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SHEAF SPACES ON FINITE CLOSED SETS

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1. *Introduction*

A sheaf space on closed sets is very similar to an ordinary sheaf space and, like ordinary sheaf spaces, they have a special relationship with sheaves. A sheaf space is a continuous local homeomorphism between two topological spaces. Sheaves and presheaves over topological spaces are contravariant functors that take a topology ordered by set inclusion as a domain category. It is known that a sheaf over an open set topology will give rise to a sheaf space and vice versa. It is usual to note that the category of all sheaves on an open set topology $\mathscr{T}$ for a space X is equivalent to the category of all sheaf spaces over X with the same topology. With this paper we modify the notion of local homeomorphism to deal with closed sets and verify that at least a restricted class of closed set sheaves are equivalent to "closed set sheaf spaces".

The notion of a closed set (pre)sheaf is a particular example of the notion of a (pre)sheaf over a category and as such is uncontroversial. The notion of a sheaf space over closed sets is itself not especially different from the usual sheaf spaces however in the absence of a general theory allowing us to forgo open set topologies we could be accused of misusing the "sheaf space" name. In fact this paper is part of a dualisation project for sheaves and toposes where the usual notion of a sheaf is "dualised" by being defined over closed sets instead of open(1). The aim of the project is to develop paraconsistent logics for categories of sheaves and so toposes. The "topos logic" of a category of sheaves over open sets is esssentially the Heyting algebra of the open sets ordered by set inclusion and so Intuitionist. Closed sets when ordered by set inclusion are a paraconsistent algebra. For categories of sheaves over closed sets this is expected to give us a paraconsistent "topos logic". In fact, it does. The principal idea of the dualisation project, that any existing topos logic amounts by a process of dualisation

(1) This idea of dualisation is distinct from the usual definition within category theory but it is intended that there be made some sort of intuitive connection.

to a description of another topos' paraconsistent logic, can be found in the forthcoming Mortensen (1994). There is also a description of closed set sheaf categories and the object of their logics, the subobject classifier Ω. The proof that the logic of these categories is indeed paraconsistent is expected to be published as a paper later in 1994. The proof consists of showing that a characteristic closed set paraconsistent algebra operator, $\dot{-}$, exists as an arrow $\Omega\times\Omega\to\Omega$ for internal poset Ω. In particular we are then able to describe the paraconsistent negation operator $\ulcorner:\Omega\to\Omega$. The proof will hold for all categories of sheaves over any closed set topology.

There is some expectation that an adequate description of a closed set sheaf space can be put to use in terms of Davey's representation constructions. Davey (1973) describes a general method for converting a subdirect product representation of an algebra to a representation of an algebra of global sections of a sheaf space. We note that Davey's construction is given in terms of open set sheaf spaces. The question of representation of paraconsistent algebras, particularly as sheaf spaces, becomes a concern for our project of "dualisation" since, as suggested above, we know that categories of closed set sheaves manifest the paraconsistent logic of the closed set topology of the base spaces. Such categories display paraconsistent algebras as part of their structure. We find for example that lattices of subobjects are paraconsistent algebras. There is a speculation that inconsistent but non-trivial theories can be represented as categories with such structure. It is further speculated that mathematical and logical objects arising from inconsistent but non-trivial mathematical and logical theory will be most naturally collected into categories of this sort. On the hypothesis of an equivalence between closed set sheaves and sheaf spaces, one way to investigate these speculations is to develop representation theorems which amount to equivalences of categories between say a category of paraconsistent algebras and a category of closed set sheaf spaces. Our object would seem to be most immediately available if our sheaf spaces were defined over closed sets. In the first instance though we must set about discovering the viability of the notion of a closed set sheaf space. In particular we will want to know how closely the theory of closed set sheaves and sheaf spaces mirrors the theory of open set sheaves and sheaf spaces. To that end we consider an equivalence of categories result for closed set sheaves and sheaf spaces.

Our aim will be to present a somewhat restricted revision of the standard constructions for the presheaf to sheaf space functor L and the sheaf space to sheaf functor Γ that can deal with structures on closed sets rather than open. While we propose to proceed along the usual line of development we shall at times be required to alter the usual proofs to accomodate the new nature of the stalk and base space topologies.

It will be advantageous to restrict the usage of the constructed functors to presheaves and sheaf spaces over spaces X with topologies $\mathscr{T}$ where any member of $\mathscr{T}$ is a finite subset of X. That is to say the usual construction will not in general work for closed sets without some restriction of this sort. We make significant use of this restriction.

We will also have occasion to restrict our constructions to presheaves F where for any closed U, the set $F(U)$ is finite. This is in response to what seems to be a deep feature of the construction of sheaf space morphisms from presheaf morphisms: given a presheaf morphism $f\colon F \to F'$ it is possible to describe a function Lf from constructed sheaf space (LF,p_F) to (LF',$p_{F'}$) but to prove that function continuous in general we will be required to accept arbitrary unions in the topology on LF. Notice too that the construction of sheaf ΓE from sheaf space (E,p) may not guarantee finite sets $\Gamma E(U)$. The particular implication for us is that while we can describe a functor Γ from the category of all sheaf spaces over X to the category of sheaves over X it will not in general compose with the functor L restricted to sheaves. So the domain of our Γ will be restricted to sheaf spaces (E,p) where E is finite. These restrictions are somewhat ad hoc but only from the point of view of creating a more general "sheafification" theory.

Note well that the above restrictions apply only for the particular construction of functors $\underline{\text{L}}$ and $\underline{\Gamma}$ included here. There should be no conclusion that this indicates which presheaves and sheaf spaces can exist on closed sets.

We will adopt the following conventions: $\mathscr{T}$ is always a closed set topology of finite subsets of X; $\underline{presh}(X,\mathscr{T})$ is the name for the category of closed set presheaves over topological space X where for any closed $U \subseteq X$ and any presheaf F the set $F(U)$ is finite; $\underline{sh}(X,\mathscr{T})$ is the category of sheaves in $\underline{presh}(X, \mathscr{T})$; $\underline{sheafsp}(X, \mathscr{T})$ is the category of sheaf spaces (E,p) over X where E is finite. Any category name given without an underbar should be taken to refer to the unrestricted categories in question. $\underline{\text{L}}$ will be a functor $\underline{presh}(X, \mathscr{T}) \to \underline{sheafsp}(X,\mathscr{T})$; $\underline{\Gamma}$ will be a functor $\underline{sheafsp}(X, \mathscr{T}) \to \underline{sh}(X,\mathscr{T})$.

We shall end by discovering that $\underline{sh}(X, \mathscr{T})$ is equivalent to $\underline{sheafsp}(X, \mathscr{T})$. Note that since part of the motivation in describing an equivalence of categories has to do with the "topos logics" of a category of closed set sheaves and since we have placed an unusual restriction on the sheaves we finally deal with, there is some sideline interest in providing $\underline{sh}(X, \mathscr{T})$ with a subobject classifier. The category $sh(X, \mathcal{T})$ always has a classifier Ω. The usual construction (Goldblatt (1984), p.369) for that Ω given $\mathcal{T}$ as described above yields for any $U \in \mathcal{T}$, finite $\Omega(U)$. This construction requires an aside showing how to avoid arbitrary unions in the construction of character arrows - see the forthcoming Mortensen (1994). We expect the sheaf Ω to be the subobject classifier for $\underline{sh}(X, \mathcal{T})$.

Our presentation follows closely that of Tennison (1975). And for the rest of this paper all topologies, sheaves, and sheaf spaces are understood to be given in terms of closed sets unless otherwise stated.

1.1 *Definition.* Any collection β of sets will be called a *basis for a closed set topology* T on a space $X=\bigcup\beta$ when we have that $b \in$ T iff b is a finite union of members of β. Any collection α is a *subbasis for closed set topology* T if the collection of all intersections of members of α is a basis for T. Plainly, any collection α can be used as a subbasis for a topology on $\bigcup\alpha$.

1.2 *Definition.* For a topological space $(X, \mathcal{T})$ set inclusion is a partial order on $\mathcal{T}$. We will use $\mathcal{T}$ to denote both a closed set topology and the poset category that has the sets of the topology as objects and all inclusions as arrows. A *presheaf over X* with topology $\mathcal{T}$ is any contravariant functor $F{:}\,\mathcal{T}^{op}\rightarrow$ SET. This will also be called a *presheaf on* $\mathcal{T}$. For any arrow $U \subseteq V$ in $\mathcal{T}$ we will use F^V_U to denote $F(U \subseteq V)$. A *sheaf on* $\mathcal{T}$ is any presheaf F that satisfies two conditions for any $U \in \mathcal{T}$ where $U=\bigcup\{U_i{:}U_i \in \mathcal{T}, i \in I\}$:

(1) if $s, s' \in F(U)$ such that $F^U_{U_i}(s) = F^U_{U_i}(s')$, all $i \in I$, then $s = s'$.

(2) if we have $\{s_i \in F(U_i){:}\ i \in I\}$ such that $F^{U_i}_{U_i \cap U_j}(s_i) = F^{U_j}_{U_i \cap U_j}(s_j)$, any $i, j \in I$, then there is some $s \in F(U)$ such that $F^U_{U_i}(s) = s_i$, any $i \in I$.

Given that $\mathcal{T}$ is a topology of closed sets the sheaf F is called a *closed set sheaf.*

A morphism of presheaves is a natural transformation of functors.

1.3 *Construction.* For any presheaf F over X with topology $\mathcal{T}$ and any $x \in X$ we define the *stalk* F_x of F at x as usual to be the direct limit of the direct system of sets and arrows $\{F(U){:}\ x \in U \in \mathcal{T}\}$ and F^U_V where $x \in V \subseteq U$ and $V \in \mathcal{T}$.

Fix $x \in X$. Let $Z = \coprod_{U \ni x} F(U)$, the disjoint union of all $F(U)$ where $x \in U \in \mathscr{T}$. We define an equivalence relation $\sim_x$ on Z where supposing $u \in F(U) \subseteq Z$ and $v \in F(V) \subseteq Z$ then $u \sim_x v$ iff there is a $W \in \mathscr{T}$ such that $x \in W \subseteq U \cap V$, and $F^U_W(u) = F^V_W(v)$. Then $F_x = \varinjlim_{U \ni x}(F(U))$ is $Z/\sim_x$ together with maps

$$F(U) \to F_x = F(U) \hookrightarrow Z \to (Z/\sim_x): s \mapsto s_x.$$

Each s_x is an equivalence class $[s]$ under $\sim_x$.

The following claims are straightforwardly true: $e \in F_x$ iff $e = s_x$ for some $s \in F(U)$ where $x \in U$. As a corollary for fixed $x \in X$ we have $F_x = \{s_x: s \in F(U), x \in U \in \mathscr{T}\}$. And for $s_x, t_x \in F_x$ with $s \in F(U)$, $t \in F(V)$, we have $s_x = t_x$ iff there is some $W \subseteq U \cap V$ with $x \in W \in \mathscr{T}$ such that $F^U_W(s) = F^V_W(t)$.

Suppose a morphism $f{:}F \to F'$ of presheaves. Recall that f is a natural transformation given by components f_U: $F(U) \to F'(U)$ for each closed $U \subseteq X$. For each $x \in X$ we have stalk morphisms f_x: $F_x \to F'_x$: $s_x \mapsto (f_U(s))_x$. Obviously for composite presheaf morphism $F \xrightarrow{f} F' \xrightarrow{g} F''$ we have $(g \cdot f)_x = g_x \cdot f_x$.

1.4 *Definition*. A map $p{:}E \to X$ between topological spaces E and X is *continuous* if for any closed subset U of X, the set $p^{-1}(U)$ is closed in E (Kelly, (1970), p.86). A map $p{:}E \to X$ is a *local homeomorphism* if for any $e \in E$ there is some open $N \subseteq E$ and some open $U \subseteq X$ such that $e \in N$, $p(e) \in U$, and the map $p|N{:}N \to U$ is a homeomorphism. The map $p|N$ is a *homeomorphism* if it is a bijection and both it and its inverse are continuous with respect to open sets. In essence, a map is a local homeomorphism if it is a homeomorphism when restricted or "localised" to an open subset of its domain. Plainly, we can describe a similar property of maps in terms of closed sets. Replace all occurences of "open" with "closed" in the definition of a local homeomorphism and a homeomorphism and we have the definition of a *closed set local homeomorphism*. A *closed set sheaf space on* X is a continuous closed set local homeomorphism $p{:}E \to X$ between the topological spaces E and X. When X is understood we shall use (E,p) to denote the sheaf space.

For continuous local homeomorphism $p{:}E \to X$ and closed $U \subseteq X$ a *section* of p over U is a continuous map $s{:}U \to E$ such that $p \cdot s = id_U$. The collection of all sections over U is denoted $\Gamma E(U)$.

A morphism of sheaf spaces $g{:}(E,p) \to (E',p')$ is a continuous map

$g{:}E \to E'$ such that $p = p' \cdot g$.

1.5 *Proposition*: any homeomorphism $p \mid N{:}N \to U$ guaranteed by p as local homeomorphism gives rise to a section $(p|N)^{-1}$: $U \to N \hookrightarrow E$.
Proof: $(p|N)^{-1}$ is by definition continuous and plainly $p \cdot (p|N)^{-1} = id_U$. ∎

1.6 *Proposition*: any section $s{:}U \to E$ is a closed map.
Proof: by hypothesis U is finite so $s(U)$ is finite. For any $e \in s(U)$ there are closed neighbourhoods M of e in E and V of $p(e)$ in X such that $p|M$: $M \to V$ is a homeomorphism. Homeomorphism $p|M$ maps $M \cap s(U)$ bijectively to $(p|\ M)(M) \cap U$ since s is a section. Since $(p|M)^{-1}$ is continuous and M closed in E, we have $(p|M)(M)$ and therefore $(p|M)(M) \cap U$ closed in X. And since $p|M$ is continuous we have $M \cap s(U)$ closed in E. Choose exactly one $p|M$ for each $e \in s(U)$ and the set $s(U)$ is the finite union of the associated sets $M \cap s(U)$. ∎

1.7 *Proposition*: any section s: $U \to E$ is a homeomorphism s: $U \to s(U)$.
Proof: since $p \cdot s = id_U$ the map s: $U \to s(U)$: $x \mapsto s(x)$ has a bijective inverse $p|s(U)$. The section s is continuous and a closed map so given p as continuous the map $p|s(U)$ is continuous. ∎

1.8 *Theorem*: For a sheaf space (E,p) with finite E the collection of sets $s(U)$, for all $s \in \Gamma E(U)$ and all $U \in \mathscr{T}$, is a basis for the topology on E.
Proof: By hypothesis all $U \in \mathscr{T}$ are finite, so all $s(U)$ must be finite subsets of E. Let M be any closed subset of E. For any $e \in M$ there is some closed neighbourhood $N \subseteq E$ such that a homeomorphism $p|N$ exists. The set $M \cap N$ is closed in E and since $p|N$ is a homeomorphism we have a section $s = (p|(M \cap N))^{-1}$ over $(p|N)(M \cap N) = p(M \cap N)$. For convenience let $W = p(M \cap N)$. Plainly, $e \in s(W) \subseteq M$. Recall that by hypothesis the space E is finite so the subset M must be finite. Choose one s for each $e \in M$ as described above and the set M is the finite union of sets $s(W)$. Since E is itself a member of the topology it follows that E is some finite union of sets $s(U)$. Since also any $s(U) \subseteq E$ the space E is the union of all $s(U)$. ∎

2. *Construction of functor* L:presh($X,\mathscr{T}$) → sheafsp($X,\mathscr{T}$).

2.1 *Construction*: Given a presheaf F on X we can construct a sheaf space (LF,p_F). Let L$F=\coprod_{x\in X} F_x$, the disjoint union of all F_x, $x \in X$. Define p_F:L$F \to X$ so that $(p_F)^{-1}(x)=F_x$, all $x \in X$.

Suppose U is closed in X and $s \in F(U)$. Define a map $\hat{s}$: $U \to$ LF: $x \mapsto s_x$. We topologize LF by accepting the collection of sets $\hat{s}(U)$, all $s \in F(U)$, all $U \in \mathscr{T}$, as a closed set subbasis. All we need show is that L$F=\bigcup\{\hat{s}(U)$: $s \in F(U)$, any $U \in \mathscr{T}\}$ This holds since where x is fixed $F_x=\{s_x$: $s \in F(U), x \in U \in \mathscr{T}\}$. We will presently establish that the sets $\hat{s}(U)$ are in fact a closed set basis for a topology.

The map p_F: L$F \to X$ is continuous with respect to this topology since for any closed $U \subseteq X$ we have $(p_F)^{-1}(U)=\coprod_{x\in U} F_x$. And that implies $(p_F)^{-1}(U)$ is the collection for all $x \in U$ of points s_x where $x \in V \in \mathscr{T}$ and $s \in F(V)$. But suppose $U \subseteq V$ and $s \in F(V)$. We always have $F^V_U(s)=F^U_U(s\,|\,U)$. So,

$$(p_F)^{-1}(U)=\bigcup\{\hat{s}(V)\colon s \in F(V),\ V \subseteq U,\ V \in \mathscr{T}\}.$$

By hypothesis U is finite so there are a finite number of $V \subseteq U$ and $(p_F)^{-1}(U)$ will be a finite union. Notice that this proof is clearly dependent upon the restricted $\mathscr{T}$. Plainly some sort of restriction on $\mathscr{T}$ or some extra hypothesis about the topology on LF will always be needed to prove that p_F is continuous with respect to closed sets.

The map p: L$F \to X$ is a local homeomorphism. Any $e \in$ LF will have some closed neighbourhood $\hat{s}(U)$. The maps $p_F\,|\,\hat{s}(U)$ and $\hat{s}$ are inverses and since p_F is continuous and $\hat{s}(U)$ is a closed set, we have continuous $p_F\,|\,\hat{s}(U)$. We prove that $\hat{s}$ is continuous by noting that $p_F\,|\,\hat{s}(U)$ will be a closed map (take closed sets of LF to closed sets of X).

2.2 *Theorem*: The collection of all sets $\hat{s}(U)$ where $\hat{s}$: $U \to$ LF: $x \mapsto s_x$ is defined iff $x \in U \in \mathscr{T}$ and $s \in F(U)$ is a basis for a topology on LF.

Proof: any collection β of sets identified as a closed set subbasis for a topology is a closed set basis for the same topology if any arbitrary intersection of members of β is a finite union of members of β. Since by hypothesis any U is finite, we have any $\hat{s}(U)$ as finite and any arbitrary intersection of sets $\hat{s}(U)$ amounts to a finite intersection. That is, suppose α is a non-finite collection of finite sets. If any members of α are disjoint, then $\bigcap\alpha=\varnothing$ and is therefore finite. Otherwise if no members of α are disjoint arbitrarily choose some $b \in \alpha$. Note that $\bigcap\alpha \subseteq b$ all $b \in \alpha$. Also, for any $x \in b$ such that $x \notin \bigcap\alpha$ there is some $b' \in \alpha$ for which $x \notin b'$. Let α' be a collection which contains b and exactly one such b' for any $x \in b$ where $x \notin \bigcap\alpha$. Then α' is a finite collection and $\bigcap\alpha' = \bigcap\alpha$.

Let $\hat{s}(U)$ be defined for $s \in F(U)$ and $\hat{t}(V)$ for $t \in F(V)$. Suppose $e \in \hat{s}(U) \cap \hat{t}(V)$, then $e = s_x = t_x$ for some $x \in U \cap V$. If $s_x = t_x$, then there is some $W \in \mathscr{T}$ such that $x \in W$ and some $r \in F(W)$ such that $r = F^U_W(s) = F^V_W(t)$. But this is true for all $x \in W$ and furthermore $r = F^W_W(r) = F^U_W(s) = F^V_W(t)$. In other words for all $x \in W$ we have $r_x = s_x = t_x$. So $\hat{r}(W) \subseteq \hat{s}(U) \cap \hat{t}(V)$. Since this is true for any $e \in \hat{s}(U) \cap \hat{t}(V)$ and $\hat{s}(U) \cap \hat{r}(V)$ is finite, that intersection is the finite union of sets $\hat{r}(W)$. ■

So far we have not needed it but when for any closed $U \subseteq X$ we have presheaf F with finite $F(U)$, we will use the underbar notation $(\underline{L}F, p_F)$ to denote the constructed sheaf space.

2.3 *Construction*: Suppose a morphism $f\colon F \to F'$ of presheaves. Suppose too that for any closed $U \subseteq X$ the set $F(U)$ is finite. Define sheaf space map

$$\underline{L}f\colon (\underline{L}F, p_F) \to (\underline{L}F', p_{F'})$$

so that for any $s_x \in \underline{L}F$, $(\underline{L}f)(s_x) = f_x(s_x) = (f_U(s))_x$. This assures us that $p_{F'} \cdot \underline{L}f = p_F$. For any $e \in \underline{L}F$ we have $\underline{L}f(e) = s'_x$ iff $e = s_x$ for some $s \in F(U)$ such that $f_U(s) = s'$. And since we have that $\underline{L}f(\hat{s}(U)) = \widehat{f_U(s)}(U)$, we find that for any closed set $\hat{s}'(U) \subseteq \underline{L}F'$ we have $(\underline{L}f)^{-1}(\hat{s}'(U)) = \bigcup\{\hat{s}(U)\colon s \in F(U), f_U(s) = s'\}$ Since we have imposed the restriction that any $F(U)$ be finite, $(\underline{L}f)^{-1}(\hat{s}(U))$ is a finite union of closed sets and therefore a closed set of $\underline{L}F$. The function $\underline{L}f$ is continuous.

It should be apparent that if we do not restrict the size of $F(U)$ it is possible that there be a non-finite number of $s \in F(U)$ for which $f_U(s) = s'$ in which case we need some extra hypothesis about the topology on LF, an alternative topology, or another construction for Lf.

This discussion yields $\underline{L}$ as a functor <u>*presh*</u>$(X, \mathscr{T}) \to$ <u>*sheafsp*</u>$(X, \mathscr{T})$ since if we suppose a composite presheaf morphism $F \xrightarrow{f} F' \xrightarrow{g} F''$, then for any $s_x \in \underline{L}F$, we have $\underline{L}(g \cdot f)(s_x) = (g_x \cdot f_x)(s_x)$ and $(\underline{L}g \cdot \underline{L}f)(s_x) = \underline{L}g(\underline{L}f(s_x)) = g_x(f_x(s_x)) = (g_x \cdot f_x)(s_x)$.

3. *Construction of functor* Γ *and* $\underline{\Gamma}$: $\underline{sheafsp}(X, \mathscr{T}) \to \underline{sh}(X, \mathscr{T})$

3.1 *Construction*. Given any sheaf space (E,p) on X we can construct a sheaf ΓE. For closed $U \subseteq X$ let

$$\Gamma E(U) = \{\text{continuous maps } U \xrightarrow{s} E \text{ such that } p \cdot s = id_U\}.$$

When $V \subseteq U$ we will set $(\Gamma E)^U_V$ to be the map $s \mapsto s|V$ and then we have a functor and thus a presheaf $\Gamma E : U \mapsto \Gamma E(U)$.

3.2 *Theorem*: ΓE is a sheaf.

Proof: Suppose $\mathscr{T} \ni U = \bigcup\{U_i : U_i \in \mathscr{T}, i \in I\}$. Suppose s, $s' \in (\Gamma E)(U)$ such that $(\Gamma E)^U_{U_i}(s) = (\Gamma E)^U_{U_i}(s')$, all $i \in I$. That is, $s|U_i = s'|U_i$, all $i \in I$. Since s and s' are functions $U \to E$ and the collection of sets U_i covers U we must have $s = s'$ as required.

Now, suppose we have $\{s_i \in (\Gamma E)(U_i): i \in I\}$ such that

$$(\Gamma E)^{U_i}_{U_i \cap U_j}(s_i) = (\Gamma E)^{U_j}_{U_i \cap U_j}(s_j), \text{ all } i, j \in I.$$

Define a map $s : U \to E$ by $s(x) = s_i(x)$ if $x \in U_i$, all $x \in U$. Since we have $p \cdot s_i = id_{U_i}$, all $i \in I$, we have $p \cdot s = id_U$. Also, for any closed $N \subseteq E$ we have

$$s^{-1}(N \cap s(U)) = \bigcup\{s_i^{-1}(N \cap s_i(U_i)): i \in I\}.$$

Since each s_i is a closed map, the sets $N \cap s_i(U_i)$ are closed in E. Since each s_i is continuous, the sets $s_i^{-1}(N \cap s_i(U_i))$ are closed in X. Since all closed sets of X are finite, $s^{-1}(N \cap s(U))$ is a finite union and therefore a closed set. So, s is continuous as required. ■

3.3 *Construction*: Suppose a sheaf space morphism $g : (E,p) \to (E', p')$. We construct the natural transformation $\Gamma g : \Gamma E \to \Gamma E'$ by specifying maps $(\Gamma g)_U : \Gamma E(U) \to \Gamma E'(U)$ for any closed $U \subseteq X$ so that for any $s \in (\Gamma E)(U)$,

$$(\Gamma g)_U(s) = g \cdot s.$$

Since both s and g are continuous $g \cdot s$ is continuous and since we know that $p' \cdot g = p$, we have $id_U = p \cdot s = (p' \cdot g) \cdot s$. So, $g \cdot s \in (\Gamma E')(U)$. And for any

$V \subseteq U$ in X the diagram

$$\begin{array}{ccccc} U & & \Gamma E(U) & \xrightarrow{(\Gamma g)_U} & \Gamma E'(U) \\ \downarrow & (\Gamma E)^U_V & \downarrow & & \downarrow (\Gamma E')^U_V \\ V & & \Gamma E(V) & \xrightarrow[(\Gamma g)_V]{} & \Gamma E'(V) \end{array}$$

commutes since for any $s \in \Gamma E(U)$ we have $(\Gamma E')^U_V((\Gamma g)_U(s)) = (g \cdot s) \mid V$ and $(\Gamma g)_V(\Gamma E)^U_V(s) = g \cdot (s \mid V)$.

These constructions yield Γ as a functor $sheafsp(X, \mathcal{T}) \to sh(X, \mathcal{T})$ since if we suppose a composite sheaf space morphism $E \xrightarrow{f} E' \xrightarrow{g} E''$ then for any closed $U \subseteq X$ and any $s \in (\Gamma E)(U)$, $(\Gamma g \cdot f)_U(s) = g \cdot f \cdot s$ and $(\Gamma g \cdot \Gamma f)_U(s) = (\Gamma g)_U((\Gamma f)_U(s)) = g \cdot (f \cdot s)$.

It remains true that in producing a sheaf ΓE from sheaf space (E,p) we have accepted and used a restricted topology on the base space X but note that we have required no restriction on E. Later we shall have need of a restricted domain Γ. Plainly if we restrict the domain to sheaf spaces (E,p) where E is finite, we can define a functor

$$\underline{\Gamma} : \underline{sheafsp}(X, \mathcal{T}) \to \underline{sh}(X, \mathcal{T}).$$

4. *Equivalence of categories for $\underline{sheafsp}(X, \mathcal{T})$ and $\underline{sh}(X, \mathcal{T})$.*

4.1 *Lemma*: For finite E the sheaf spaces (E,p) and $(L\underline{\Gamma}E, p_{\underline{\Gamma}E})$ are isomorphic.

Proof: We construct two morphisms k: $E \to L\underline{\Gamma}E$ and k': $L\underline{\Gamma}E \to E$ which are shown to be inverse closed maps (and so both continuous).

Consider any $e \in E$. Let s: $U \to E$ be some section with $e \in s(U)$. Let $N = s(U)$, then s has inverse $p \mid N$. Let s': $U' \to E$ be another section with $e \in s'(U')$. Let $N' = s'(U')$, then s' has inverse $p \mid N'$. Note that $s \in (\underline{\Gamma}E)(U)$ and $s' \in (\underline{\Gamma}E)(U')$. Let $W = p(N \cap N')$. Since $(p \mid N)(N \cap N') = p(N \cap N') = (p \mid N')(N \cap N')$ we have $s \mid W = s' \mid W$. Now both s and s' are closed maps so $N \cap N'$ is closed in E. And s is continuous so $s^{-1}(N \cap N') = W$ is closed in X. So, we have $(\underline{\Gamma}E)^U_W(s) = (\underline{\Gamma}E)^{U'}_W(s')$. And since $p(e) \in W$ we have $s \sim_{p(e)} s'$. Therefore we can define a map

$$k\colon E \to L\underline{\Gamma}E\colon e \mapsto s_{p(e)}$$

where s is any section of p for which $e \in cod(s)$.

Suppose s is a section over U. For any $e \in s(U)$ we have $k(e) = s_{p(e)}$ so $k\,(s(U)) = \{s_{p(e)}\colon e \in s(U)\}$. Since $p(s(U)) = U$ we have $k(s(U)) = \{s_{p(e)}\colon p(e) \in U\} = \hat{s}(U)$. So, k is a closed map.

Any element of $L\underline{\Gamma}E$ is s_x for some $s \in (\underline{\Gamma}E)(U)$ where $x \in U$. Any $s \in (\underline{\Gamma}E)(U)$ is a section of p. Suppose $s \in (\underline{\Gamma}E)(U)$ and $s' \in (\underline{\Gamma}E)(U')$. We have $s \sim_x s'$ iff $x \in U, U'$ and there is some W such that $x \in W$ and $(\underline{\Gamma}E)^U_W(s) = (\underline{\Gamma}E)^{U'}_W(s')$ which means $s|W = s'|W$ and, in particular, $s(x) = s'(x)$. So we may define an arrow

$$k'\colon L\underline{\Gamma}E \to E\colon s_x \mapsto s(x).$$

The sets $\hat{s}(U)$ are a basis for the topology on $L\underline{\Gamma}E$ and since $k'\,(\hat{s}(U)) = \{s(x)\colon x \in U\} = s(U)$ we have k' as a closed map.

The map k' is the inverse of k since for a section s with $e \in cod(s)$, $s(p(e)) = e$. So, $k' \cdot k = id_E$. Alternatively, the map k is the inverse of k' since $s_x \in L\underline{\Gamma}E$ only if $x \in dom(s)$ so plainly $s(x) \in cod(s)$ and since s is a section $p(s(x)) = x$. So, $k \cdot k'$ is the identity arrow for $L\underline{\Gamma}E$. ■

Note that technically the maps k and k' are maps $k\colon E \to \underline{L\Gamma}E$ and $k'\colon \underline{L\Gamma}E \to E$ since if E is finite as assumed, then for any closed $U \subseteq X$ the sets $\underline{\Gamma}E(U)$ must also be finite.

4.2 *Lemma*: Presheaf F over X and $\Gamma\underline{L}F$ are isomorphic iff F is also a sheaf where for any closed $U \subseteq X$ we have finite $F(U)$.

Proof: we produce a natural isomorphism $h\colon F \dot{\to} \Gamma\underline{L}F$ where for any closed $U \subseteq X$, we have a bijection $h_U\colon F(U) \to (\Gamma\underline{L}F)(U)$. For any $s \in F(U)$ we have a map $\hat{s} \in \Gamma\underline{L}F(U)$. We set $h_U(s) = \hat{s}$. For closed $V \subseteq U$ in X consider the diagram

$$\begin{array}{ccccc}
U & & \Gamma F(U) & \xrightarrow{\;h_U\;} & (\Gamma\underline{L}F)(U) \\
\downarrow & & \downarrow{\scriptstyle F^U_V} & & \downarrow{\scriptstyle (\Gamma\underline{L}F)^U_V} \\
V & & F(V) & \xrightarrow[\;h_V\;]{} & (\Gamma\underline{L}F)(V)
\end{array}$$

For any $s \in F(U)$ we have $h_U(s) = \hat{s}$. Then, $(\Gamma\underline{L}F)^U_V(\hat{s}) = \hat{s}|V$. Now, $F^U_V(s) = s|V$ and $h_V(s|V) = \widehat{s|V}$. Since $F^U_V(s) = F^V_V(s|V)$ we have $s \sim_x (s|V)$, so $s_x = (s|V)_x$, and as a result $\hat{s}|V = \widehat{s|V}$. So the diagram commutes as required and map h is a natural transformation. Note we require the restriction upon F in order that we may define arrows $\Gamma\underline{L}F^U_V$.

The function h_U is injective since $\hat{s} = \{s_x: x \in U\} = \{t_x: x \in U\} = \hat{t}$ iff for all $x \in U$ we have $s_x = t_x$ iff for all $x \in U$ there is some $W' \subseteq U$ such that $F^U_{W'}(s) = F^U_{W'}(t)$. The set U will be finitely covered by the sets W' so if F is a sheaf and that last condition holds, then $s = t$.

Now if $e \in \Gamma\underline{L}F(U)$, then e is some section $U \rightarrow \underline{L}F$ of p_F. As such e is a closed map and $e(U)$ is closed in $\underline{L}F$. Given what we know about the basis for the topology on $\underline{L}F$ we have a finite union of closed sets $e(U) = \bigcup\{\hat{s}_i(U_i): s_i \in F(U_i), U_i \in \mathscr{T}, i \in I\}$ (Note well: the subscript i is an index and $s_i \in F(U_i)$ should not be mistaken for some $s_x \in \underline{L}F$). Implied by this we have $U = \bigcup\{U_i: i \in I\}$ and $F^{U_i}_{U_i \cap U_j}(s_i) = F^{U_j}_{U_i \cap U_j}(s_j)$, all $i, j \in I$. We can characterise e as the map where for all $x \in U$ if $x \in U_i$, then $e(x) = s_i(x)$. Since the sets U_i cover U there is exactly one $s \in F(U)$ for which $F^U_{Ui}(s) = s_i$. So for any U_i and any $x \in U_i$ we have $F^U_{Ui}(s) = s_i = F^{U_i}_{U_i}(s_i)$, that is $(s_i)_x = s_x$. So e is identical to the map $\hat{s}$. The function h_U is surjective. ∎

Note that technically $h: F \twoheadrightarrow \Gamma\underline{L}F$ is more accurately the transformation $h: F \twoheadrightarrow \underline{\Gamma\underline{L}}F$ since $\underline{L}F$ is finite by construction.

4.3 *Theorem:* Given our restrictions on the topology $\mathscr{T}$ on space X, the functor $\underline{L}$ restricted to sheaves and the functor $\underline{\Gamma}$ are an equivalence of categories $\underline{sh}(X, \mathscr{T})$ and $\underline{sheafsp}(X, \mathscr{T})$.

Proof: for this proof read $\underline{L}|\underline{sh}(X, \mathscr{T})$ for $\underline{L}$. Functors $\underline{L}$ and $\underline{\Gamma}$ are an equivalence of categories if there are natural isomorphims $\underline{L\Gamma} \cong id_{sheafsp}$ and $\underline{\Gamma L} \cong id_{sh}$. We have already established isomorphisms $k_E: E \rightarrow \underline{L\Gamma}E$ for sheaf space E and $h_F: F \rightarrow \underline{\Gamma L}F$ for sheaf F, so we need only show that the isomorphisms are natural. Suppose $f: F \rightarrow F'$ is a morphism of sheaves. Consider the following diagram in $\underline{sh}(X)$:

$$\begin{array}{ccccc} F & & F & \xrightarrow{h_F} & \Gamma\underline{L}F \\ f\big\downarrow & & f\big\downarrow & & \big\downarrow \Gamma\underline{L}(f) \\ F' & & F' & \xrightarrow[h_{F'}]{} & \Gamma\underline{L}F' \end{array} \qquad (1)$$

The diagram commutes if the component diagrams for each $U \in \mathcal{T}$ commute. This holds since for any $s \in F(U)$ we have $(h_F)_U(s) = \hat{s}$ and $(\Gamma\underline{L}f)_U(\hat{s}) = \underline{L}f\cdot\hat{s}$ which is a map $U \to \underline{L}F'$ where for any $x \in U$ we have $(\underline{L}f\cdot\hat{s})(x) = (f_U(s))_x$. In other words $\underline{L}f\cdot\hat{s} = \widehat{f_U(s)}$ And that is exactly $(h_{F'})_U(f_U(s))$.

Suppose g: $(E,p) \to (E', p')$ is a sheaf space morphism where both E and E' are finite. Consider the following diagram in $\underline{sheafsp}(X)$:

$$\begin{array}{ccccc} E & & E & \xrightarrow{k_E} & \Gamma\underline{L}E \\ g\big\downarrow & & g\big\downarrow & & \big\downarrow \Gamma\underline{L}(g) \\ E' & & E' & \xrightarrow[k_{E'}]{} & \Gamma\underline{L}E' \end{array} \qquad (2)$$

For any $e \in E$, we have $k_E(e) = s_{p(e)}$ for some section s of p with $e \in cod(s)$. Assume s is a section over U. Now,

$$\begin{aligned} \underline{L\Gamma}g(k_E(e)) &= \underline{L\Gamma}g(s_{p(e)}) \\ &= (\underline{\Gamma}g)_{p(e)}(s_{p(e)}) \\ &= ((\underline{\Gamma}g)_U(s))_{p(e)} \\ &= (g\cdot s)_{p(e)}. \end{aligned}$$

And, $k_{E'}(g(e)) = s'_{p'(g(e))}$ for some section s' of p' for which $g(e) \in cod(s')$. The map $g\cdot s$ is exactly one such section. Note that both g and s are continuous so $g\cdot s$ is continuous, and $p = p'\cdot g$ so $p'\cdot g\cdot s = id_U$. And plainly $g(e) \in (g\cdot s)(U)$ when $e \in s(U)$. So, we can assert $k_{E'}(g(e)) = (g\cdot s)_{p'(g(e))}$, and since $p'(g(e)) = p(e)$ we have $\underline{L\Gamma}g(k_E(e)) = k_{E'}(g(e))$ as required. ■

4.4 *In summary*:

Given the usual methods, a restricted class of sheaves over closed sets is provably equivalent to a restricted class of sheaf spaces over closed sets.

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