Logique & Analyse **139-140** (1992), 223-238

FIXED POINT THEOREMS FOR INCONSISTENT AND INCOMPLETE FORMATION OF LARGE CATEGORIES

Joshua COLE & Chris MORTENSEN

ABSTRACT: The method of fixed points is used to show that an unrestricted comprehension scheme for large categories can be described in either inconsistent or incomplete theories of several background logics; thus simplifying and generalising a result of Feferman.

1. *Introduction*

Defects in the current foundations for category theory are well known and have occupied thinkers about foundations since Eilenberg and Mac Lane proposed the theory of categories in 1945. This is not to say that foundational problems have seriously interrupted the progress of mathematicians who actually use categories in their day-to-day work. The popularity of category constructs in the practice of mathematics continues to grow despite any truly satisfactory resolution to the original foundational problems. It must however be conceded that the categories of mathematical practice are generally small or locally small and so are well accounted for by the current foundations.

Just as it is natural and healthy for discussions about architectural achievements to focus on aesthetics and functionality of design, so is the current interest in the applications of categories normal and productive. Engineering talk about the properties of bricks, mortar, steel and timber seem tedious by comparison. However when the architect becomes more adventurous with their applications of the material, the cautious among us begin to wonder about the received wisdoms concerning bricks and mortar and we turn back to the engineers for reassurance. Focus should again begin to move back onto logicians. Questions about the foundations of category theory have become all the more pertinent with the recent surge in popularity of the theory.

This paper is a reaction to an approach to the foundations of category theory by Solomon Feferman in [2]. Feferman presents a theory of partial operations and classifications which are kinds of intentional characterisations of the concepts of characteristic functions and sets.

Classifications have a general comprehension axiom which facilitates the formation of the general kinds of collections required for the practice of unrestricted category theory by mathematicians. Feferman models this axiom in an underlying logic which is classical and S4 modal.

Our aim in this paper is to attempt to simplify the modelling in an intuitionist logic and then to dualise the intuitionist model to get a model in a paraconsistent logic. Our approach stems from a recognition of the well known isomorphism between S4 theories and theories in intuitionist logic, the observation that such intuitionist logics are the logics of open sets and that there are paraconsistent logics which are the logics of the corresponding closed subsets. We will observe that these topological constructions are defective in their treatments of double negation, leading to a reformulation in a different dual pair of logics.

The motivation for this paper is a desire to recommend paraconsistent theories as equally viable as intutitionist ones for the practice of mathematics.

2. *Preliminaries*

Feferman sets out to model a comprehension axiom for classifications:

$$\exists f\, \forall a_1 \ldots a_n \exists c[fa_1 \ldots a_2 \simeq c \;\wedge\; (\forall x(x \eta c \leftrightarrow \Box \phi_c x)) \;\wedge\; (\forall x(x \bar{\eta} c \leftrightarrow \Box \neg \phi_c x))]$$

in a logic which is essentially S4.

Ignoring the possible misordering of the quantifiers and setting aside the idea of the characteristic (partial) function f, the idea seems to be to have two predicates, η and $\bar{\eta}$ to be intuitively understood as functioning like the predicates $\in$ and $\notin$ in set theory. ϕ_c is some property which, by virtue of the comprehension scheme, gives rise to a classification c. $\Box\phi x$ is read intuitively as saying it can be verified that the property ϕ holds of object x. As a result, while η and $\bar{\eta}$ in some sense function like $\in$ and $\notin$, it is not the case that:

$$(\forall x)(\forall y)(x \bar{\eta} y \leftrightarrow \neg x \eta y)$$

This is because some classifications y are partial, which means for some x,

$(x\eta y \vee x\bar{\eta}y)$ fails and consequently so does $(x\bar{\eta}y \leftrightarrow \neg x\eta y)$. As an example consider the property ϕx defined as $x\eta x$. An attempt to verify ϕa amounts to a check whether $a\eta a$. But to verify $a\eta a$ requires first a demonstration that ϕa holds, which is what we were trying to demonstrate in the first place. So attempts to verify some properties are circular and cannot be completed.

Restricting the properties ϕx that give rise to an associated classification c_ϕ to those for which $\Box\phi x$ holds has the effect of bringing about a kind of incompleteness in the resulting theory in the sense that

$$(\exists y)(\exists x)\neg(x\eta y \vee x\bar{\eta}y)$$

Our approach is to be open about this incompleteness by dropping the predicate $\bar{\eta}$ from the language and just using the sentence operator $\neg$ instead. We also drop $\Box$ from the language and achive its original purpose by using a 3-valued logic and allowing some sentences of the form ϕx to take the middle truth value. The resultant theory will be incomplete in the usual sense that $A \vee \neg A$ fails for some sentence A.

To be more precise, let L be a logic with a set of truth values X with designated values $\nabla \subset X$. Let the consequence relation $\models_L$ be defined as follows. $A \models_L B$ iff for all valuations v on X if $v(A) \in \nabla$ then $v(B) \in \nabla$. A set of sentences Th_L is an L-theory iff it is closed w.r.t. $\models_L$ and closed w.r.t. conjunctions.

DEFINITION: An L-theory Th_L is *incomplete* iff for some sentence A neither $A \in Th_L$ nor $\neg A \in Th_L$.

DEFINITION: Th_L is *inconsistent* iff for some sentence A both $A \in Th_L$ and $\neg A \in Th_L$.

3. *A J3 Theory*

In this section the aim is to show how a modified comprehension axiom for classifications:

(*C*) $\quad (\exists y_\phi)(\forall x)(x\eta y_\phi \leftrightarrow \phi x)$

can be modelled in the intuitionist logic J3. That is, we will construct a theory Th_{J3} whose underlying logic is J3 such that $C \in Th_{J3}$.

Like Feferman we denote the language of our theory by $\mathsf{L}\ (\simeq, \eta)$ which is a basic language $\mathsf{L}\ (\simeq)$ extended by the addition of the binary predicate η (but not $\bar{\eta}$ or $\Box$). We assume the existence of a simplier theory $Th_{\simeq}$ in the simpler language $\mathsf{L}\ (\simeq)$. The negation operator will be denoted by the symbol $\neg$.

J3 is a logic with a set of three truth values $\{F, N, T\}$, a set of designated values $\nabla_{J3} = \{T\}$ and negation $\neg$ defined such that $\neg T = F$, $\neg N = F$ and $\neg F = T$.

J3 is a topological logic. This means that we can consider the truth values F, N and T to be sets in some topology and the operators $\wedge$, $\vee$, $\neg$, $\forall$, $\exists$ to be defined in terms of set operators in the topology.

Consider a partially ordered set of worlds W:

$$w^* \leq w$$

and the set of hereditary subsets X:

$$\Lambda,\ \{w\},\ \{w^*, w\}$$

X is a set of open sets and $\langle W, X\rangle$ is a topological space. A valuation is a function $v{:}\mathsf{L}\ (\simeq, \eta) \rightarrow X$.

Compound sentences containing operators $\wedge$, $\vee$, $\neg$, $\forall$, $\exists$ are evaluated by recursively applying the following rules:
For any valuation v,

(i) $v(A \wedge B) = v(A) \cap v(B)$

(ii) $v(A \vee B) = v(A) \cup v(B)$

(iii) $v(\neg A) = \mathsf{I}\ (\overline{v(A)}) =$ The largest open set S such that $S \subseteq \overline{v(A)}$

(iv) $v((\forall x)Fx) = \cap \{y\text{: for some term } t,\ v(Ft) = y\}$

(v) $v((\exists x)Fx) = \cup \{y\text{: for some term } t,\ v(Ft) = y\}$

We let F denote Λ, N denote $\{w\}$ and T denote $\{w^*, w\}$ and tell the usual intutionist story regarding designated truth values. That is, let $\nabla_{J3} = \{T\}$.

Following Feferman we define a model for the comprehension axiom using a transfinite inductive definition. That is we define a valuation ν such that $\nu(C) \in \nabla_{J3}$ i.e. $\nu(C) = T$.

The required valuation ν will be named ν_f in the construction that follows. ν_f will be defined in terms of a previous valuation ν_{f-1} which will in turn be defined in terms of ν_{f-2}, etc. So in order to define the valuation ν_f we require definitions of a series of valuations ν_1, ν_2, ν_3, ... Our definition begins with a basis ν_1.

BASIS: For all sentences $A \in \mathsf{L}\ (\simeq, \eta)$, ν_1 is defined:

(B1) $\nu_1(A) = \{w^*, w\}$ if $A \in \mathsf{L}\ (\simeq)$ and $A \in Th_{\simeq}$
(B2) $\nu_1(A) = \Lambda$ if $A \in \mathsf{L}\ (\simeq)$ but $A \notin Th_{\simeq}$
(B3) $\nu_1(A) = \{w\}$ otherwise
(B4) Remaining compound sentences are then evaluated by recursive applications of the rules (i) - (v).

Because this is a transfinite inductive definition the induction step has two parts, one for successor steps and one for limit steps.

INDUCTION STEP: ν_α is defined:
If α is a successor ordinal then

(I1) $\nu_{\alpha+1}(a\eta c_\phi) = \nu_\alpha\ (\phi a)$
(I2) Remaining compound sentences are then evaluated by recursive applications of the rules (i) - (v).

If α is a limit ordinal then

(I3) $\nu_\alpha\ (A) = \bigcup_{\beta<\alpha} \nu_\beta(A)$

This completes the definition of a transfinite series of valuations ν_1, ν_2, ν_3, ... The valuation ν_f is a special valuation in this series which is a fixed point. That is a valuation such that for every sentence S, $\nu_f(S) = \nu_{f+1}(S)$.

Before proving that such a valuation is implicit in our definition of the transfinite series of valuations, we prove some interim results.

DEFINITION: $\nu_\alpha \leq \nu_\beta$ iff
$\{A \in \mathsf{L}\ (\simeq,\eta):\nu_\alpha(A)=T\} \subseteq \{A \in \mathsf{L}\ (\simeq,\eta):\nu_\beta(A)=T\}$
and $\{A \in \mathsf{L}\ (\simeq,\eta):\nu_\alpha(A)=F\} \subseteq \{A \in \mathsf{L}\ (\simeq,\eta):\nu_\beta(A)=F\}$.

or alternatively:

For every sentence A if $v_\alpha(A)=T$ then $v_\beta(A)=T$ and if $v_\alpha(A)=F$ then $v_\beta(A)=F$.

THEOREM (Monotonicity): For valuations v_α, v_β if $\alpha \leq \beta$ then $v_\alpha \leq v_\beta$.

PROOF: By induction on the number of connectives in an arbitrary sentence A.

If A is an atomic sentence then the theorem trivially holds because the rules for building up any valuation only change the values of compound sentences. So $v_\alpha(A) = v_\beta(A)$.

Assume that for sentences P, Q the theorem holds. We show that the theorem holds for sentences $P \wedge Q$, $P \vee Q$, $\neg P$, $(\forall x)P$, $(\exists x)P$. Three of the five demonstrations are given here:

(a) Assume $v_\alpha(P \wedge Q) = T$. By clause (i) of the evaluation procedure for valuations it follows that $v_\alpha(P) = T$ and $v_\beta(Q) = T$. From the induction hypothesis we have $v_\beta(P) = T$ and $v_\beta(Q) = T$. By again employing clause (i) of the evaluation procedure for valuations we conclude that $v_\beta(P \wedge Q) = T$ as required. A similar argument can be given assuming $v_\alpha(P \wedge Q) = F$ and concluding that $v_\beta(P \wedge Q) = F$

(c) Assume $v_\alpha(\neg P) = T$. By clause (iii) of the evaluation procedure for valutions it follows that $v_\alpha(P) = F$. By the induction hypothesis it follows that $v_\beta(P) = F$. Again from (iii) we have that $v_\beta(\neg P) = T$ as required. A similar argument can be given assuming $v_\alpha(\neg P) = F$ and concluding that $v_\beta(\neg P) = F$.

(e) Assume $v_\alpha((\exists x)P) = T$. By clause (v) of the evaluation procedure for valuations it follows that for some term t, $v_\alpha((\exists x)P[x/t]) = T$. By the induction hypothesis we have that for some term t, $v_\beta((\exists x)P[x/t]) = T$. By clause (v) of the evaluation procedure for valuations we conclude that $v_\beta((\exists x)P) = T$ as required. □

THEOREM (Fixed Point): This definition generates a fixed point. That is a valuation v_f such that for every sentence $A \in \mathsf{L}(\simeq,\eta)$, $v_f(A) = v_{f+1}(A)$.

PROOF: By the previous theorem, we have that this method generates a sequence of valuations: $v_1 \leq v_2 \leq v_3 \leq \ldots$.

Once a sentence gets assigned a value T or F by a valuation it retains that value in all later valuations. Now, there are only denumerably many sentences in the language L $(\simeq, \eta)$. The set of ordinals of the second number class is non-denumerable. So for some λ of the second number class $v_\lambda = v_{\lambda+1}$. □

THEOREM (v_f is a model for *C*): $v_f((\exists y_\phi)(\forall x)(x\eta y_\phi \leftrightarrow \phi x)) \in \nabla_{J3}$
PROOF:
Left to right:

Assume for arbitrary a and for some c_ϕ that $v_f(a\eta c_\phi)=T$. Let α be the least ordinal such that $v_\alpha(a\eta c_\phi)=T$. α must be a successor ordinal. By the method of construction it must be the case that $v_{\alpha\text{-}1}(\phi a)=T$. Since $\alpha\text{-}1 \leq f$ it follows from montonicity that $v_f(\phi a)=T$.

If we assume for arbitrary a that $v_f(a\eta c_\phi)=F$ it can be shown by a similar argument that $v_f(\phi a)=F$.

Assume for arbitrary a that $v_f(a\eta c_\phi)=N$. Assume also that $v_f(\phi a)=T(F)$, then by the method of construction $v_{f+1}(a\eta c_\phi)=T(F)$. But v_f is a fixed point so $v_f(a\eta c_\phi)=T(F)$ which contradicts the first assumption. So $v_f(\phi a)=N$.

Right to left:

Assume for arbitrary a that $v_f(\phi a)=T$. By the method of construction it follows that $v_{f+1}(a\eta\ c_\phi)=T$. But $v_f=v_{f+1}$ since v_f is a fixed point. So $v_f(a\eta\ c_\phi)=T$.

By a similar argument it can be shown that if $v_f(\phi a)=F$ then $v_f(a\eta c_\phi)=F$.

Assume for arbitrary a that $v_f(\phi a)=N$. Assume also that $v_f(a\eta c_\phi)=T(F)$. Let α be the least ordinal such that $v_\alpha(a\eta c_\phi)=T(F)$. α is a successor ordinal. By the method of construction $v_{\alpha\text{-}1}(\phi a)=T(F)$. By the monotonicity theorem and the fact that $\alpha\text{-}1 \leq f$ it follows that $v_f(\phi a)=T(F)$ which contradicts the first assumption. So $v_f(a\eta c_\phi)=N$.

So for arbitrary a, $v_f(a\eta c_\phi) = v_f(\phi a)$ and hence by the definition of $\leftrightarrow$, for arbitrary a, $v_f(a\eta c_\phi \leftrightarrow \phi a)=T$. So $v_f((\exists y_\phi)(\forall x)(x\eta y_\phi \leftrightarrow \phi x)) = T$. $T \in \nabla_{J3}$. □

DEFINITION: Let R denote the classification defined by the property $\neg x\eta x$.

THEOREM: $v_f(R\eta R) = N$.
PROOF: We show that it is impossible for $v_f(R\eta R) = T$ or F.

Assume $v_f(R\eta R) = T$. Let α be the least ordinal such that $v_\alpha(R\eta R) = T$. α must be a successor ordinal and $v_{\alpha\text{-}1}(\neg R\eta R) = T$. By (iii) $v_{\alpha\text{-}1}(R\eta R) = F$. By monotonicity, it follows that $v_f(R\eta R) = F$ which contradicts the original assumption.

Assume $v_f(R\eta R) = F$. Then $v_f(\neg R\eta R) = T$. By (I1) $v_{f+1}(R\eta R) = T$. But since v_f is a fixed point $v_f = v_{f+1}$. So $v_f(R\eta R) = T$, contradicting the original assumption.

So $R\eta R$ does not take on truth value T or F in any valuation v_α. It therefore retains its original valuation of N at v_f. □

DEFINITION: Let $\overline{R}$ denote the classification defined by the property $x\eta x$.

THEOREM: $v_f(\overline{R}\eta\overline{R}) = N$.
PROOF: Again we show that it is impossible for $v_f(\overline{R}\eta\overline{R}) = T$ or F. We do this by showing there can be no least ordinal α such that $v_\alpha(\overline{R}\eta\overline{R}) = T(F)$.

Let α be the least ordinal such that $v_\alpha(\overline{R}\eta\overline{R}) = T(F)$. α is a successor ordinal and $v_{\alpha-1}(\overline{R}\eta\overline{R}) = T(F)$ contradicting the original assumption that α be the least such ordinal. □

DEFINITION: Define the theory $Th_{J3} = \{A \in \mathsf{L}(=,\eta) : v_f(A) \in \nabla_{J3}\}$

THEOREM (Incompleteness): For some sentence P, neither $P \in Th_{J3}$ nor $\neg P \in Th_{J3}$.
PROOF: $v_f(R\eta R) = N = \{w\}$. By (iii) $v_f(\neg R\eta R) = \Lambda$. $\{w\}, \Lambda \not\subseteq \nabla_{J3}$. So $R\eta R \notin Th_{J3}$ and $\neg R\eta R \notin Th_{J3}$. □

In this section we have given a definition of a sequence of valuations v_1, v_2, v_3, ... in the topological logic J3. We have proved that one of these valuations is a fixed point v_f which defines a theory Th_{J3}. This theory is incomplete and contains our modified axiom of comprehension C.

4. *A P3 Theory*

Next we show how a similar paraconsistent model for the modified comprehension axiom can be obtained by exploting the topological nature of the previous construction. Its dual closed set construction is a model for C in the logic P3.

P3 is a logic with a set of three truth values $\{F,B,T\}$, a set of designated values $\nabla_{P3} = \{B,T\}$ and negation $\ulcorner$ defined such that $\ulcorner T = F$, $\ulcorner B = T$ and $\ulcorner F = T$.

Notice that we can transform the J3 lattice into the P3 lattice by turning it upside down. This is achieved by exchanging $\cup$ and $\cap$. A problem with this though is that the bottom value (F) is the only designated truth value, which is absurd. Paraconsistentists have settled on a more satisfactory dualisation of ∇_{J3} as $\nabla_{P3} = \{X : X$ is a truth value and $X \notin \nabla_{J3}$. That is $\nabla_{P3} = \{\Lambda, \{w\}\}$.

A construction isomorphic to this one can be obtained by considering the closed subsets of W instead the open ones and retaining the original ordering, unions and intersections. A new definition of negation needs to be supplied

Instead of considering the open subsets of our original set of worlds W, we now turn our consideration to the closed subsets.

Retain the original partially ordered set of worlds W:

$$w^* \leq w$$

and now consider the set of anti-hereditary subsets Y:

$$\Lambda, \{w^*\}, \{w^*, w\}$$

Y is a set of closed subsets and $\langle W, Y \rangle$ is a topological space.

This time we define valuations

$$v_1,\ v_2,\ v_3,\ \ldots, v_f : \mathsf{L}\ (\simeq, \eta) \rightarrow Y$$

Let F denote Λ, B denote $\{w^*\}$ and T denote $\{w^*, w\}$. $\nabla_{P3} = \{B, T\}$

To change the underlying logic J3 to P3, we rename the truth value N as B and add B to the set of designated values ∇_{P3}.

The rules (i) - (v) for evaluating the valuations of compound sentences carry through unchanged with the exception of (iii). Closed set negation is denoted by the symbol $\ulcorner$ and is defined:

(iii) $v(\ulcorner A) = \mathsf{C}\ (\overline{v(A)}) =$ The smallest closed set S such that $\overline{v(A))} \subseteq S$

This means that $\ulcorner T = F$, $\ulcorner B = T$ and $\ulcorner F = T$, as desired. (B3) is changed to:

(B3) $v_1(A) = \{w^*\}$ otherwise

DEFINITION: $Th_{P3} = \{A \in \mathsf{L}\ (\simeq,\eta) : v_f(A) \in \nabla_{P3}\}$

The monotonicity theorem carries through with minor modification in clause (c) where $\neg$ is changed to $\ulcorner$. The fixed point theorem carries through with a modification changing N to B. $C \in Th_{P3}$ follows similarly. □

DEFINITION: Let R denote the classification defined by the property $\ulcorner x\eta x$.

THEOREM: $v_f(R\eta R) = B$

PROOF: Replace N by B in the previous version of this proof. □

THEOREM (Inconsistency): For some sentence P, both $P \in Th_{P3}$ and $\ulcorner P \in Th_{P3}$

PROOF: $v_f(R\eta R) = B = \{w^*\}$ By the new (iii) $v_f(\ulcorner R\eta R) = \{w^*,w\}$. $\{B, T\} \in \nabla_{P3}$. So $R\eta R$ and $\ulcorner R\eta R \in Th_{P3}$. □

Thus the topological dual of the previous intuitionist construction is a paraconsistent one. Both are equally viable. They are essentially different perspectives of the same construction.

5. *Double Negation and Routley-* Negation*

In our J3 theory $\neg N=F$ and $\neg F=T$. As a result, $\neg\neg N=T$. Recall that sentences like $R\eta R$ and $\overline{R}\eta\overline{R}$ are assigned the truth value N in the final fixed point valuation v_f. This provides a neat solution to Russell-type paradoxes in our theory of classifications, but as a further consequence of the topological nature of the negation operator $\neg$ it is also the case that $v(\neg\neg R\eta R) = T$ and $v(\neg\neg\overline{R}\eta\overline{R})=T$. This is unsatisfactory since N is an undesignated truth value while T is designated so that $\neg\neg A \supset A$ is not in the theory.

Similarly, in the P3 theory $\ulcorner\ulcorner B=F$ so that while $v(R\eta R)=B$, $v(\ulcorner\ulcorner R\eta R)=F$. B is designated in P3 but F is not, so $A \supset \ulcorner\ulcorner$A fails.

The logic underlying Feferman's theory is classical so that double negation holds in it. A better reconstruction is therefore one which affirms double negation. If we want double negation to behave as required for sentences which are assigned the middle truth value, the definition of the negation operator has to be arranged so that the middle truth value is a fixed point under its own operation. That is, in an intuitionist theory we want the negation of N to be N also, and in a paraconsistent theory we want the negation of B to be B.

Again we consider the set of worlds W:

$$w^* \leq w$$

We adopt the Herditary Condition of relevant semantics:

If $x \leq y$ and $x \in I(A)$ then $y \in I(A)$.

Routley *-negation (denoted by the symbol $\sim$) is defined as follows:

Let $w^{**} = w$

(i) $w \in I(A)$ iff $w^* \notin I(\sim A)$
(ii) $w^* \in I(A)$ iff $w^{**} \notin I(\sim A)$ iff $w \notin I(\sim A)$

There are four cases we need to consider: (a) $\sim\Lambda$, (b) $\sim\{w^*\}$, (c) $\sim\{w\}$, (d) $\sim\{w,w^*\}$.

(a) Assume $I(A) = \Lambda$. That is $w^* \notin I(A)$ and $w \notin I(A)$. Then by (ii) it follows that $w \in I(\sim A)$ and by (i) $w^* \in I(\sim A)$. That is, $I(\sim A) = \{w,w^*\}$.

(b) Assume $I(A) = \{w^*\}$. That is $w^* \in I(A)$ and $w \notin I(A)$. Then it follows that $w \notin I(\sim A)$ and $w^* \in I(\sim A)$. That is $I(\sim A) = \{w^*\}$.

(c) Assume $I(A) = \{w\}$. That is $w^* \notin I(A)$ and $w \in I(A)$. Then it follows that $w \in I(\sim A)$ and $w^* \notin I(\sim A)$. That is $I(\sim A) = \{w\}$.

(d) Assume $I'(A) = \{w,w^*\}$. That is $w^* \in I(A)$ and $w \in I(A)$. Then it follows that $w \notin I(\sim A)$ and $w^* \notin I(\sim A)$. That is $I(\sim A) = \Lambda$.

So there are two fixed points under negation here: $\sim\{w^*\} = \{w^*\}$ and $\sim\{w\} = \{w\}$.

THEOREM: $w \in I(A), I(\sim A)$ iff $w^* \notin I(A), I(\sim A)$
PROOF: $w \in I(A)$ iff $w^* \notin I(\sim A)$. And $w \in \mathrm{I}'(\sim A)$ iff $w^* \notin I'(A)$. □

Notice that so far the hereditary condition has not been employed. In the set of worlds W this condition says, since $w^* \leq w$ if $w^* \in I(A)$ then $w \in I(A)$. There are two cases to consider: (a) $w^* = w$ and (b) $w^* \neq w$. Now if $w^* = w$ then $\{w^*\} = \{w\} = \{w,w^*\}$ and we have just the Boolean algebra $[\{w\},\Lambda]$. So our present concern is with the second case where $w^* \neq w$ and $w^* \leq w$ so $w^* < w$.

THEOREM: (Consistency of w^*) There is no sentence A such that $w^* \in I(A)$ and $w^* \in I(\sim A)$.
PROOF: Assume there is a sentence P such that $w^* \in I(P)$ and $w^* \in I(\sim P)$. Then by the hereditary condition $w \in I(P)$. From the definition of negation it follows that $w^* \notin I(\sim P)$ which contradicts the second assumption. □

THEOREM: (Completeness of w) For every sentence A, $w \in I(A)$ or $w \in I(\sim A)$.
PROOF: Assume there is a sentence P such that $w \notin I(P)$ and $w \notin I(\sim A)$. Then by the definition of negation $w^* \in I(\sim P)$. From the hereditary condition it follows that $w \in I(\sim P)$ which contradicts the second assumption. □

Applying Routley-* negation and the hereditary condition on our original paritally ordered set of worlds $W = w^* \leq w$ gives us a choice of two 3-element algebras suitable as underlying logics for a model of the modified comprehension axiom: $[\Lambda \subseteq \{w^*\} \subseteq \{w^*,w\}]$ and $[\Lambda \subseteq \{w\} \subseteq \{w^*,w\}]$ where the middle value in each is a fixed point under negation.

We take this pair of logics and regard them as duals of each other, where one is an intuitionist logic and the other is a paraconsistent one. In addition to the Routley definition of *-negation we also adopt the *-operation which dualises theories.

DEFINITION: Let Th be a theory defined by an interpretation I and set of designated values ∇. That is $Th = \{A:I(A) \in \nabla\}$.

DEFINITION: $Th^* = \{A:I(\sim A) \notin \nabla\}$.

Now if we let ∇ in Th be the singleton $\{T\}$ as intutionists insist, then we can deduce the nature of ∇ in the dual theory Th^* which will be paraconsistent.

THEOREM: If ∇ in Th is $\{T\}$ then ∇ in Th^* is $\{B, T\}$.
PROOF: $Th = \{A: I(A) \in \{T\}\}$. $Th^* = \{A: I(\sim A) \notin \{T\}\}$. So $Th^* = \{A: I(\sim A) \in \{F,N\}\}$. By the definition of $\sim$ we have $Th^* = \{A: I(A) \in \{N,T\}\}$. The convention is to label the middle truth value in a paraconsistent logic B instead of N so $Th^* = \{A:I(A) \in \{B,T\}\}$. Thus ∇ in Th^* is the set $\{B,T\}$. □

In the following two sections we detail the construction of a further two theories again containing the comprehension axiom C. This time they will be Routley-* duals of each other and negation will be Routley-* negation.

6. *A K3 Theory*

K3 is a logic with a set of three truth values $\{T, N, F\}$, a set of designated values $\nabla_{K3} = \{T\}$ and negation $\sim$ defined such that $\sim T = F$, $\sim N = N$ and $\sim F = T$.

We define valuations

$$v_1, v_2, v_3, \ldots, v_f: \mathsf{L}(\simeq, \eta) \rightarrow \{\Lambda, \{w^*\}, \{w^*, w\}\}$$

Let F denote Λ, N denote $\{w^*\}$, and T denote $\{w^*, w\}$. Let the set of designated truth values $\nabla_{K3} = \{T\}$.

The rules (i) - (v) are as for J3 except for (iii). We denote Routley-* negation by the symbol $\sim$ and define it as before.

The definitions of valutions $v_1, v_2, v_3, \ldots, v_f$ are as before and the fixed point theorem carries through.

DEFINITION: Define $Th_{K3} = \{A \in \mathsf{L}(\simeq, \eta)\colon v_f(A) \in \nabla_{K3}\}$

The theory Th_{K3} contains the comprehension axiom C and is incomplete like J3 because $v_f(R\eta R) = N = v_f(\sim R\eta R)$ and $N \notin \nabla_{K3}$. However this time, because of the new definition of negation, $v_f(\sim\sim R\eta R) = N$ as desired.□

7. *An RM3 Theory*

RM3 is a logic with a set of three truth values $\{T, B, F\}$, a set of designated values $\nabla_{RM3} = \{B, T\}$ and negation $\sim$ defined such that $\sim T = F$, $\sim B = B$ and $\sim F = T$.

We define valuations

$$v_1, v_2, v_3, \ldots, v_f : \mathsf{L}(\simeq, \eta) \rightarrow \{\Lambda, \{w\}, \{w^*, w\}\}$$

Let F denote Λ, B denote $\{w\}$, and T denote $\{w^*, w\}$. Let the set of designated truth values $\nabla_{RM3} = \{B,T\}$.

We evoke the same proven strategy to arrive at a theory Th_{RM3} which models the comprehension axiom C, but unlike Th_{K3} is inconsistent because $v_f(R\eta R) = B = v_f(\sim R\eta R)$ and $B \in \nabla_{RM3}$. $v_f(\sim\sim R\eta R) = \mathrm{B}$. □

8. *Conclusion*

We have seen that Feferman's original construction lends itself to reconstructions in the toplogical logic duals J3 and P3 as well as in the lattice logic duals K3 and RM3. All reconstructions utilise some sort of fixed point method for arriving at the model of the comprehension axiom, as did Feferman in his original modal logic setting. This suggests that the fixed point method for constructing models of axioms has broad application. The method appears not to make any special demands on its underlying logic, although if it is used on an infinite domain it does presume some fairly strong properties of the metatheory's "set" theory.

So what about the foundations of category theory? It must be acknowledged that the problem of finding an adequate foundation for category theory is rather unsatisfactorily solved by Feferman; and so far as the present approach is just a reconstruction of the Feferman idea, the same can be said of it.

Feferman's theory of partial operations and classifications provides a foundation for category theory in that it is a theory that makes possible the formation of set-like entities, classifications, from "arbitrary" property-like entities, operations. Feferman goes some way towards giving a consistent account of naive set theory, and it is something like naive set theory which is needed in foundations for a fully general (well founded) category theory. However, as Feferman notes himself, unlike naive set theory his theory contains no extensionality axiom. No attempt has been made to model an extensionality axiom in any of our reconstructions. We do note however that modelling an extensionality axiom does appear to be possible. Brady uses a similar methodology to model both a comprehension axiom and extensionality axiom in [1].

A disappointment with the Feferman paper is that his comprehension axiom for classifications is less than wholly general. Whereas with the standard foundations for category theory a distinction must be made between small and large categories, the Feferman foundations require a similar distinction to be made between partial and total categories. For example, the new foundations allow the construction of the category of all total categories, but not the category of *all* categories, partial and total.

This is characteristic of the common approaches to foundations. With NBG foundations it is legal to form the category of all small categories, but not all categories small and large. The Grothendiek method of universes allows the formation of all categories in a universe U, but not all categories both in and outside U. It does allow the formation of the category of all categories in U and *some* outside U, but its objects will all be categories in a larger universe U' and one cannot include as objects categories outside U'.

Inside such foundational frameworks one is given glimpses of what it is like to have true freedom to categorially abstract, but in each case a seemingly arbitrary boundary is drawn over which one is forbidden to legally step. Such boundaries owe their existence to an almost pathelogical fear of contradiction on the parts of past foundationalists. And such fear is not without justification. Contradiction in the classical setting is a fearsome creature with a rapacious nature. No sooner is its presence permitted than the entire system is swamped by it, in that every sentence and its negation take on the status of theoremhood. But it is well known that this nature can be tamed, even harnessed, with a shift in logic.

The comprehension scheme used in the four theories given here is truly unrestrictive. The strategy in all four cases has been to abandon the classical logic as background logic for the theory and to assign a third truth value to "contradictory" sentences. Aside from the behaviour of negation, the difference between intuitionist theories Th_{J3}, Th_{K3} and the paraconsistent Th_{P3}, Th_{RM3} is just an attitude to this third truth value. Intutionists regard it as undesignated like *False* whereas paraconsistentists regard it as designated like *True*. In all cases the existence of contradiction is contained in small regions.

We believe that foundations allowing a truly general category theory can only be achieved by abandoning some of the notions of classical logic. The Feferman foundations originally appeared classical and quite general. We have shown that it is possible to capture many of the original ideas of the Feferman approach in theories which are intuitionistic or paraconsistent. The shift away from classical logic allowed us to use an unrestricted comprehension axiom and to simplify the presentation.

The University of Adelaide

References

[1] Brady, R., 'The Consistency of the Axioms of Abstraction and Extensionality in a Three-Valued Logic, *Notre Dame Journal of Formal Logic* 12 (1971), 447-453.
[2] Feferman, S., 'Categorical Foundations and Foundations of Category Theory' in Butts and Hintikka (eds.), *Logic, Foundations of Mathematics and Computability Theory* 149-169, Reidel, Dordrecht-Holland, 1977.