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ON THE EXPRESSIBILITY OF PROPOSITIONS

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Abstract

In possible world semantics *propositions* are defined as sets of possible worlds. A proposition P is said to be *expressible* in a formal language L iff there exists a set of formulas Γ of L such that the possible worlds satisfying Γ are precisely those contained in P. It is well known that not every proposition is expressible in a given language L; in other words there exists a gap of expressibility between propositions and formulas. As shown herein, this gap can cause problems in modelling belief dynamics. Motivated by these problems in this article we study the expressibility of propositions. More precisely we investigate conditions under which the expressibility gap reaches its lower bound, and we show that even under these conditions there exist infinitely many propositions that are not expressible in a given language L, unless L contains only finitely many logical equivalence classes.

Keywords: Formal Logic, Belief Change, Knowledge Representation.

1. *Introduction*

In recent years the characterization of *propositions* that has been most popular among philosophers is the one adopted in possible worlds semantics, namely to identify propositions with sets of possible worlds. The basic idea is to identify a proposition P with the set of all possible worlds where P is true. For example, the proposition "Alexander the Great was Greek" is defined as the set of all possible worlds at which Alexander the Great was indeed Greek. As noted by Lewis[6], "... The idea goes back at least to Clarence I. Lewis[5], in which the set of worlds is called the 'comprehension' of the proposition; and to Rudolf Carnap[1], in which propositions are taken as sets of state descriptions, and state descriptions are said to represent Leibniz' possible worlds or Wittgenstein's possible states of affairs".

Sets of possible worlds can also be assigned to formulas of a given formal language L. More precisely, to any formula $\varphi \in L$ we assign the set of all possible worlds where φ is true. We denote this set by $[\varphi]$. Being a set of possible worlds, $[\varphi]$ is a proposition.(1) We call $[\varphi]$ the proposition *expressed* by the formula φ. Generalizing these definitions, we denote the set of all possible worlds satisfying a set of formulas Γ, by $[\Gamma]$, and we call this set the proposition expressed by Γ. We shall say that a proposition P is *expressible* in a formal language L, iff there exists a subset Γ of L, such that P is expressed by Γ.(2)

It is well known that not every proposition is expressible in a given language L; in other words, there exists a gap of expressibility between propositions and (sets of) formulas. In this article we investigate conditions under which this expressibility gap reaches its "lower bound", and we provide a characterization of the languages for which the gap is eliminated at its lower end. The study was prompted by the anomalies that the expressibility gap causes in modelling *belief dynamics*, anomalies which however have been largely neglected by workers in the areas. To motivate the forthcoming discussion, we shall briefly discuss these problems before presenting our results.

2. *Problems in Modelling Belief Dynamics*

In this section the concept at focus is the process by which a rational agent, call him Orpheus, changes his beliefs about the current state of a *dynamic world*, in response to the occurrence of events. The familiar blocks world [2], is a typical example of a dynamic world, and events such as "stack block A on block B" are the kind of events we consider herein. Formalizing the notion of a dynamic world in its full generality is beyond the scope of this article (refer to [8] for such a formalization). For our purposes it suffices to work within the simple framework described below.

We consider a dynamic world which at any given time point is at a particular *world state*. As time progresses the world changes states by the occurrence of *events*. We denote by W and E, respectively, the set of all

(1) We assume that, not only every proposition is defined as a set of possible worlds, but also that every set of possible worlds defines a proposition.

(2) Notice that according to our definition of expressibility, if a proposition P is not expressible in a language L, then there is no finite or even *infinite* set of formulas of L that expresses P.

world states, and the set of all events, of the dynamic world under consideration. For an event $e \in E$ and a world state $w \in W$, we denote by $\mathfrak{R}(e,w)$ the world state resulting from the occurrence of e at w. We assume that the beliefs of our agent Orpheus about the world, are represented as formulas of an object language L. For simplicity let us assume that L is a propositional language and moreover that Orpheus is deductively omniscient (he knows all the logical consequences of his beliefs), and therefore his belief state at any time point is represented by a theory of L.(3) Since the world states in W are the entities where the truth value of the formulas of L is evaluated, we shall temporarily (only for this section), identify world states with possible worlds. Consequently sets of world states will often be treated as propositions, and for any $\Gamma \subseteq$ L, $[\Gamma]$ now denotes the set of all worlds states where Γ is true.

Having set up the stage, let us now consider the process by which Orpheus changes his beliefs about the current world state, due to the occurrence of events. More precisely, suppose that Orpheus' beliefs about the current world state is represented by the theory T, and at this point the event e occurs causing the world to change states. Responding to the occurrence of e, Orpheus needs to update his beliefs, changing from T to a new belief state which we denote by $\mathfrak{C}(e,T)$. We are interested in the reasoning process that Orpheus uses to arrive at the new belief state $\mathfrak{C}(e,T)$.

Firstly we observe that, given that Orpheus' current belief state is T, any world state satisfying T is equally likely, as far as Orpheus is concerned, to be the current world state. More precisely, suppose that $[T] = \{w_1, w_2, w_3, ...\}$. Then for all that Orpheus knows the current world state could be any of $w_1, w_2, w_3, ...$ This being the case, Orpheus presumably reasons as follows about the state resulting from the occurrence of e (see [10], [4], [8], [9]): "For all I know, the current world state could be any of $w_1, w_2, w_3, ...$ If the current state is w_1 then the resulting state is $\mathfrak{R}(e,w_1)$, if the current state is w_2 then the resulting state is $\mathfrak{R}(e,w_2)$, if the current state is w_3 then the resulting state is $\mathfrak{R}(e,w_3)$, ... However, as I am uncertain about the current world state, all I can say about the resulting state is that it will be one of the states $\mathfrak{R}(e,w_1)$, $\mathfrak{R}(e,w_2)$, $\mathfrak{R}(e,w_3)$, ...". Let us denote the set $\{\mathfrak{R}(e,w_1), \mathfrak{R}(e,w_2), \mathfrak{R}(e,w_3), ...\}$ by $\mathfrak{R}(e, [T])$, i.e. $\mathfrak{R}(e, [T]) = \{\mathfrak{R}(e,w_i): w_i \in [T]\}$. According to the above line of reasoning, $\mathfrak{R}(e,[T])$ is the set of all world states compatible with Orpheus' beliefs about the state resulting

(3) A theory T of L is any subset of L closed under logical entailment.

from the occurrence of e. Therefore the new belief state $\mathfrak{C}(e,T)$ should be such that the world states compatible with $\mathfrak{C}(e,T)$ are precisely the elements of $\mathfrak{R}(e,[T])$. This gives us the following implicit definition of $\mathfrak{C}(e,T)$:

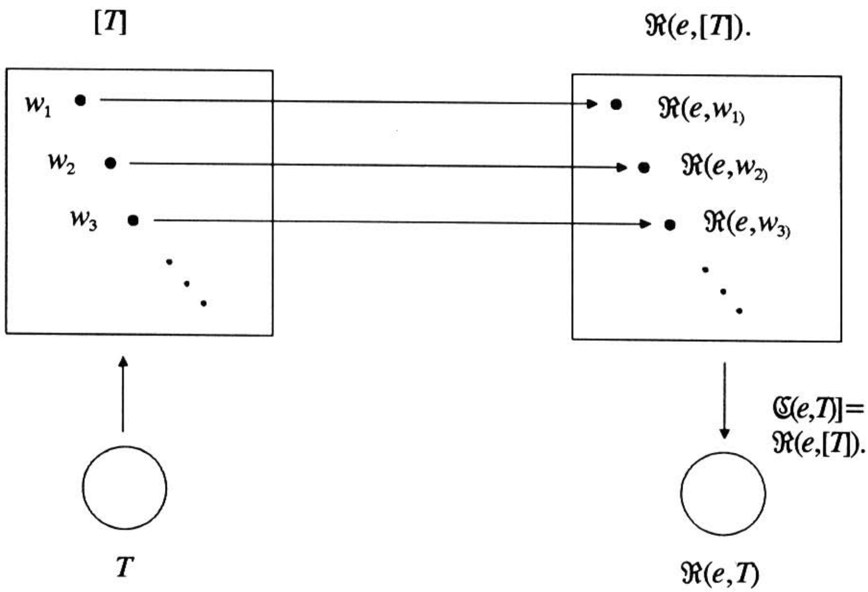

Figure 1: Updating Belief States

(𝒫ℜ) $[\mathfrak{C}(e,T)] = \mathfrak{R}(e,[T])$.

Notice that the new belief state $\mathfrak{C}(e,T)$ is not defined directly from the old belief state T, but instead the transition from T to $\mathfrak{C}(e,T)$ occurs in three steps: First one moves from T to $[T]$, then from $[T]$ to $\mathfrak{R}(e,[T])$, and finally $\mathfrak{C}(e,T)$ is defined from $\mathfrak{R}(e,[T])$ in terms of the condition (𝒫ℜ) (Figure 1) This last step however from $\mathfrak{R}(e,[T])$ to $\mathfrak{C}(e,T)$ is problematic. We have no guarantee that there exists a theory $\mathfrak{C}(e,T)$ of L such that $[\mathfrak{C}(e,T)] = \mathfrak{R}(e,[T])$. Indeed, taking world states to be possible worlds, condition (𝒫ℜ) defines $\mathfrak{C}(e,T)$ as the theory of L that expresses the proposition $\mathfrak{R}(e,[T])$. However, due to the gap of expressibility that occurs between propositions and formulas, it is not safe to assume that such a theory always exists.

This problem with belief dynamics has not received the appropriate attention by workers in the area. This is mainly because the definition commonly adopted for $\mathfrak{C}(e,T)$, is slightly different from the one given above, yet different enough to avoid the problem caused by the expressibility gap. More precisely, according to this second definition, the new belief state $\mathfrak{C}(e,T)$ is no longer defined as the theory of L that expresses $\mathfrak{R}(e,[T])$, but rather as the theory of L consisting of all formulas that are true at every world state in $\mathfrak{R}(e,[T])$ [3], [10], [4]. Let us denote by Th(w) the set of all formulas that are true at a world state w, i.e. $\mathrm{Th}(w) = \{\varphi \in L: w \models \varphi\}$. Then the second definition of $\mathfrak{C}(e,T)$ is given by the following condition:

$$(\mathcal{PR})' \qquad \mathfrak{C}(e,T) = \bigcap_{w \in \mathfrak{R}(e,[T])} \mathrm{Th}(w).$$

We shall refer to the first and second definition of $\mathfrak{C}(e,T)$, as the *extensional* and *intensional* definition of belief dynamics respectively. It is not hard to verify that when $\mathfrak{R}(e,[T])$ is expressible in L, the two definitions give rise to precisely the same theory $\mathfrak{C}(e,T)$. Yet, when $\mathfrak{R}(e,[T])$ is not expressible in L, the two definitions clearly differ: The former collapses under the inexpressibility of $\mathfrak{R}(e,[T])$, while the later gives us the theory $\bigcap_{w \in \mathfrak{R}(e,[T])} \mathrm{Th}(w)$. This theory however represents only part of what Orpheus can infer about the state resulting from the occurrence of e (namely the part that can be expressed by formulas of L). Indeed, consider the proposition $P = [\bigcap_{w \in \mathfrak{R}(e,[T])} \mathrm{Th}(w)]$ expressed by the theory we derive from the intensional definition of belief dynamics. Clearly, $\mathfrak{R}(e,[T])$ is a subset of P, and given that $\mathfrak{R}(e,[T])$ is not expressible in L, it follows that $\mathfrak{R}(e,[T])$ is a *proper* subset of P, which in turn makes P a proposition strictly weaker from $\mathfrak{R}(e,[T])$. Therefore, while Orpheus is capable, based on the reasoning process described above, to confine the set of candidate resulting states to $\mathfrak{R}(e,[T])$, the belief state $\bigcap_{w \in \mathfrak{R}(e,[T])} \mathrm{Th}(w)$ ascribed to Orpheus by the intensional definition of belief dynamics, allows further world states as candidates (namely all states in $P - \mathfrak{R}(e,[T])$), thus undermining Orpheus' true capabilities. In other words, there is some loss of information when Orpheus' conclusions about the state resulting from e, are expressed as a theory of L according to the intensional definition of belief dynamics. This loss of information is perhaps best illustrated by the scenario described below.

Continuing with our example, let us assume that following e, another event e' occurs. Based on the reasoning process described above we conclude that Orpheus' inferences about the state resulting from e' are described by the proposition $\mathfrak{R}(e',\mathfrak{R}(e,[T]))$ (Figure 2). Let us assume that both the proposition $\mathfrak{R}(e',\mathfrak{R}(e,[T]))$ and $\mathfrak{R}(e',[\mathfrak{C}(e,T)])$ are expressible in L, and let T' be the theory of L that expresses the former proposition, i.e. $[T'] = \mathfrak{R}(e',\mathfrak{R}(e,[T]))$. Let us also assume that e' is such that for any two world states w, w', $\mathfrak{R}(e',w) \neq \mathfrak{R}(e',w')$ whenever $w \neq w'$. Then given that $\mathfrak{R}(e,[T])$ is not expressible in L, it is not hard to verify that T' is a proper superset of the theory $\mathfrak{C}(e', \mathfrak{C}(e,T)) = \bigcap_{w \in \mathfrak{R}(e',[\mathfrak{C}(e,T)])} \mathrm{Th}(w)$, at which we arrive following the intensional definition of belief dynamics. All the formulas in $T' - \mathfrak{C}(e', \mathfrak{C}(e,T))$ represent facts about the state resulting from e', that Orpheus is capable of deducing, which however have been lost in the process of representing propositions as theories of L.

Summarizing the discussion of this section we make the following observations. When belief states are represented by theories of a formal language L, the gap of expressibility that exists between propositions and formulas, makes problematic the modelling of belief dynamics. In particular, the extensional definition of belief dynamics collapses whenever it faces an inexpressible proposition, while the intensional definition (which is the one most commonly used) is only an approximation to this process, capturing progressively less and less about the state of the world, as more and more inferences vanish into the expressibility gap.

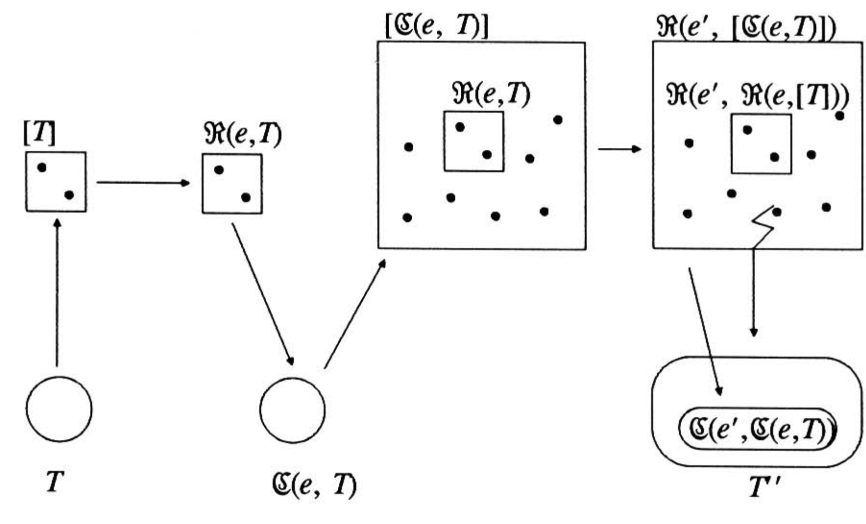

Figure 2: Loss of Information due to Lack of Expressibility

These anomalies in belief dynamics motivated a study on the expressibility of propositions, the results of which are reported in this article. Before presenting our results however, we need to fix some notation and terminology.

3. *Preliminaries*

We define a *standard logic* to be an ordered pair $\langle L, \vdash \rangle$, where L is a nonempty countable formal language closed under the propositional connectives,(4) and $\vdash$ is a consequence relation defined over L, that satisfies the following conditions:

(i) If φ is a truth-functional tautology, then $\vdash \varphi$.
(ii) If $\vdash \varphi \rightarrow \psi$ and $\vdash \varphi$, then $\vdash \psi$ (modus ponens).
(iii) $\vdash$ is consistent, that is, there exists a $\varphi \in L$ such that $\nvdash \varphi$.
(iv) $\vdash$ satisfies the Deduction Theorem, that is, $\{\varphi_1, \varphi_2, \ldots, \varphi_n\} \vdash \psi$ iff $\vdash (\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n) \rightarrow \psi$.
(v) $\vdash$ is compact.

Let $\langle L, \vdash \rangle$ be a standard logic, and let Γ be a subset of L. We denote by $Cn(\Gamma)$ the closure of Γ under $\vdash$, i.e. $Cn(\Gamma) = \{\varphi \in L: \Gamma \vdash \varphi\}$. We shall say that Γ is *consistent* iff $Cn(\Gamma) \neq L$. Often we shall also refer to a formula $\varphi \in L$ as being consistent, meaning that the set $\{\varphi\}$ is consistent. A *theory* T of L is any subset of L closed under $\vdash$, i.e. $T = Cn(T)$. A theory K of L is *complete* iff for every formula $\varphi \in L$, $\varphi \in K$ or $\neg\varphi \in K$. We shall denote the set of all consistent complete theories of L by K_L. For a set of formulas Γ of L, we shall say that T is an *extension* of Γ iff T is a theory of L that includes Γ, i.e. $\Gamma \subseteq T = Cn(T)$. If for an extension T of Γ, $Cn(\Gamma)$ is a proper subset of T, then we shall say that T is a *proper extension* of Γ. If an extension K of Γ is a consistent complete theory of L, we shall say that K is a *consistent complete extension* of Γ. Notice that from the assumptions we made for standard logics, it follows (from Zorn's Lemma) that every consistent set of formulas $\Gamma \subseteq L$ has a consistent complete extension. This is a property that we shall use quite often in our proofs.

(4) In other words, if $\varphi, \psi \in L$ then $\neg\varphi$, $\varphi \vee \psi$, $\varphi \wedge \psi$, $\varphi \rightarrow \psi$, $\varphi \leftrightarrow \psi$ are also formulas of L.

4. *Assessing the Expressibility Gap*

With the definitions of the previous section we are now ready to proceed with our study on the expressibility of propositions. The results presented herein apply to the entire range of standard logics.

Given an arbitrary standard logic $\langle L, \vdash \rangle$, the general question we shall be addressing is whether there exist propositions that are not expressible in L, and if so, "how many" are they. However before we can answer this question we need to identify the "domain" of propositions; in other words we need to specify the set W of all possible worlds from which the propositions in question are formed.

It does not much matter what possible worlds are, as long as they either verify or falsify any formula of L, and there are enough of them so that the relation between $\vdash$ and the set W of all possible worlds, satisfies certain desirable logical properties (e.g. soundness and completeness of $\vdash$ with respect to W). More precisely, we will assume that the set W of all possible worlds associated with a standard logic $\langle L, \vdash \rangle$, satisfies the following two conditions:

(i) For every formula $\varphi \in L$, and any possible world $w \in W$, either $w \models \varphi$ or $w \models \neg\varphi$.
(ii) For any set of formulas $\Gamma \subseteq L$ and any formula $\varphi \in L$, $\Gamma \vdash \varphi$ iff for all $w \in W$, $w \models \Gamma$ entails $w \models \varphi$.

We shall call a set of possible worlds W that satisfies the above two conditions, a *universe* for the logic $\langle L, \vdash \rangle$. Our results on the expressibility of propositions in L will be stated relative to some universe for $\langle L, \vdash \rangle$.

Notice that the conditions (i) and (ii) do not exclude the case of two distinct possible worlds verifying precisely the same set of formulas. Consequently it is not hard to see that for every standard logic $\langle L, \vdash \rangle$, there exist universes for $\langle L, \vdash \rangle$, with arbitrarily large cardinality. Moreover, the larger the universe, the more the propositions that can be formed from it, the smaller the proportion of the propositions that are expressible in L. Consequently, for any given standard logic $\langle L, \vdash \rangle$, the expressibility gap can became arbitrarily large as it grows with the size of the chosen universe.

Since there is not much to say about how large the expressibility gap can be, we shall instead focus on its "lower boundary", analysing how small this gap can get. Clearly, for any standard logic $\langle L, \vdash \rangle$, the expressibility gap reaches its lower end when the universe W out of which propositions are formed, is minimal, in the sense that no proper subset of it is a universe for $\langle L, \vdash \rangle$. It is not hard to verify that a universe W for $\langle L, \vdash \rangle$ is minimal, iff for every consistent complete theory K of L, there exists precisely one member w of W such that $K = \text{Th}(w)$.(5) From this it follows that for each minimal universe W, Th is a one-to-one correspondence between W and the set K_L of consistent complete theories of L. This again implies that all minimal universes of $\langle L, \vdash \rangle$ are isomorphic with respect to $\models$. Therefore the expressibility gap is the same (subject to isomorphism) across all minimal universes for $\langle L, \vdash \rangle$. This facilitates our study on the lower boundary of the expressibility gap since for any given standard logic $\langle L, \vdash \rangle$, we only need to analyse the expressibility of propositions at one representative minimal universe W. Given the one-to-one correspondence between any minimal universe and K_L, for the sake of simplicity we shall identify the representative minimal universe W with K_L, identifying each possible world $w \in W$ with the theory Th(w) that it satisfies. Under this convention a proposition is now taken to be a set of consistent complete theories of L, while for a set of formulas $\Gamma \subseteq L$, $[\Gamma]$ denotes the set of all consistent complete theories of L that include Γ. This convention will allow us to detach our study from semantical considerations and evaluate the expressibility of propositions in purely syntactic terms.

It turns out that the lower boundary of the expressibility gap varies from standard logic to standard logic, and for some logics the gap is in fact eliminated. We shall call such logics *intensionally strong*. More precisely, we shall say that a standard logic $\langle L, \vdash \rangle$ is intensionally strong iff every proposition P in 2^{K_L} (the powerset of K_L) is expressible in L.(6) If a standard logic is not intensionally strong, we shall say that the logic is *intensionally weak*. The major objective of this article is twofold. Firstly, to provide a characterization of the class of intensionally weak logics. Secondly, for every intensionally weak logic, to evaluate the "size" of the expressibility gap at its lower boundary.

As a first step to this end consider Lemma 4.1 presented below. The notion of a consistent proposition that appears in this lemma is defined quite simply as a nonempty proposition.

(5) Recall that for a possible world w, Th(w) denotes the set of all formulas in L that are true at w, i.e. $\text{Th}(w) = \{\varphi \in L: w \models \varphi\}$.

(6) Notice that in the present context, a proposition $P \in 2^{K_L}$ is expressible in L iff there exists a set Γ of formulas of L, such that the consistent complete extensions of Γ are precisely the elements of P.

Lemma 4.1 Let $\langle \mathrm{L}, \vdash \rangle$ *be a standard logic, and let* $P \in 2^{K_{\mathrm{L}}}$ *be a consistent proposition. Then P is expressible in* L *iff* $P = [\cap P]$.(⁷)

The proof of Lemma 4.1 is straightforward and we shall not include it here. Lemma 4.1 simply says that given a proposition P in $2^{K_{\mathrm{L}}}$, if any set of formulas of L can express P, then $\cap P$ is such a set. Given Lemma 4.1 it is not hard to verify that a standard logic $\langle \mathrm{L}, \vdash \rangle$ is intensionally weak iff there exists a theory T of L such that for two distinct propositions P and P' in 2^{K_L}, $\cap P = \cap P' = T$. We call such a theory *ambiguous*. More precisely, let $\langle \mathrm{L}, \vdash \rangle$ be a standard logic, and let T be a theory of L. We define $[\![T]\!]$ to be the set of all propositions P in $2^{K_{\mathrm{L}}}$ such that $\cap P = T$, i.e. $[\![T]\!] = \{ P \in 2^{K_{\mathrm{L}}} : \cap P = T \}$.(⁸) We shall say that a theory of L is ambiguous iff $[\![T]\!]$ contains more than one element. If T is not ambiguous, we shall say that T is *specific*. Then, as mentioned above, from Lemma 4.1 we derive the following corollary.

Corollary 4.1 Let $\langle \mathrm{L}, \vdash \rangle$ *be a standard logic. Then* $\langle \mathrm{L}, \vdash \rangle$ *is intensionally weak iff there exists a theory of* L *that is ambiguous.*

Having reduced intensional weakness to the existence of ambiguous theories, we shall continue our expedition seeking a characterization of ambiguous theories. The following two lemmas will be very useful in our journey.

Lemma 4.2 Let $\langle \mathrm{L}, \vdash \rangle$ *be a standard logic and let T be a consistent theory of* L. *Then* $[T] \in [\![T]\!]$ *and for every* $P \in [\![T]\!]$, $P \subseteq [T]$, *i.e.* $[T]$ *is the maximum element in* $[\![T]\!]$ *with respect to set inclusion.*

Proof.

Part I: $[T] \in [\![T]\!]$.
Clearly $T \subseteq \cap [T]$. We show that the converse is also true. Let φ be a formula of L such that $\varphi \notin T$. Then $\neg\varphi$ is consistent with T and therefore there exists a consistent complete theory K of L such that $T \cup \{\neg\varphi\} \subseteq K$.

(⁷) Notice that since P is a nonempty set of consistent complete theories of L, $\cap P$ is a theory of L.

(⁸) Notice the difference between $[T]$ and $[\![T]\!]$: The former is a proposition (a set of consistent complete theories), while the later is a collection of propositions (a collection of sets of consistent complete theories). In fact it is not hard to verify that $[T] \in [\![T]\!]$.

Then $K \in [T]$ and $\varphi \notin K$, from which we derive that $\varphi \notin \cap [T]$.

Part II: For every $P \in [\![T]\!]$, $P \subseteq [T]$.
Let P be any element of $[\![T]\!]$. Then $\cap P = T$ and therefore every element of P is a consistent complete extension of T. On the other hand $[T]$ is by definition the set of all consistent complete extensions of T and therefore $P \subseteq [T]$. ■

Lemma 4.3 Let $\langle \mathrm{L}, \vdash \rangle$ *be a standard logic,* T *a consistent theory of* L, *and* P *a subset of* $[T]$. *Then* $P \in [\![T]\!]$ *iff for every* $\varphi \in \mathrm{L}$ *such that* $\varphi \notin T$, *there exists an element* K *of* P *containing* $\neg\varphi$.

Proof.

Part I: If $P \in [\![T]\!]$ *then for every* $\varphi \in \mathrm{L}$ *such that* $\varphi \notin T$, *there exists an element* K *of* P *containing* $\neg\varphi$.

Assume that $P \in [\![T]\!]$. Since T is consistent, $P \neq \varnothing$. Moreover let $\varphi \in$ L be such that $\varphi \notin T$. Then $\varphi \notin \cap P$ and therefore there exists an element K of P not containing φ. Since all the elements of P are consistent complete theories it follows that $\neg\varphi \in K$.

Part II: If for every $\varphi \in \mathrm{L}$ *such that* $\varphi \notin T$, *there exists an element* K *of* P *containing* $\neg\varphi$, *then* $P \in [\![T]\!]$.

Assume that for every $\varphi \in \mathrm{L}$ such that $\varphi \notin T$, there exists an element K of P containing $\neg\varphi$. Since T is consistent, there exists a $\varphi \in \mathrm{L}$ such that $\varphi \notin T$. Therefore by the above assumption there exists an element K of P that contains $\neg\varphi$. Consequently $P \neq \varnothing$. Moreover, since $P \subseteq [T]$, every element of P includes T, and therefore $T \subseteq \cap P$. We show that the converse is also true. Let φ be a formula of L such that $\varphi \notin T$. Then there exists a $K \in P$ such that $\neg\varphi \in K$ and since the elements of P are consistent complete theories, $\varphi \notin K$ and consequently $\varphi \notin \cap P$. This proves that $T = \cap P$ and therefore $P \in [\![T]\!]$. ■

Based on the above two lemmas, we can now prove the following result that will bring us very close to a characterization of ambiguous theories. However before we present this result we need one more definition. Let $\langle \mathrm{L}, \vdash \rangle$ be a standard logic and let T, T' be two theories of L. We shall say that

T' is a *finite extension* of T iff there exists a formula $\varphi \in L$ such that $T' = Cn(T \cup \{\varphi\})$.

Lemma 4.4 Let $\langle L, \vdash \rangle$ *be a standard logic,* T *a consistent theory of* L, *and* P *an element of* $[\![T]\!]$. *Then* P *is minimal in* $[\![T]\!]$ *with respect to set inclusion iff every element of* P *is a finite extension of* T.

Proof.

Part I: If P *is minimal in* $[\![T]\!]$ *with respect to set inclusion, then every element of* P *is a finite extension of* T.

Assume that P is minimal in $[\![T]\!]$ with respect to set inclusion, and let $K^{\#}$ be an arbitrary element of P. If $K^{\#}$ is the only element of P then $T = K^{\#}$ and clearly $K^{\#}$ is a finite extension of T. Assume therefore that $(P - \{K^{\#}\}) \neq \emptyset$. Since P is minimal in $[\![T]\!]$, $(P - \{K^{\#}\}) \notin [\![T]\!]$. Therefore by Lemma 4.3, there exists a $\varphi \in L$ such that $\varphi \notin T$, and for every $K \in (P - \{K^{\#}\})$, $\neg\varphi \notin K$. On the other hand, since $P \in [\![T]\!]$, again by Lemma 4.3, there is a member of P containing $\neg\varphi$, and since none of the theories in $(P - \{K^{\#}\})$ contain $\neg\varphi$, the member of P containing $\neg\varphi$ is bound to be $K^{\#}$. Therefore $K^{\#}$ contains $\neg\varphi$ and moreover $K^{\#}$ is the only member of P containing $\neg\varphi$. The next step is to show that $K^{\#} = Cn(T \cup \{\neg\varphi\})$. Clearly $Cn(T \cup \{\neg\varphi\}) \subseteq K^{\#}$, so all we need to prove is that $K^{\#} \subseteq Cn(T \cup \{\neg\varphi\})$. Assume on the contrary that for some $\psi \in K^{\#}$, $\psi \notin Cn(T \cup \{\neg\varphi\})$. Then clearly $(\neg\varphi \rightarrow \psi) \notin T$. From this and Lemma 4.3 we derive that there exists a member K of P containing $\neg(\neg\varphi \rightarrow \psi)$ and consequently, $\neg\varphi \in K$ and $\neg\psi \in K$. This however leads us to a contradiction. Indeed, since $\psi \in K^{\#}$ and $K^{\#}$ is consistent, $\neg\psi \notin K^{\#}$ and therefore $K^{\#} \neq K$. On the other hand however $\neg\varphi \in K$, contradicting the fact that $K^{\#}$ is the only member of P containing $\neg\varphi$. This proves that $K^{\#} = Cn(T \cup \{\neg\varphi\})$. Therefore $K^{\#}$ is a finite extension of T, and since $K^{\#}$ was chosen arbitrarily, it follows that every element of P is a finite extension of T.

Part II: If every element of P *is a finite extension of* T, *then* P *is minimal in* $[\![T]\!]$ *with respect to set inclusion.*

Assume that every element of P is a finite extension of T. We prove that P is minimal in $[\![T]\!]$ with respect to set inclusion by showing that every proper subset of P is not in $[\![T]\!]$. Let P' be an arbitrary proper subset of

P. Then there exists a $K^{\#} \in P$ such that $K^{\#} \notin P'$. Since $K^{\#} \in P$, $K^{\#}$ is a finite extension of T and therefore there exists a $\varphi \in$ L such that $K^{\#} = Cn(T \cup \{\varphi\})$. We claim that for every $K \in P'$, $\varphi \notin K$. Assume on the contrary that for some $K' \in P'$, $\varphi \in K'$. Since $K' \in P' \subseteq P$ and $\cap P = T$, it follows that $T \subseteq K'$. Therefore, $Cn(T \cup \{\varphi\}) \subseteq K'$ and consequently $K^{\#} \subseteq K'$. Given that $K^{\#}$ and K' are consistent complete theories, from $K^{\#} \subseteq K'$ we derive that $K^{\#} = K'$, which however contradicts the initial assumption that $K^{\#} \notin P'$. This proves that for every $K \in P'$, $\varphi \notin K$. On the other hand, since $K^{\#} = Cn(T \cup \{\varphi\})$ and $K^{\#}$ is consistent, it follows that $\neg\varphi \notin T$, and therefore Lemma 4.3 implies that $P' \notin [\![T]\!]$. ∎

From Lemma 4.4 we can now derive a characterization of ambiguous theories. More precisely, let $\langle$L, $\vdash\rangle$ be a standard logic and let T be a theory of L. We shall say that T is *almost complete* iff every consistent complete extension of T is a finite extension of T. Lemma 4.5 below associates the notion of almost completeness with that of ambiguity.

Lemma 4.5 Let $\langle$L, $\vdash\rangle$ *be a standard logic and let T be a consistent theory of* L. *Then T is specific (not ambiguous) iff T is almost complete.*

Proof.

Part I: If T is specific then T is almost complete.

Assume that T is specific. By Lemma 4.2, $[T] \in [\![T]\!]$ and therefore since T is specific, $[T]$ is the only element of $[\![T]\!]$ and consequently $[T]$ is minimal in $[\![T]\!]$. Then by Lemma 4.4 every element of $[T]$ (which by definition is the set of all consistent complete extensions of T) is a finite extension of T.

Part II: If T is almost complete then T is specific.

Assume that T is almost complete. Then every element of $[T]$ is a finite extension of T and since by Lemma 4.2 $[T] \in [\![T]\!]$, it follows by Lemma 4.4 that $[T]$ is minimal in $[\![T]\!]$ with respect to set inclusion. Then given that any element of $[\![T]\!]$ is a subset of $[T]$ (Lemma 4.2), it follows that $[T]$ is the only element of $[\![T]\!]$ and consequently T is specific. ∎

Based on Lemma 4.5 and Corollary 4.1 we derive the following result that brings us very close to a characterization of intensional weakness.

Lemma 4.6 Let ⟨L, ⊢⟩ *be a standard logic. Then* ⟨L, ⊢⟩ *is intensionally weak iff there exists a consistent complete theory of* L *that is not finitely axiomatizable.*(9)

Proof.

Part I: If ⟨L, ⊢⟩ *is intensionally weak then there exists a consistent complete theory of L that is not finitely axiomatizable.*

Assume that ⟨L, ⊢⟩ is intensionally weak. Then by Corollary 4.1 there exists an ambiguous theory T of L. Since T is ambiguous, by Lemma 4.5, T is not almost complete and therefore there exists a consistent complete theory K that includes T, such that K is not a finite extension of T, i.e. $K \neq Cn(T \cup \{\varphi\})$, for every $\varphi \in$ L. From this it follows that K is a consistent complete theory of L that is not finitely axiomatizable. Indeed, assume on the contrary that for a finite set of formulas Γ, $K = Cn(\Gamma)$, and let ψ be the conjunction of all the formulas in Γ. Then, given that K is a consistent complete extension of T, we have that $K = Cn(\{\psi\}) = Cn(T \cup \{\psi\})$, which of course contradicts the fact that K is not a finite extension of T.

Part II: If there exists a consistent complete theory of L *that is not finitely axiomatizable then* ⟨L, ⊢⟩ *is intensionally weak.*

Assume that there exists a consistent complete theory K of L that is not finitely axiomatizable, and let $T^{\#}$ be the theory of L that we derive from the closure under ⊢ of the empty set, i.e. $T^{\#} = \{\varphi \in \mathrm{L}: \vdash \varphi\}$. Clearly, K is a consistent complete extension of $T^{\#}$, and since K is not finitely axiomatizable, from the definition of $T^{\#}$ it follows that K is not a finite extension of $T^{\#}$. Therefore $T^{\#}$ is not almost complete, and consequently by Lemma 4.5, $T^{\#}$ is ambiguous. Then by Corollary 4.1 we derive that ⟨L, ⊢⟩ is intensionally weak. ■

We now need to take only one final step before we provide our charac-

(9) A theory T of L is finitely axiomatizable iff there exists a finite set of formulas $\Gamma \subseteq$ L such that $T = Cn(\Gamma)$.

terization of intensional weakness. At this last step we shall relate the existence in a standard logic ⟨L, ⊢⟩ of a non finitely axiomatizable consistent complete theory, with the number of logical equivalence classes into which the consequence relation ⊢ partitions L. More precisely, let ⟨L, ⊢⟩ be a standard logic. Define ≡ to be the following binary relation in L: For every $\varphi, \psi \in$ L, $\varphi \equiv \psi$ iff $\vdash \varphi \leftrightarrow \psi$. It is not hard to verify that ≡ is an equivalence relation on L, and consequently it partitions L into equivalence classes, which we shall call the *logical equivalence classes* of ⟨L, ⊢⟩.

Lemma 4.7 Let ⟨L, ⊢⟩ *be a standard logic. Then* ⟨L, ⊢⟩ *has infinitely many logical equivalence classes iff there exists a consistent complete theory of* L *that is not finitely axiomatizable.*

Proof. Proving that the existence of a non finitely axiomatizable consistent complete theory entails the existence of infinitely many logical equivalent classes is straightforward. We shall therefore focus on the converse, namely on proving that the existence of infinitely many logical equivalence classes entails the existence of a non finitely axiomatizable consistent complete theory.

Assume on the contrary that L has infinitely many logical equivalence classes and yet every consistent complete theory of L is finitely axiomatizable. Based on these assumptions we shall make a number of observations from which we shall eventually derive a contradiction.

Observation 4.1 The set K_L of all consistent complete theories of L *is infinite.*

Proof. Assume on the contrary that K_L is finite. It is not hard to see that for any two theories T, T' of L, if $T \neq T'$ then $[T] \neq [T']$. Consequently, the number of different theories of L is no greater that the number of different subsets of K_L. Since K_L is finite, the power set of K_L is also finite and consequently, L has only finitely many distinct theories. This however contradicts our initial assumption of L having infinitely many logical equivalence classes since to every logical equivalence class there corresponds a distinct theory (simply take the closure under ⊢ of a representative of the class). □

We can say more than simply that K_L is infinite. Since every consistent complete theory of L is finitely axiomatizable, for every $K \in K_L$ there is a $\xi \in$ L such that $K = Cn(\{\xi\})$, and given that L has only countably many formulas, it follows that K_L is *countable*. Let $K_0, K_1, K_2, \ldots$ be an enumeration of the consistent complete theories of L, i.e. $K_L = \{K_0, K_1, K_2, \ldots\}$. For every $i \in \aleph$ (where $\aleph = \{0, 1, 2, \ldots\}$ is the set of natural numbers), we define ξ_i to be a formula in L such that $K_i = Cn(\{\xi_i\})$:

(1) For every $i \in \aleph$, $K_i = Cn(\{\xi_i\})$.

We define Ξ to be the set of all ξ_i for $i \in \aleph$, i.e. $\Xi = \{ \xi_i : i \in \aleph \}$.

Let $i,j \in \aleph$ be such that $i \neq j$. Then $K_i \neq K_j$ and therefore by (1), $\xi_i \not\vdash \xi_j$, and since K_i is complete it follows that $\neg\xi_j \in K_i$. Consequently the following is true:

(2) For every $i, j \in \aleph$, if $i \neq j$ then $\xi_i \vdash \neg\xi_j$.

For a set of formulas Γ of L we define $\vartheta(\Gamma)$ to be the set $\vartheta(\Gamma) = \{\neg\xi_i : \xi_i \in \Xi \text{ and } K_i \notin [\Gamma]\}$.

Observation 4.2 For every consistent formula $\varphi \in$ L, $Cn(\{\varphi\}) = Cn(\vartheta(\{\varphi\}))$.

Proof. Let φ be an arbitrary consistent formula of L. We first show that $[\varphi] = [\vartheta(\{\varphi\})]$. Let K_m be an arbitrary member of $[\varphi]$. Then by definition, $\neg\xi_m \notin \vartheta(\{\varphi\})$ and therefore by (2), $\xi_m \vdash \neg\xi_j$ for every $\neg\xi_j \in \vartheta(\{\varphi\})$, from which we derive that $K_m \in [\vartheta(\{\varphi\})]$. Conversely, let K_m be an arbitrary member of $[\vartheta(\{\varphi\})]$. Then by (1), $\neg\xi_m \notin \vartheta(\{\varphi\})$ and consequently $K_m \in [\varphi]$. This proves that $[\varphi] = [\vartheta(\{\varphi\})]$ and therefore $Cn(\vartheta(\{\varphi\})) = Cn(\{\varphi\})$. □

Observation 4.3 For every consistent formula $\varphi \in$ L, $\vartheta(\{\varphi\})$ *is finite.*

Proof. Let φ be an arbitrary consistent formula of L. By Observation 4.2, $Cn(\vartheta(\{\varphi\})) = Cn(\{\varphi\})$ and therefore by compactness, there exists a finite number of formulas $\neg\xi_{i1}, \neg\xi_{i2}, \ldots, \neg\xi_{in}$ in $\vartheta(\{\varphi\})$ such that $\neg\xi_{i1} \wedge \neg\xi_{i2} \wedge \ldots \wedge \neg\xi_{in} \vdash \varphi$. We show that $\neg\xi_{i1}, \neg\xi_{i2}, \ldots, \neg\xi_{in}$ are the only formulas in $\vartheta(\{\varphi\})$. Assume on the contrary that for some $\neg\xi_j \in \vartheta(\{\varphi\})$, $\neg\xi_j \notin \{\neg\xi_{i1}, \neg\xi_{i2}, \ldots, \neg\xi_{in}\}$. Since $\neg\xi_j \in \vartheta(\{\varphi\})$, by Observation 4.2,

$\neg\xi_j \in Cn(\{\varphi\})$ and consequently, $\neg\xi_{i1} \wedge \neg\xi_{i2} \wedge \ldots \wedge \neg\xi_{in} \vdash \neg\xi_j$. Then by the Deduction Theorem we derive that $\xi_j \vdash \xi_{i1} \vee \xi_{i2} \vee \ldots \vee \xi_{in}$, which however in combination with (2) and the fact that $\xi_j \notin \{\xi_{i1}, \xi_{i2}, \ldots, \xi_{in}\}$, leads us to a contradiction. This proves that $\mathfrak{S}(\{\varphi\}) = \{\neg\xi_{i1}, \neg\xi_{i2}, \ldots, \neg\xi_{in}\}$ and consequently $\mathfrak{S}(\{\varphi\})$ is finite. □

Observation 4.4 Let T be a consistent theory of L. *Then T is finitely axiomatizable iff* $\mathfrak{S}(T)$ *is finite.*

Proof. Assume that T is finitely axiomatizable, and let Γ be a finite set of formulas such that $Cn(\Gamma) = T$. Define φ to be the conjunction of all the formulas in Γ. Clearly $T = Cn(\{\varphi\})$ from which we derive that $[T] = [\varphi]$, which again implies that $\mathfrak{S}(T) = \mathfrak{S}(\{\varphi\})$. Then from Observation 4.3 it follows that $\mathfrak{S}(T)$ is finite. For the converse notice that, given Observation 4.2, it follows that, $T = Cn(\bigcup_{\varphi \in T} Cn(\{\varphi\})) = Cn(\bigcup_{\varphi \in T} Cn(\mathfrak{S}(\{\varphi\}))) = Cn(\bigcup_{\varphi \in T} \mathfrak{S}(\{\varphi\})) = Cn(\mathfrak{S}(T))$. Consequently, if $\mathfrak{S}(T)$ is finite then T is finitely axiomatizable. □

Define Ev and Od to be the sets $Od = \{\neg\xi_i\colon \xi_i \in \Xi$ and i is odd $\}$ and $Ev = \{\neg\xi_i\colon \xi_i \in \Xi$ and i is even$\}$, i.e. $Od = \{\neg\xi_1, \neg\xi_3, \neg\xi_5, \ldots\}$ and $Ev = \{\neg\xi_0, \neg\xi_2, \neg\xi_4, \ldots\}$. By the definition of Od and Ev, it is not hard to verify the following:

Observation 4.5 Let $K \in K_{\mathrm{L}}$ *be a consistent complete theory of* L. *Then K is an extension of Cn(Od) iff K is not an extension of Cn(Ev).*

We now need only one final observation to derive a contradiction:

Observation 4.6 Let T and T' be two theories of L *such that for any consistent complete theory* $K \in K_{\mathrm{L}}$, *K is an extension of T iff K is not an extension of T'. Then both T and T' are finitely axiomatizable.*

Proof. Assume on the contrary that one of the two theories, say T, is not finitely axiomatizable. From this assumption we shall derive that every finite subset of $T \cup T'$ is consistent. Let Δ be an arbitrary finite subset of $T \cup T'$ and let Γ, Γ' be the finite subsets of T and T' respectively, whose union makes up Δ, i.e. $\Gamma = \Delta \cap T$ and $\Gamma' = \Delta \cap T'$. Since T is not finitely axiomatizable there exists a formula $\varphi \in T$ such that $\Gamma \nvdash \varphi$. Consequently the set $\Gamma \cup \{\neg\varphi\}$ has a consistent complete extension, call it K. Clearly K is not an extension of T, and therefore K is an extension of T'. Then K is a consistent complete extension of $\Gamma \cup \Gamma' = \Delta$. Hence Δ is consistent. Since Δ was chosen arbitrarily, it follows that every finite subset of $T \cup T'$ is consistent and therefore by compactness, $T \cup T'$ is consistent. This again implies that there exists a consistent complete theory K' of L, that is an extension of both T and T', which of course leads us to a contradiction.□

Combining the above observations, we can now derive a contradiction. More precisely, by Observation 4.5 and Observation 4.6, both $Cn(Od)$ and $Cn(Ev)$ are finitely axiomatizable. On the other hand however, both $\Im(Cn(Od))$ and $\Im(Cn(Ev))$ are clearly infinite sets and therefore, from Observation 4.4 we derive a contradiction. This completes the proof of Lemma 4.7

Lemma 4.7 has another, much shorter proof, which however relies on results from Boolean algebras.(10) We sketch this proof below.

Consider the Lindenbaum algebra of the standard logic $\langle L, \vdash \rangle$. Given the conditions (i)-(v) in the definition of a standard logic, one can show that the Lindenbaum algebra of $\langle L, \vdash \rangle$ is a Boolean algebra. The elements of this algebra are the logical equivalence classes of $\langle L, \vdash \rangle$, and moreover the principle ultrafilters of the algebra correspond precisely to the finitely axiomatizable consistent complete theories of L. Then Lemma 4.7 follows directly from a well known result in Boolean algebras [7], according to which a Boolean algebra has a non-principle ultrafilter iff it is infinite. ■

From Lemma 4.6 and Lemma 4.7 we directly derive the following theorem that constitutes the central result of this article.

Theorem 4.1 Let $\langle L, \vdash \rangle$ *be a standard logic. Then* $\langle L, \vdash \rangle$ *is intensionally weak iff it has infinitely many logical equivalence classes.*

From Theorem 4.1 it follows that there are not many standard logics for which the expressibility gap is eliminated at its lower boundary. In fact, given Theorem 4.1 it is not hard to verify that the only intensionally strong standard logics are those that collapse to a propositional calculus with finitely many propositional variables. As another indication of the wide range of intensionally weak logics, consider the following result.

(10) This second proof was suggested by the anonymous referees.

Theorem 4.2 Every first order logic with equality is intensionally weak.

Proof. Let $\langle L, \vdash \rangle$ be an arbitrary first order logic with equality and let x_1, x_2, x_3, ... be the variables in L (every first order language has a countablely infinite set of variables). Define ψ_n to be the formula of L stating that "there are at most n distinct elements". Formally, for every natural number $n \geq 1$, we define ψ_n to be

$$\exists x_1 \forall x_2 \,(x_2 = x_1) \qquad \text{for } n = 1$$
$$\exists x_1 \exists x_2 \ldots \exists x_n \forall x_{n+1} (x_{n+1} = x_1 \vee x_{n+1} = x_2 \vee \ldots \vee x_{n+1} = x_n) \text{ for } n \geq 2$$

The only symbols appearing in the formula ψ_n are the variables x_1, x_2, ..., x_{n+1}, the left and right parentheses, the identity relation and the logical operators, and therefore for every natural number $n \geq 1$, ψ_n is a formula of L. Clearly, for any two natural numbers m, $n \geq 1$, if $m \neq n$ then $\not\vdash \psi_m \leftrightarrow \psi_n$. Therefore for $m \neq n$, ψ_m and ψ_n belong to different logical e-quivalence classes. Hence $\langle L, \vdash \rangle$ has infinitely many logical equivalence classes and therefore by Theorem 4.1, $\langle L, \vdash \rangle$ is intensionally weak. ■

With Theorem 4.1 we have accomplished the first of our two objectives, namely to characterize the class of intensionally weak standard logics. Recall that our second objective is to evaluate for a given intensionally weak standard logic $\langle L, \vdash \rangle$, how "big" is the expressibility gap at its lower end. More precisely, intensional weakness entails the existence of at least one proposition in 2^{K_L} that is not expressible in L. This however tells us very little about "how many" propositions in 2^{K_L} are not expressible in L. The following result will help us estimate the "size" of the expressibility gap at its lower end.

Lemma 4.8 Let $\langle L, \vdash \rangle$ *be a standard logic, and let* A_L *be the set of all ambiguous theories of* L. *Then there is no maximal element in* A_L *with respect to set inclusion.*

Proof. Assume on the contrary that A_L has a maximal element with respect to set inclusion. Then there exists a theory $T^{\#}$ of L such that $T^{\#}$ is ambiguous, and every proper extension of $T^{\#}$ is specific([11]). Since $T^{\#}$ is ambiguous, it follows that $T^{\#}$ is consistent and therefore $[T^{\#}] \neq \varnothing$. Let $K \in [T^{\#}]$ be an arbitrary consistent complete extension of $T^{\#}$. Since $T^{\#}$ is ambiguous, by Lemma 4.5, $T^{\#}$ is not almost complete and therefore $T^{\#} \neq K$. Then there exists a formula $\varphi \in K$ such that $\varphi \notin T^{\#}$. Let T be the theory $T = Cn(T^{\#} \cup \{\varphi\})$. Clearly K is a consistent complete extension of T, and therefore T is consistent. Moreover, T is a proper extension of $T^{\#}$, which makes T specific and by Lemma 4.5, almost complete. Then there exists a formula ψ such that $K = Cn(T \cup \{\psi\})$, and given that $T = Cn(T^{\#} \cup \{\varphi\})$ we derive that $K = Cn(T^{\#} \cup \{\varphi \wedge \psi\})$. Consequently, K is a finite extension of $T^{\#}$. Since K was chosen arbitrarily, it follows that every consistent complete extension of $T^{\#}$ is a finite extension of $T^{\#}$ and therefore $T^{\#}$ is almost complete. Then Lemma 4.5 implies that $T^{\#}$ is specific, which of course contradicts our initial assumption of $T^{\#}$ being a maximal ambiguous theory of L. ■

Based on Lemma 4.8 we can now prove the following theorem.

Theorem 4.3 For every intensionally weak standard logic $\langle \mathrm{L}, \vdash \rangle$, *there exist infinitely many propositions in* 2^{K_L} *that are not expressible in* L.

Proof. Let $\langle \mathrm{L}, \vdash \rangle$ be an intensionally weak standard logic. Then by Corollary 4.1, there exists a theory of L that is ambiguous. Since L has one ambiguous theory, from Lemma 4.8 it follows that L has infinitely many ambiguous theories. Moreover, it is not hard to verify that for any two ambiguous theories T, T' of L, if $T \neq T'$ then $[\![T]\!] \cap [\![T']\!] = \varnothing$. Finally, from Lemma 4.1 is follows that for every ambiguous theory T of L, all propositions in $([\![T]\!] - \{[T]\})$ (which by definition is nonempty), are not expressible in L. Combining the above we derive that there exist infinitely many propositions in 2^{K_L} that are not expressible in L. ■

([11]) Recall that T' is a proper extension of a set of formulas Γ iff T' is a theory of L and $Cn(\Gamma)$ is a proper subset of T'.

5. *Conclusion*

In this article we have analysed the gap of expressibility that occurs between propositions (defined as sets of possible worlds) and (sets of) formulas of a formal language. The study has mainly been motivated by the problems that this gap causes in modelling belief dynamics. There are essentially two major results reported in this article. The first is a characterization of the class of standard logics for which the expressibility gap does not dissolve even under the strongest restrictions on possible worlds (intensionally weak standard logics). According to this first result, the only intensionally strong standard logics are those whose consequence relation partitions the formulas into finitely many logical equivalence classes. A consequence of this is that all first order logics with equality are intensionally weak. According to the second main result of this article, for every intensionally weak standard logic $\langle L, \vdash \rangle$, there exist infinitely many propositions that are not expressible in L, even under the strongest restrictions on possible worlds.

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