Cf. the counter example in Priest [1991], p. 363, fn. 4, for the contraposition of the ε-axiom.

with definite description; but what about *indefinite* description such as the ε-operator?(29) Are there logical principles which provoke a clash between with the non-unique character of ε-terms on the one hand and the characteristics of the denotation predicate on the other?

Applying classical first order logic to the denotation axiom

$$den(\ulcorner s \urcorner, t) \to s = t$$

provided in Priest [1983], p. 162, (4), yields

$$\exists x\, \forall y\, (den(\ulcorner s \urcorner, x) \to x = y)$$

for every closed term s. This can be regarded as expressing the uniqueness of the denotation relation; it is quite disastrous in conjunction with ε-terms, even with the ε-rule restricted to a rule of deduction, as can be seen immediately employing the following fixed point:

$$\phi = \varepsilon x\, \forall y\, (den(\ulcorner \phi \urcorner, x) \to x = y)$$

The above formula enables us to proceed as follows:

$$\dfrac{den(\ulcorner \phi \urcorner, \phi) \qquad \dfrac{\dfrac{\exists x\, \forall y\, (den(\ulcorner \phi \urcorner, x) \to x = y)}{\forall y\, (den(\ulcorner \phi \urcorner, \phi) \to \phi = y)}}{den(\ulcorner \phi \urcorner, \phi) \to \phi = 0}}{\phi = 0}$$

Analogously, we obtain $\phi = 1$; transitivity yields $0=1$.

Paraconsistent logic must be safe from this or similar intrusions of classical principles. A look at a somewhat normalized deduction in the

(29) Before you think about avoiding possible trouble linked to the indefinite description operator ε by taking refuge in a definite description operator such as μ on the basis of the least number principle as in Priest [1987]: be aware that the usual proofs of it require contraction. (See, for instance, Hilbert and Bernays [1934], p. 284 f; the crucial step is from the bottom of p. 284 to the top of p. 285 which amounts to $A \vee B \to A \vee (B \wedge \neg A)$.) In the context of the present paper, this means two things: first of all, as a contraction abstinent logician I will not accept oppositional contradictions based on the use of the least number principle as genuine dialetheias as long as I am not shown a way to prove the least number principle without contraction; secondly, I think it is likely that the kind of problem hinted at in the remainder of the paper is only shifted somewhere else.

classical sequential calculus of $\exists x\, \forall y\, (den(\ulcorner s \urcorner, x) \rightarrow x = y)$ is indeed reassuring:

$$\begin{array}{c}
den(\ulcorner s \urcorner, a) \Rightarrow s = a \\ \hline
den(\ulcorner s \urcorner, a) \Rightarrow s = a, a = b \\ \hline
\Rightarrow s = a, den(\ulcorner s \urcorner, a) \rightarrow a = b \\ \hline
\Rightarrow s = a, \forall y\, (den(\ulcorner s \urcorner, a) \rightarrow a = y) \\ \hline
\Rightarrow s = a, \exists x\, \forall y\, (den(\ulcorner s \urcorner, x) \rightarrow x = y) \\ \hline
\Rightarrow \exists x\, \forall y\, (den(\ulcorner s \urcorner, x) \rightarrow x = y), s = a \\ \hline
den(\ulcorner s \urcorner, s) \Rightarrow \exists x\, \forall y\, (den(\ulcorner s \urcorner, x) \rightarrow x = y), s = a \\ \hline
\Rightarrow \exists x\, \forall y\, (den(\ulcorner s \urcorner, x) \rightarrow x = y), den(\ulcorner s \urcorner, s) \rightarrow s = a \\ \hline
\Rightarrow \exists x\, \forall y\, (den(\ulcorner s \urcorner, x) \rightarrow x = y), \forall y\, (den(\ulcorner s \urcorner, s) \rightarrow s = y) \\ \hline
\Rightarrow \exists x\, \forall y\, (den(\ulcorner s \urcorner, x) \rightarrow x = y), \exists x\, \forall y\, (den(\ulcorner s \urcorner, x) \rightarrow x = y) \\ \hline
\Rightarrow \exists x\, \forall y\, (den(\ulcorner s \urcorner, x) \rightarrow x = y)
\end{array}$$

This deduction breaks down for Priest's entailment connective $\rightarrow$, the reason being that the inferences which put together the $\rightarrow$-formula in the early parts of the deduction are not available for Priest's entailment connective. But who knows how many more possibilities there are? Without a non-triviality proof dealing with ε-terms in conjunction with the denotation predicate remains a somewhat risky business.

7. *Summary*.

The situation regarding Priest's claim that there are genuine dialetheias remains curiously undecided. Paraconsistent logicians can produce results which satisfy their dialetheist ambitions; but apparently only with the help of methods which are not part of logic proper and/or employ contraction, and, therefore, the 'law of the excluded middle' in some sense. This means that the claim that there are genuine dialetheias amongst the logical paradoxes has not yet been established; but it also means that if you open yourself up to the possibility of dialetheias, by using paraconsistent logic with ε-terms and some semantical tools, you find some extraordinary specimens, such as a term which equals itself and at the same time differs from itself,(30) without apparent danger of running into triviality; something a contraction abstinent logician has to, and probably wants to,

(30) The reader might feel reminded of Hegel's "unity of identity and difference"

stay away from. An unsettled problem for the paraconsistent logician is that although there is no apparent threat of triviality, there is still the danger of results such as $0 = 1$ when dealing with ε-terms in conjunction with semantic tools.

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