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# DISCRETE TENSE LOGIC WITH BEGINNING AND ENDING TIME: AN INFINITE HIERARCHY OF COMPLETE AXIOMATIC SYSTEMS *

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## 1. *Introduction*

The purpose of the present paper is to give semantically sound and complete axiomatizations of all members in a certain infinite hierarchy of systems of *discrete* (linear) tense logic with *beginning* and *ending time*. In the semantics of any such system one conceives of *time* as being simply the closed interval [-m,n] of integers determined by a pair m,n of natural numbers (so that it will always be the case that zero belongs to the time of such a system). As far as the syntax of our tense-logics is concerned, it is most closely related to that of two systems developed by Dana Scott in

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1964 and 1965 [see Prior (1967), ch. iv, §3, pp. 66-70 as well as Appendix A,§§7.1 and 7.3]: it combines connectives for the *next* and the *last* moment in discrete time (say, "tomorrow" and "yesterday", respectively) with the classical Priorean tense operators G, F, H and P (for, respectively, "always in the future", "once in the future", "always in the past" and "once in the past"), as in the Scott 1965 system; and with the Aristotelian omnitemporal operators "always" ("at all times") and "once" ("at some time"), as in the Scott 1964 system. In addition to these eight tense-logical connectives, our logics —referred to as the systems $\mathcal{T}$m,n with m,n $\geq$ 0 in the sequel— have a counterpart of the Kamp (1971) operator "now" (or "today"), which is understood in a "non-pleonastic" sense to refer to the *present* moment.

The systems $\mathcal{T}$m,n differ, however, from all existing treatments of tense logic known to me in the following vitally important respect: in addition to the syntactic resources just mentioned, our $\mathcal{T}$m,n-language has a special set of what I call *systematic frame constants*, for which straightforward truth conditions are laid down in the semantics (§4 *infra*) and which play a highly important, characteristic role in our axiomatization. In spite of this being so, I should point out that there seem to exist in the literature two forerunners of our systematic frame constants in tense logic, viz. (i) the Prior (1967) discussion of so called "world-state propositions" [see especially ch. v, §§1-7 and Appendix B, §3], and (ii) the Gabbay (1981) discussion of what he pertinently calls "the problem of irreflexivity in tense logics", on the basis of which he arrives at his famous *irreflexivity rule*, first presented and investigated in that paper [see also my earlier contribution Åqvist (1992) for some remarks on Gabbay (1981)]. The task of examining in detail the relationship of my present systems to those interesting sources of inspiration is quite complex, however, and must be left for another occasion.

In order to get the "flavour" of my systematic frame constants, let us quickly consider how to *read* formulae, or sentence schemata, involving them. The two axioms in A3(d) [§5 *infra*], which express the idea that time has an end as well as a beginning, are as follows, with appended readings:

A3(d).

$a_n \rightarrow e\bot$ : "if we are at the end of time, then tomorrow anything whatsoever will be the case".

$a_{-m} \rightarrow w\bot$ : "if we are at the beginning of time, then yesterday anything whatsoever was the case".

The axiom A1(g) expresses the idea that we simply *are in time*, and reads:

A1(g). $(a_{-m} \vee a_{-m+1} \vee \ldots \vee a_{-1} \vee a_0 \vee a_1 \vee \ldots \vee a_{n-1} \vee a_n)$:
"Either we are at the beginning of time, or we are at the moment right after the beginning, or..., or we are at the moment immediately preceding the present moment, or we are at the present moment, or we are at the moment right after the present one, or..., or we are at the moment immediately preceding the end of time, or we are at the end of time".

Again, the first axiom schema in A3(f) can then be understood as follows:

A3(f)[1st]. $\neg a_n \rightarrow (ewA \leftrightarrow A)$:
"if we are not [i.e. somewhere else than] at the end of time, then it will be tomorrow that it was the case yesterday that A if and only if A".

A final, slightly more complicated example. The series of axiom schemata in A4(c), expressing the truth conditions for sentences of the form GA [as well as their "systematic ambiguity", if you like], are read as follows:

A4(c).

$a_{-m} \rightarrow (GA \leftrightarrow eA \wedge e^2A \wedge \ldots \wedge e^{n+m}A)$:
"if we are at the beginning of time, then it *will always* be that A iff tomorrow A and the day after tomorrow A and... and the (n+m-1)st day after tomorrow A".

$a_0 \rightarrow (GA \leftrightarrow eA \wedge e^2A \wedge \ldots \wedge e^nA)$:
"if we are at the present moment, then it *will always* be that A iff tomorrow A and the day after tomorrow A and... and the (n-1)st day after tomorrow A".

$a_{n-1} \rightarrow (GA \leftrightarrow eA)$:
"if we are at the moment right before the end of time, then it *will always* be that A iff tomorrow [i.e. at the next moment = the end of time] A".

$a_n \rightarrow GA$:
"if we are at the end of time, then anything whatsoever *will always* be the case".

The above examples of readings ("we are at such and such a moment") of our systematic frame constants seem to agree nicely with the following assertion made by Prior (1967) on p. 188f.:

> A world-state proposition in the tense-logical sense is simply an *index of an instant*;... (his italics).

Moreover, our axiom schemata A1(a) and (b) again agree nicely with Prior's requirement that a world-proposition be true at *one instant only* [see Prior (1967) e.g. pp. 83, 189].

The plan of this paper is as follows. After having presented the syntax of the systems $\mathcal{T}m,n$ in §2, their model-theoretic semantics in §§3-4, and their proof-theory in §5, we start making preparations for the Completeness Theorem (given in §9) by proving various Lemmata in §6. Then, we proceed to a fundamental result on so called canonical $\mathcal{T}m,n$-structures in §7, and to an equally important "Coincidence Lemma" on such structures in §8. These materials suffice to yield the desired conclusion that the proof theory for our systems $\mathcal{T}m,n$ is semantically sound and complete relatively to their model theory, as asserted by the two Theorems of §9. I should point out as well that my tense-logics $\mathcal{T}m,n$ are formulated as Hilbert style axiomatic systems, and that the completeness proof is straightforward Henkin style, using maximal consistent sets of formulae.

Finally, a few comparisons and contrasts. Consider the following series of "possible" theorem schemata that have been proposed and discussed in the tense-logical literature, where we use G, F, H, P in their normal *strict*, non-Diodorean sense, e for "tomorrow", w for "yesterday", and $\boxminus$, $\hat{\Diamond}$ for the Aristotelian "always" and "once", respectively — just as in §2 *infra*.

| | |
|---|---|
| (1) $GA \rightarrow eA$; $HA \rightarrow wA$ | cf. T1, Y1 in the Scott 1965 system; see Prior (1967), pp. 67 and 178 |
| (2) $\neg e \neg A \leftrightarrow eA$; $\neg w \neg A \leftrightarrow wA$ | cf. T2, Y2 in Scott 1965; *ibid.* |
| (3) $GA \rightarrow FA$; $HA \rightarrow PA$ | cf. A6a, A6b in Burgess (1979) §1, and A4a, A4b in Burgess (1984) §0.3 |
| (4) $GA \vee FGA$; $HA \vee PHA$ | cf. A5a, A5b in Burgess (1979) §1, and A3a, A3b in Burgess (1984) §0.3 |

| | |
|---|---|
| (5) ewA↔A; weA↔A | cf. the 5th postulate in the Scott 1964 system; see Prior (1967) pp. 67, 178 |
| (6) A ∧ GA → PGA; A ∧ HA → FHA | cf. A8a, A8b in Burgess (1979) §1, and A6b, A6a in Burgess (1984) §0.3 |
| (7) A ∧ GA → wGA; A ∧ HA → eHA | cf. A21.1, A22.1 in Åqvist & Hoepelman (1981) §10 |
| (8) ⊟ (GA→A)→(GA→HA) | cf. the simplification of schema 101 given in Bull (1968), p. 27 n. 3 |

We easily verify that the two schemata in (1) are valid (§4) and provable (§5) in our systems. In the special case where we are at the end/beginning/ of time, GA/HA/ and eA/wA/ will both be true for *any* wff (formula) A, so this special case cannot constitute a counterexample to the validity of (1).

The situation is quite different with respect to (2). Here, that special case affords the following counterexample to the validity of the right-to-left implications in (2). Suppose that we are at the end/beginning/ of time. Then, again, anything of the form eA/wA/ will be true, so that anything of the form ¬eA/¬wA/ will be false, e.g. ¬e¬A/¬w¬A/. This observation also explains why we have no counterexample to the left-to-right implications in (2), which are indeed valid without restriction [see axiom schemata A3(b) in §5 *infra*]. But the right-to-left implications in (2) can only be asserted on the condition that we are *not* at the end/beginning/ of time, as is done in our axiom schemata A3(c) [§5 *infra*].

The schemata in (3) are since long ago well known to express non-ending-time and non-beginning-time principles [see also Prior (1967), Appendix A, e.g. §§ 5.2, 5.5 and 5.6]; hence, they should most definitely *not* be valid or provable in any tense-logic for beginning and ending time. But how do we avoid (3) becoming provable in the presence of (1) and the *unrestricted* (2)?

The schemata in (4) are again well known to express ending-time and beginning-time principles [see also Prior (1967), ch. iv, §6, pp. 72-74]; so they must be valid in our systems $\mathcal{T}$m,n and are indeed readily proved in the latter.

For similar reasons as in the case of (2), the schemata in (5) cannot be accepted as valid without restriction. Suppose again that we are at the end/beginning/ of time. Then anything of the form eA/wA/ is true, e.g. ewA/weA/. But from this fact it cannot be inferred that A itself be true at the end/beginning/ of time. This is a counterexample to the left-to-right implications in (5), explaining why the two axiom schemata in A3(f) [§5 *infra*] are formulated as they are. (On the other hand, the right-to-left implications in (5) are unrestrictedly valid and easily seen to be provable in all $\mathcal{T}m,n$ systems).

Interestingly, there are also counterexamples to the validity in our systems of the Burgess schemata in (6) *supra*. Suppose now that we are at the beginning/end/ of time and that A ∧ GA/A ∧ HA/ is true there: this still cannot prevent anything of the form HA/GA/from being true there as well, e.g. HF¬A/GP¬A/, which is equivalent to the negation of PGA/FHA/. In other words, PGA/FHA/ cannot fail to be false at the beginning/end/ of time, regardless of what else be true there. Hence the counterexamples in our logics to the schemata in (6). Note, then, that the present argument does not apply to the schemata in (7), which are valid and provable in every system $\mathcal{T}m,n$. It would work against (7), if the succedents of its implications were strengthened to ¬w¬GA and ¬e¬HA, respectively. But we now know such a strengthening to be inadmissible, and we know why this is so.

The Bull-Prior schema (8) —like (6) claimed to express the discreteness of time— remains valid in our systems.

## 2. *Syntax of the systems $\mathcal{T}m,n$ of discrete tense logic with beginning and ending time*

The *vocabulary* (morphology, alphabet, language) of the systems $\mathcal{T}m,n$ (≥ 0) is identical to that of my system $\mathcal{T}$ dealt with in Åqvist (1992): it is a structure made up of the following disjoint *basic syntactic categories*:

(i) An at most denumerable set $\{p_0, p_1, p_2, \ldots\}$ of *propositional variables*.

(ii) *Propositional constants*: ⊤ (*verum*) and ⊥ (*falsum*) for, respectively, tautologyhood and contradictoriness ("absurdity") as well as a family $\{a_i\}$ ($i \in Z$) of *systematic frame constants*, which are indexed by the set Z of all integers (positive, negative, and zero).

(iii) The *Boolean sentential connectives* ¬, ∧, ∨, →, ↔ for, respectively, negation, conjunction, disjunction, material implication and material equivalence.

(iv) Four groups of *one-place tense-logical operators*, viz.
  (a) o ("now"/or "today"/, i.e. "it is now/today/ the case that")
  (b) e ("tomorrow"), w ("yesterday")
  (c) G ("always in the future"), F ("some time in the future"), H ("always in the past"), P ("some time in the past")
  (d) $\boxminus$ ("always"), $\diamondminus$ ("once").

For a fuller discussion of the readings of these operators, see Section 2 of my Åqvist (1992), where, *inter alia*, certain suggestive spatial metaphors are invoked (e for "at the next point *east* of here", w for "at the last point *west* of here", etc.).

Again, for any natural numbers m,n ≥ 0, we recursively define the set $W_{m,n}$ of *wffs* (i.e. well formed formulae) of the system $\mathcal{T}$m,n in such a way that $W_{m,n}$ will have as members (i) all propositional variables, (ii) $\top$, $\bot$ and every systematic frame constant in the finite set
$\{a_{-m}, a_{-m+1}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-1}, a_n\}$;
moreover, $W_{m,n}$ is required to be closed under every connective (operator) in the categories (iii) and (iv) *supra*.

Note here that, although the *vocabulary* of the systems $\mathcal{T}$m,n is common to them and held constant, the sets $W_{m,n}$ of their *wffs* will vary along with variations of m,n over the natural numbers.

## 3. *Semantics for $\mathcal{T}$m,n: frames*

Consider the system $\mathcal{T}$m,n, for any m,n ≥ 0. By a $\mathcal{T}$m,n-*frame* or, briefly, a *frame* we understand an ordered quintuple (U, $(w_0, u_0, e_0)$, (m,n), E, W), where

(i) U is a non-empty, finite set (heuristically, of *moments* in time);

(ii) $w_0$, $u_0$, $e_0$ are designated members of U (heuristically, $w_0$ is the *western limit* of U, $e_0$ is the *eastern limit* of U, and $u_0$ is the *present* moment in U);

(iii) m,n are the natural numbers under consideration;

(iv) E and W are partial functions defined on U satisfying the following conditions:

(C1E) For all $u \in U$ except $e_0$, $Eu$ is in U; there is no $u \in U$ such that $u = Ee_0$ (E is a function *from* $U - \{e_0\}$ *into* U, but $Ee_0 \notin U$).

(C1W) For all $u \in U$ except $w_0$, $Wu$ is in U; there is no $u \in U$ such that $u = Ww_0$ (W is a function *from* $U - \{w_0\}$ *into* U, but $Ww_0 \notin U$).

(C2E) For all $u,v \in U$ except $e_0$, if $u \neq v$, then $Eu \neq Ev$ (E is *one-one* in $U - \{e_0\}$).

(C2W) For all $u,v \in U$ except $w_0$, if $u \neq v$, then $Wu \neq Wv$ (W is *one-one* in $U - \{w_0\}$).

(C3) For all $u,v \in U$ such that $u \neq e_0$ and $v \neq w_0$, $Eu = v$ iff $Wv = u$ (E and W are the *inverses* of each other in the set $U - \{w_0, e_0\}$, the existence of inverses being guaranteed by (C2E) and (C2W)).

(C4) For all u in $U - \{w_0, e_0\}$, $EWu = WEu$ (E,W *commute* in $U - \{w_0, e_0\}$.

(C5) For all u in $U - \{w_0, e_0\}$, $Eu \neq Wu$ (E,W are *disjoint* in $U - \{w_0, e_0\}$.

(C6) $w_0 = W^m u_0$ and $e_0 = E^n u_0$ (*location of limits* relatively to $u_0$).

(C7) For each u in U we have that
either $u = u_0$
or $u = E^k u_0$, for some natural number k with $0 < k \leq n$,
or $u = W^k u_0$, for some natural number k with $0 < k \leq m$
(finite *accessibility* from $u_0$).

(C8E) For each u in U except $e_0$, $E^k u \neq u$ for all natural numbers k

$$\text{with } 0 < k \leq \begin{cases} n+m, & \text{if } u=w_0=W^{m}u_0 \text{ (by(C6))} \\ n+m-1, & \text{if } u=W^{m-1}u_0 \\ . & \\ . & \\ n(=n+m-m), & \text{if } u=W^{m-m}u_0=u_0 \\ n-1, & \text{if } u=Eu_0 \\ . & \\ . & \\ 1(=n-(n-1)), & \text{if } u=E^{n-1}u_0 \end{cases}$$

(E-analogue of *Peano*'s fourth axiom).

(C8W) For each u in U except $w_0$, $W^k u \neq u$ for all natural numbers k

$$\text{with } 0 < k \leq \begin{cases} m+n, & \text{if } u=e_0=E^{n}u_0 \text{ (by(C6))} \\ m+n-1, & \text{if } u=E^{n-1}u_0 \\ . & \\ . & \\ m(=m+n-n), & \text{if } u=E^{n-n}u_0=u_0 \\ m-1, & \text{if } u=Wu_0 \\ . & \\ . & \\ 1(=m-(m-1)), & \text{if } u=W^{m-1}u_0 \end{cases}$$

(W-analogue of *Peano*'s fourth axiom).

*Explanation*. In conditions (C6), (C7) and (C8E)-(C8W) we understand the "exponents" in such a way that $E^k = \underbrace{E....E}_{k\ times}$ and $W^k = \underbrace{W....W}_{k\ times}$ (for any natural number $k \geq 0$). Obviously, this operation can be defined recursively in the usual fashion.

*Remark.* Although (C4) and (C5) can be seen to be redundant in the present definition of a $\mathcal{T}$m,n-frame, we find it instructive to state them already in this context.

## 4. *Semantics for $\mathcal{T}$m,n: models and truth conditions, validity and satisfiability*

Let (U, ($w_0$, $u_0$, $e_0$), (m,n, E, W) be any $\mathcal{T}$m,n-frame. By a *valuation on* such a frame we mean any function V which to each propositional variable $p_i$ (i = 0, 1, 2, ...) assigns a subset $V(p_i)$ of U, and by a *model* (for $\mathcal{T}$m,n) any ordered pair

$$\mathcal{M} = ((U, (w_0, u_0, e_0), (m,n), E, W), V)$$

the first term of which is a frame and the second a valuation on that frame. Let $\mathcal{M}$ be any model for $\mathcal{T}$m,n. We now define the concept of *truth at* a point u ∈ U *in* $\mathcal{M}$ for any wff A [in symbols: $\mathcal{M},u \models A$] by the following recursion:

$$\mathcal{M},u \models p_i \text{ iff } u \in V(p_i) \text{ (for any propositional variable } p_{i\,(=0,1,2,...)})$$

$$\mathcal{M},u \models \top$$

$$\text{not: } \mathcal{M},u \models \bot$$

Let [-m,n] be the *closed* interval of integers determined by -m and n, i.e. the set {-m, -m+1, ..., -1, 0, 1, ..., n-1, n}. For our systematic frame constants in $W_{m,n}$ we then adopt the truth condition:

$$\mathcal{M},u \models a_i \ (i \in [-m,n]) \text{ iff } \begin{cases} u = E^i u_0, & \text{if } 0 \leq i \leq n \\ u = W^{-i} u_0, & \text{if } 0 > i \geq -m \end{cases}$$

If A is a Boolean compound, our recursive definition goes on as usual. If A is of the form §B with § a one-place tense-logical operator, then we stipulate (still using 'u', 'v' as variables over U):

$\mathcal{M},u \models oB$ iff $\mathcal{M},u_0 \models B$

$$\mathcal{M},u \models eB \text{ iff } \begin{cases} \mathcal{M}, Eu \models B, \text{ if } u \neq e_0 \\ \text{for all v such that } Eu=v: \mathcal{M},v \models B, \text{ if } u=e_0 \end{cases}$$

$$\mathcal{M},u \models wB \text{ iff } \begin{cases} \mathcal{M}, Wu \models B, \text{ if } u \neq w_0 \\ \text{for all v such that } Wu=v: \mathcal{M},v \models B, \text{ if } u=w_0 \end{cases}$$

$$\mathcal{M},u \models GB \text{ iff } \begin{cases} \mathcal{M},E^k u \models B, \text{ for all } k: 0<k\leq \left[\begin{array}{l} n+m, \text{if } u=w_0=W^m u_0 \\ n+m-1, \text{if } u=W^{m-1}u_0 \\ \cdot \\ \cdot \\ n\ (=n+m-m), \text{if } u=u_0 \\ n-1, \text{if } u=Eu_0 \\ \cdot \\ \cdot \\ 1\ (=n-(n-1)), \text{if } u=E^{n-1}u_0 \end{array}\right. \\ \\ \text{for all v such that } Eu=v: \mathcal{M},v \models B, \text{ if } u=e_0=E^n u_0 \end{cases}$$

$$
\mathcal{M},u \models HB \text{ iff } \begin{cases} \mathcal{M},W^k u \models B, \text{for all } k{:}\, 0<k\leq \left[\begin{array}{l} m+n, \text{if } u=e_0=E^n u_0 \\ m+n-1, \text{if } u=E^{n-1}u_0 \\ . \\ . \\ m\ (=m+n-n), \text{if } u=u_0 \\ m-1, \text{if } u=Wu_0 \\ . \\ . \\ 1\ (=m-(m-1)), \text{if } u=W^{m-1}u_0 \end{array}\right. \\ \\ \text{for all } v \text{ such that } Wu=v{:}\ \mathcal{M},v \models B, \text{ if } u=w_0=W^m u_0 \end{cases}
$$

The truth conditions for wffs of the forms FB and PB are the "dual" ones, obtained from those for GB and HB just given by replacing the two occurrences of "all" by occurrences of "some". Finally, we have

$\mathcal{M},u \models \boxminus B$ iff for all v in U: $\mathcal{M},v \models B$
$\mathcal{M},u \models \Diamond B$ iff for some v in U: $\mathcal{M},v \models B$

The locution '$\mathcal{M},u \models A$' means that the wff A is *true at* the point u *in* the $\mathcal{T}$m,n-model $\mathcal{M}$. We say that a wff A is $\mathcal{T}$m,n-*valid* iff $\mathcal{M},u \models A$ for all $\mathcal{T}$m,n-models $\mathcal{M}$ and all points u in U. Also, we say that a set $\Gamma$ of wffs (in $W_{m,n}$) is $\mathcal{T}$m,n-*satisfiable* iff there exists a $\mathcal{T}$m,n-model $\mathcal{M}$ and a member u of U such that for all wffs in $\Gamma$: $\mathcal{M},u \models A$. Clearly, for any m,n $\geq 0$, A is $\mathcal{T}$m,n-valid iff the singleton $\{\neg A\}$ is not $\mathcal{T}$m,n-satisfiable.

## 5. *On the proof theory of discrete tense logic with beginning and ending time: the axiomatic systems $\mathcal{T}m,n$*

For any natural numbers $m,n \geq 0$, the system $\mathcal{T}m,n$ is determined by the following rule of inference, rule of proof, and axiom schemata:

*Rule of inference*

R0 (*modus ponens*) $\dfrac{A,\ A \rightarrow B}{B}$

*Rule of proof*

R2 (universal necessitation) $\dfrac{A}{\boxminus A}$

*Remark.* For the distinction between a *rule of inference* and a *rule of proof*, see e.g. Åqvist (1992), §5 and Sundholm (1983).

*Axiom schemata*

A0. All truth-functional tautologies (over $W_{m,n}$).

A1. (a) $oe^i(H\neg a_i \wedge a_i \wedge G\neg a_i)$, if $0 \leq i \leq n$
(b) $ow^{-i}(H\neg a_i \wedge a_i \wedge G\neg a_i)$, if $0 > i \geq -m$

(c) $a_i \rightarrow ea_{i+1}$, if $-m \leq i < n$
(d) $a_i \rightarrow wa_{i-1}$, if $-m < i \leq n$

(e) $a_i \wedge A \rightarrow oe^iA$, if $0 \leq i \leq n$
(f) $a_i \wedge A \rightarrow ow^{-i}A$, if $0 > i \geq -m$

(g) $a_{-m} \vee a_{-m+1} \vee \ldots \vee a_{-1} \vee a_0 \vee a_1 \vee \ldots \vee a_{n-1} \vee a_n$.
(h) $a_i \rightarrow \neg a_j$, for all integers $i,j \in [-m,n]$ with $i \neq j$.

*Remark.* In A1(a)-(f) i is assumed to be any integer in $[-m,n]$. And, as to the "exponents", we assume that, for each natural number $k \geq 0$, $e^k/w^k/$ is the k-termed sequence $\underbrace{e.....e}_{k\ \text{times}}$ $/\underbrace{w.....w}_{k\ \text{times}}/$ of occurrences of the operator e/w/. (Define $e^k/w^k/$ recursively, if you like).

A2. (a) $\boxminus a \rightarrow oA$
(b) $oA \leftrightarrow \neg o \neg A$

(c) $o(A \rightarrow B) \rightarrow (oA \rightarrow oB)$
(d) $oA \rightarrow \boxminus oA$.

A3. (a) $\boxminus A \rightarrow eA$; $\boxminus A \rightarrow wA$
(b) $\neg e \neg A \rightarrow eA$; $\neg w \neg A \rightarrow wA$
(c) $\neg a_n \rightarrow (eA \rightarrow \neg e \neg A)$; $\neg a_{-m} \rightarrow (wA \rightarrow \neg w \neg A)$
(d) $a_n \rightarrow e\bot$; $a_{-m} \rightarrow w\bot$
(e) $e(A \rightarrow B) \rightarrow (eA \rightarrow eB)$; $w(A \rightarrow B) \rightarrow (wA \rightarrow wB)$
(f) $\neg a_n \rightarrow (ewA \leftrightarrow A)$; $\neg a_{-m} \rightarrow (weA \leftrightarrow A)$

A4. (a) $\boxminus A \rightarrow GA$; $\boxminus A \rightarrow HA$
(b) $FA \leftrightarrow \neg G \neg A$; $PA \leftrightarrow \neg H \neg A$

(c)

$$a_{-m} \rightarrow (GA \leftrightarrow eA \wedge e^2 A \wedge \ldots \wedge e^{n+m} A) \qquad [a_{-m} \rightarrow (GA \leftrightarrow \bigwedge_{k=1}^{n+m} e^k A)]$$

$$a_{-m+1} \rightarrow (GA \leftrightarrow eA \wedge e^2 A \wedge \ldots \wedge e^{n+m-1} A) \qquad [a_{-m+1} \rightarrow (GA \leftrightarrow \bigwedge_{k=1}^{n+m-1} e^k A)]$$

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$$a_{-1} \rightarrow (GA \leftrightarrow eA \wedge e^2 A \wedge \ldots \wedge e^{n+1} A) \qquad [a_{-1} \rightarrow (GA \leftrightarrow \bigwedge_{k=1}^{n+1} e^k A)]$$

$$a_0 \rightarrow (GA \leftrightarrow eA \wedge e^2 A \wedge \ldots \wedge e^{n} A) \qquad [a_0 \rightarrow (GA \leftrightarrow \bigwedge_{k=1}^{n} e^k A)]$$

$$a_1 \rightarrow (GA \leftrightarrow eA \wedge e^2 A \wedge \ldots \wedge e^{n-1} A) \qquad [a_1 \rightarrow (GA \leftrightarrow \bigwedge_{k=1}^{n-1} e^k A)]$$

.

.

$$a_{n-1} \rightarrow (GA \leftrightarrow eA) \qquad [a_{n-1} \rightarrow (GA \leftrightarrow \bigwedge_{k=1}^{n-(n-1)} e^k A)]$$

$$a_n \rightarrow GA$$

(d)

$$a_n \to (HA \leftrightarrow wA \wedge w^2A \wedge \ldots \wedge w^{m+n}A) \qquad [a_n \to (HA \leftrightarrow \bigwedge_{k=1}^{m+n} w^kA)]$$

$$a_{n-1} \to (HA \leftrightarrow wA \wedge w^2A \wedge \ldots \wedge w^{m+n-1}A) \qquad [a_{n-1} \to (HA \leftrightarrow \bigwedge_{k=1}^{m+n-1} w^kA)]$$

.

.

.

$$a_1 \to (HA \leftrightarrow wA \wedge w^2A \wedge \ldots \wedge w^{m+1}A) \qquad [a_1 \to (HA \leftrightarrow \bigwedge_{k=1}^{m+1} w^kA)]$$

$$a_0 \to (HA \leftrightarrow wA \wedge w^2A \wedge \ldots \wedge w^{m}A) \qquad [a_0 \to (HA \leftrightarrow \bigwedge_{k=1}^{m} w^kA)]$$

$$a_{-1} \to (HA \leftrightarrow wA \wedge w^2A \wedge \ldots \wedge w^{m-1}A) \qquad [a_{-1} \to (HA \leftrightarrow \bigwedge_{k=1}^{m-1} w^kA)]$$

.

.

.

$$a_{-m+1} \to (HA \leftrightarrow wA) \qquad [a_{-m+1} \to (HA \leftrightarrow \bigwedge_{k=1}^{m-(m-1)} w^kA)]$$

$$a_{-m} \to HA$$

A5. (a) $\boxminus A \to A$
(b) $\diamondminus A \leftrightarrow \neg \boxminus \neg A$
(c) $GA \wedge A \wedge HA \to \boxminus A$.

*Remark.* In the axiom schemata A4(c) and (d), we adopt and illustrate within square brackets a notation for *finite conjunctions* that should be familiar and self-explanatory.

As usual, the above axiom schemata and rules determine syntactic notions of $\mathcal{T}m,n$-*provability* and $\mathcal{T}m,n$-*deducibility* as follows. We say that a wff A is $\mathcal{T}m,n$-*provable* [in symbols: $\vdash_{\mathcal{T}m,n} A$, or just $\vdash A$, when there is no risk of confusion] iff A belongs to the smallest subset of $W_{m,n}$, which (i) contains every instance of A0, A1(a)-(h), ..., A5(a)-(c) as its members, and which (ii) is closed under the rule of inference R0 and the rule of proof R2. Again, we say that the wff A is $\mathcal{T}m,n$-*deducible from* the set $\Gamma$ ($\subseteq W_{m,n}$) of *assumptions* [in symbols: $\Gamma \vdash_{\mathcal{T}m,n} A$] iff there are wffs $B_1, \ldots, B_k$

in Γ, for some natural number $k \geq 0$, such that $\vdash_{\mathcal{S}m,n} (B_1 \wedge \ldots \wedge B_k) \rightarrow A$ (i.e. the wff $(B_1 \wedge \ldots \wedge B_k) \rightarrow A$ is to be $\mathcal{S}$m,n-provable in the sense of the preceding definition).

Moreover, letting $\Gamma \subseteq W_{m,n}$, we say that Γ is $\mathcal{S}$m,n-*inconsistent* iff $\Gamma \vdash_{\mathcal{S}m,n} \bot$, and $\mathcal{S}$m,n-*consistent* otherwise (i.e. iff $\Gamma \nvdash_{\mathcal{S}m,n} \bot$). Finally, we say that Γ is *maximal consistent* (in $\mathcal{S}$m,n) iff Γ is $\mathcal{S}$m,n-consistent and, for each wff A in $W_{m,n}$, either $A \in \Gamma$ or $\neg A \in \Gamma$; where this latter condition is known as requiring Γ to be *negation-complete*.

## 6. *Some lemmata on the axiomatic systems $\mathcal{S}$m,n*

6.1. LEMMA (Useful Properties of $\mathcal{S}$m,n-deducibility and $\mathcal{S}$m,n-provability).

Let Γ,Δ be any subsets of $W_{m,n}$ and let A,B be any members of $W_{m,n}$. Then, by our definitions of $\Gamma \vdash A$ and $\vdash A$ above (dropping the indices m,n), those relations satisfy *inter alia* the following properties:

(P0) If $\Gamma \vdash A$ and $\Gamma \vdash A \rightarrow B$, then $\Gamma \vdash B$.
(P2) If $\vdash A$, then $\vdash \boxminus A$.
(P3) $\vdash A$ iff $\varnothing \vdash A$.
(P4) If $A \in \Gamma$, then $\Gamma \vdash A$.
(P5) If $\Gamma \vdash A$, then $\Gamma \cup \Delta \vdash A$.
(P6) If $\Gamma \vdash A$, then there is a *finite* subset $\Gamma_0$ of Γ such that $\Gamma_0 \vdash A$.

*Proof.* Immediate by the definitions of $\mathcal{S}$m,n-deducibility and $\mathcal{S}$m,n-provability. Note (i) that the property (P2), corresponding to the rule of proof R2, *cannot* be strengthened so as to have the import: if $\Gamma \vdash A$, then $\Gamma \vdash \boxminus A$, and (ii) that, in the terminology of Bull & Segerberg (1984), the property (P6) is known as syntactic (or proof-theoretical) *compactness*. See my Åqvist (1992), §6, Lemma 6.1, where we also point out that (P6) does not hold in the infinitary system $\mathcal{S}\mathcal{T}$ dealt with in that paper.

6.2. LEMMA (Deduction Theorem for $\mathcal{S}$m,n).

Let Γ, A, B be as in the hypothesis of Lemma 6.1. If $\Gamma \cup \{A\} \vdash B$, then $\Gamma \vdash A \rightarrow B$.

*Proof.* Again immediate by the definition of $\mathcal{T}$m,n-deducibility.

6.3. LEMMA (Derived Rules of Proof).

Let § be any of the six operators o, e, w, G, H and $\boxminus$. Then, in any system $\mathcal{T}$m,n, the following are *derived* rules of *proof*:

DR2 (§-necessitation) $\dfrac{A}{\S A}$

DR3 (§-monotonicity) $\dfrac{A \to B}{\S A \to \S B}$

DR4 (§-congruence) $\dfrac{A \leftrightarrow B}{\S A \leftrightarrow \S B}$

*Proof.*
*Ad* DR2. Use R2 together with A2(a), A3(a) and A4(a) in the respective cases of § = o, e, w, G, H. For the case § = $\boxminus$, just use R2 alone.
*Ad* DR3. Use DR2 as just derived together with axiom schemata A2(c) and A3(e) in the cases § = o and § = e, w. For the cases § = G, § = H and § = $\boxminus$, use T9, T10 and T18 in the next Lemma, respectively.
*Ad* DR4. Immediate from DR3.

6.4. LEMMA (Useful Theorem Schemata).

All instances of the following *theorem schemata* are $\mathcal{T}$m,n-provable.

T0. $a_{-m} \to (e^kA \to \neg e^k\neg A)$, for all k with $0 < k \leq m+n$;
$a_n \to (w^kA \to \neg w^k\neg A)$, for all k with $0 < k \leq n+m$

T1. $oA \leftrightarrow \boxminus oA$

T2. $oA \leftrightarrow (o\neg A \to \bot)$

T3. $\boxminus A \to oe^nA$; $\boxminus A \to ow^mA$

T4. $oe^nA \leftrightarrow \neg oe^n\neg A$; $ow^mA \leftrightarrow \neg ow^m\neg A$

T4.1. $oe^nA \leftrightarrow (oe^n\neg A \to \bot)$; $ow^mA \leftrightarrow (ow^m\neg A \to \bot)$

T5. $\neg oe^n\bot$; $\neg ow^m\bot$

T6. $\neg a_n \to (e\top \to \neg e\bot)$; $\neg a_{-m} \to (w\top \to \neg w\bot)$

$$\text{T7.}\quad \begin{cases} a_{-m} \rightarrow (e^{k}\, \neg a_{-m} \rightarrow (e^{k} a_{-m} \rightarrow \bot)), & \text{for all } k\text{: } 0<k\leq n+m \\ a_{-m+1} \rightarrow (e^{k}\, \neg a_{-m+1} \rightarrow (e^{k} a_{-m+1} \rightarrow \bot)), & \text{for all } k\text{: } 0<k\leq n+m-1 \\ \cdot \\ \cdot \\ \cdot \\ a_{0} \rightarrow (e^{k}\, \neg a_{0} \rightarrow (e^{k} a_{0} \rightarrow \bot)), & \text{for all } k\text{: } 0<k\leq n \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1} \rightarrow (e\, \neg a_{n-1} \rightarrow (e a_{n-1} \rightarrow \bot)). \end{cases}$$

$$\text{T8.}\quad \begin{cases} a_{n} \rightarrow (w^{k}\, \neg a_{n} \rightarrow (w^{k} a_{n} \rightarrow \bot)), & \text{for all } k\text{: } 0<k\leq m+n \\ a_{n-1} \rightarrow (w^{k}\, \neg a_{n-1} \rightarrow (w^{k} a_{n-1} \rightarrow \bot)), & \text{for all } k\text{: } 0<k\leq m+n-1 \\ \cdot \\ \cdot \\ \cdot \\ a_{0} \rightarrow (w^{k}\, \neg a_{0} \rightarrow (w^{k} a_{0} \rightarrow \bot)), & \text{for all } k\text{: } 0<k\leq m \\ \cdot \\ \cdot \\ \cdot \\ a_{-m+1} \rightarrow (w\, \neg a_{-m+1} \rightarrow (w a_{-m+1} \rightarrow \bot)). \end{cases}$$

T9. $G(A \rightarrow B) \rightarrow (GA \rightarrow GB)$

T10. $H(A \rightarrow B) \rightarrow (HA \rightarrow HB)$

T11.

$$
\begin{cases}
a_{-m} \rightarrow (FA \leftrightarrow \bigvee_{k=1}^{n+m} e^{k}A) \\
a_{-m+1} \rightarrow (FA \leftrightarrow \bigvee_{k=1}^{n+m-1} e^{k}A) \\
\cdot \\
\cdot \\
\cdot \\
a_{-1} \rightarrow (FA \leftrightarrow \bigvee_{k=1}^{n+1} e^{k}A) \\
a_{0} \rightarrow (FA \leftrightarrow \bigvee_{k=1}^{n} e^{k}A) \\
a_{1} \rightarrow (FA \leftrightarrow \bigvee_{k=1}^{n-1} e^{k}A) \\
\cdot \\
\cdot \\
\cdot \\
a_{n-1} \rightarrow (FA \leftrightarrow eA) \\
a_{n} \rightarrow \neg F \neg A
\end{cases}
$$

T12.

$$
\begin{cases}
a_{n} \rightarrow (PA \leftrightarrow \bigvee_{k=1}^{m+n} w^{k}A) \\
a_{n-1} \rightarrow (PA \leftrightarrow \bigvee_{k=1}^{m+n-1} w^{k}A) \\
\cdot \\
\cdot \\
\cdot \\
a_{0} \rightarrow (PA \leftrightarrow \bigvee_{k=1}^{m} w^{k}A) \\
\cdot \\
\cdot \\
\cdot \\
a_{-m+1} \rightarrow (PA \leftrightarrow wA) \\
a_{-m} \rightarrow \neg P \neg A
\end{cases}
$$

(The notation for *finite disjunctions*, employed in T11 and T12, should be familiar and self-explanatory).

T13. $PGA \rightarrow A$; $FHA \rightarrow A$

T14. $GA \rightarrow GGA$; $HA \rightarrow HHA$

T15. $(HA \wedge A \wedge GA) \rightarrow GHA$; $(HA \wedge A \wedge GA) \rightarrow HGA$

T16. $\boxminus A \leftrightarrow (HA \wedge A \wedge GA)$

T17. $\diamondminus A \leftrightarrow (PA \vee A \vee FA)$

T18. $\boxminus (A \rightarrow B) \rightarrow (\boxminus A \rightarrow \boxminus B)$

T19. $\boxminus A \rightarrow \boxminus \boxminus A$

T20. $\diamondminus \boxminus A \rightarrow A$

T21. $\diamondminus \boxminus A \rightarrow \boxminus \boxminus A$

*Proof.* Elementary, but tedious. A detailed proof is available from the author of this paper.

6.5. LEMMA (Scott's Rule for $\mathscr{T}$m,n; terminology of Bull & Segerberg (1984)).

Let $\Gamma$ be a set of wffs ($\subseteq W_{m,n}$) and let A be a wff ($\in W_{m,n}$). Let § be any of the six operators o, e, w, G, H, $\boxminus$ (all of which are "necessity modalities" in an obvious sense). Then: if $\Gamma \vdash A$, then $\{\S B: B \in \Gamma\} \vdash \S A$.
*Proof.* Suppose that $\Gamma \vdash A$. By the definition of $\mathscr{T}$m,n-deducibility we have, for some natural number $k \geq 0$ and some wffs $B_1, \ldots, B_k$ in $\Gamma$,

$$\vdash (B_1 \wedge \ldots \wedge B_k) \rightarrow A$$

which result is by A0 equivalent to

$$\vdash B_1 \rightarrow (B_2 \rightarrow \ldots (B_k \rightarrow A) \ldots).$$

Consider first the case where $k = 0$. This means that $\vdash A$. Hence, by the rule DR2 of §-necessitation (or by R2, when $\S = \boxminus$), we obtain that $\vdash \S A$. Hence, by property (P5) in Lemma 6.1 [with $\Delta = \{\S B: B \in \Gamma\}$], the desired result is immediate.
Consider next the case where $k \geq 1$. By DR2 (or by R2), we then get

$$\vdash \S(B_1 \rightarrow (B_2 \rightarrow \ldots (B_k \rightarrow A) \ldots))$$

Hence, using axiom schemata A2(c) and A3(e) in the cases $\S = o$ and

§ = e, w and theorem schemata T9, T10 and T18 in the cases § = G, H, ⊟, we obtain, after a sufficient number of applications of these schemata,

$$\vdash \S B_1 \rightarrow (\S B_2 \rightarrow \ldots (\S B_k \rightarrow \S A) \ldots)$$

which result is by A0 equivalent to

$$\vdash (\S B_1 \wedge \S B_2 \wedge \ldots \wedge \S B_k) \rightarrow \S A.$$

Since each $\S B_i$ ($1 \leq i \leq k$) is such that $B_i \in \Gamma$, this last result obviously amounts to the desired one. Q.E.D.

*Corollary*. Suppose that $\Gamma \vdash A$. Then $\{ow^m B: B \in \Gamma\} \vdash ow^m A$ as well as $\{oe^n B: B \in \Gamma\} \vdash oe^n A$.

*Proof.* By repeated applications of Lemma 6.5, leaving the details to the reader.

6.6. LEMMA (Familiar Properties of Maximal $\mathscr{T}$m,n-Consistent Sets).

Let Γ be any maximal consistent (in $\mathscr{T}$m,n) set of wffs. Then, for all wffs A, B:

(1) If $\vdash A$, then $A \in \Gamma$.
(2) If $A \in \Gamma$ and $A \rightarrow B \in \Gamma$, then $B \in \Gamma$ (Γ closed under R0).
(3) $\top \in \Gamma$ and $\bot \notin \Gamma$.
(4) $\neg A \in \Gamma$ iff $A \notin \Gamma$.
(5) $A \wedge B \in \Gamma$ iff $A \in \Gamma$ and $B \in \Gamma$.
(6) $A \vee B \in \Gamma$ iff $A \in \Gamma$ or $B \in \Gamma$ (or both).
(7) $A \rightarrow B \in \Gamma$ iff if $A \in \Gamma$ then $B \in \Gamma$.
(8) $A \leftrightarrow B \in \Gamma$ iff $A \in \Gamma$ if and only if $B \in \Gamma$.
(9) $a_i \in \Gamma$, for exactly one i in the closed interval [-m,n] of integers.

*Proof.* Clauses (1)-(8) are established as usual. To deal with (9), we first observe that the disjunction A1(g) is in Γ (by (1)), so that, by (6), at least one of its disjuncts must be in Γ. Hence the existence part of (9). The uniqueness of "that" disjunct is then immediate by axiom schema A1(h) *supra*.

6.7. DEFINITION (The Lindenbaum Extension of a $\mathcal{A}$m,n-Consistent Set).

Let $\Gamma$ be any $\mathcal{A}$m,n-consistent set of wffs, and let $\langle A_0, A_1, \ldots\rangle$ be an enumeration of the set $W_{m,n}$ of all wffs of the system $\mathcal{A}$m,n. Define an infinite sequence $\langle \Gamma_k \rangle_{0 \leq k < \omega}$ by the recursion:

$$\Gamma_0 = \Gamma$$

$$\Gamma_{k+1} = \begin{cases} \Gamma_k \cup \{A_k\}, & \text{if this is consistent} \\ \Gamma_k \cup \{\neg A_k\}, & \text{otherwise.} \end{cases}$$

Then, define

$$\Gamma_\omega = \bigcup_{0 \leq k < \omega} \Gamma_k \; (= \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \ldots)$$

where, for future reference, we call $\Gamma_\omega$ the *Lindenbaum extension* of $\Gamma$.

6.8. THEOREM (Lindenbaum's Lemma for $\mathcal{A}$m,n).

Let $\Gamma$ be a $\mathcal{A}$m,n-consistent set of wffs. Then, as just defined, $\Gamma_\omega$ is maximal consistent (in $\mathcal{A}$m,n).

*Proof.* Familiar. If not, see Åqvist (1992), §8. The proof there given can be considerably simplified, since our present systems $\mathcal{A}$m,n, unlike $\mathcal{A}$, lack infinitary rules of inference.

## 7. *Canonical $\mathcal{A}$m,n-structures: a fundamental result*

7.1. DEFINITION. For any natural numbers m,n $\geq$ 0, let $U^{\mathcal{A}m,n}$ be the set of all maximal consistent (in $\mathcal{A}$m,n) sets of wffs. Let x be a fixed element of $U^{\mathcal{A}m,n}$. We now define the *canonical* $\mathcal{A}$m,n-*structure generated by* x as the structure

$$\mathcal{M}^x = ((U, (w_0, u_0, e_0), (m,n), E, W), V)$$

where

(i) $U = \{u \in U^{\mathcal{T}m,n}$: for each wff A, if $\boxminus A \in x$, then $A \in u\}$
(ii) $u_0 = \{A: oA \in x\}$
$w_0 = \{A: ow^m A \in x\}$
$e_0 = \{A: oe^n A \in x\}$
(iii) m,n are the natural numbers under consideration
(iv) E and W are the functions on U defined by setting,
for each $u \in U$: $Eu = \{A: eA \in u\}$, and
$Wu = \{A: wA \in u\}$.
(v) $V(p_i) = \{u \in U: p_i \in u\}$ (for all propositional variables $p_i$, where $i = 0, 1, 2, \ldots$).

*Remark*. Let us recall here our previous definitions (make them recursive, if you like) of the "powers" $E^k$, $W^k$ and $e^k$, $w^k$ of the functions/operators E, W and e, w [given in the *Explanation* in §3 *supra* and in the *Remark* under the axiom schemata A1(a)-(h) in §5, respectively]. In the light of them we obtain from clauses (ii) and (iv) in Definition 7.1 just laid down results like the following:

$E^k u = \{A: e^k A \in u\}$
$W^k u = \{A: w^k A \in u\}$
$E^k u_0 = \{A: oe^k A \in x\}$
$W^k u_0 = \{A: ow^k A \in x\}$

for all $u \in U$ and all natural numbers k ($\geq 0$). We leave the verification to the reader and go on to state, and prove, the following fundamental result concerning generated canonical $\mathcal{T}$m,n-structures.

7.2. THEOREM. As just defined, $\mathcal{M}^x$ is a $\mathcal{T}$m,n-model and its first term $(U, (w_0, u_0, e_0), (m,n), E, W)$ is a $\mathcal{T}$m,n-frame.

*Proof*. The crucial part of this Theorem is the second one, for, once we have proved the first term of $\mathcal{M}^x$ to be a frame, it is obvious that $\mathcal{M}^x$ as a whole is a model (leaving the reference to $\mathcal{T}$m,n tacit in the sequel). What we have to do, then, is essentially the following. First, we must show that, as defined, the designated sets $u_0$, $w_0$ and $e_0$ are members of U. Secondly, we must show that, as defined, the functions E and W satisfy the conditions (C1E), (C1W) through (C8E), (C8W) in the definition of a $\mathcal{T}$m,n-frame.

*(I) $u_0$, $w_0$ and $e_0$ all belong to U.* Starting with the case of $u_0$, our task is to prove that {A: oA ∈ x} is

(a) consistent,
(b) maximal in the sense of being negation-complete, and
(c) "x-adequate" in the sense that for all wffs A, if ⊟ A ∈ x, then A ∈ {A: oA ∈ x} (= $u_0$) [cf. clause (i) in the above definition of the canonical 𝒯m,n-structure *generated by* x].

*Ad* (a). Assume that {A: oA ∈ x} is not consistent, i.e. that

| | |
|---|---|
| 1. {A: oA ∈ x} ⊢ ⊥ | counterassumption |
| 2. {oA: A ∈ {A: oA ∈ x}} ⊢ o⊥ | from 1 by Lemma 6.5 with § = o |
| 3. {oA: oA ∈ x} ⊢ o⊥ | simplifying 2 |
| 4. x ⊢ o⊥ | from 3 by the fact that {oA: oA ∈ x} ⊆ x |
| 5. ⊢o⊤, ⊢o¬⊥, ⊢ ¬o⊥ | DR2-DR4 with § = o, A2(b) |
| 6. x ⊢ ¬o⊥ | immediate from the third item in 5 |

where 4 and 6 contradict the consistency of x. Hence, $u_0$ (= {A: oA ∈ x}) is consistent.

*Ad* (b). Assume that {A: oA ∈ x} is not negation-complete, i.e. that

| | |
|---|---|
| 1. $\left\{\begin{array}{l} oB \notin x[B \notin u_0] \\ o\neg B \notin x[\neg B \notin u_0] \end{array}\right\}$ for some wff B | counterassumption |
| 2. ¬oB ∈ x and ¬o¬B ∈ x | from 1 by the maximal consistency of x ∈ U |
| 3. oB ∈ x | from 2 [2nd conjunct] by A2(b) |

where 3 and the first conjunct in 2 contradict the consistency of x. Hence, $u_0$ is maximal.

*Ad* (c). We are to show that if ⊟ A ∈ x, then A ∈ $u_0$, i.e. oA ∈ x. So,

| | |
|---|---|
| 1. ⊟ A ∈ x | assumption |
| 2. ⊟ A → oA ∈ x | axiom schema A2(a), x max cons |
| 3. oA ∈ x | x closed under R0, from 1, 2 |

where 3 is our desired conclusion. This completes the proof that $u_0$ belongs to U.

We next proceed to the case of the "western limit" $w_0$.

*Ad* (a). Assume that $\{A: ow^m A \in x\}$ is not consistent, i.e. that

| | | |
|---|---|---|
| 1. | $\{A: ow^m A \in x\} \vdash \bot$ | counterassumption |
| 2. | $\{ow^m A: A \in \{A: ow^m A \in x\}\} \vdash ow^m \bot$ | from 1 by the Corollary of Lemma 6.5 |
| 3. | $\{ow^m A: ow^m A \in x\} \vdash ow^m \bot$ | simplifying 2 |
| 4. | $x \vdash ow^m \bot$ | by 3 and the fact that $\{ow^m A: ow^m A \in x\} \subseteq x$ |
| 5. | $\vdash \neg ow^m \bot$ | by T5 of Lemma 6.4 |
| 6. | $x \vdash \neg ow^m \bot$ | immediate from 5 (by (P3), (P5) in Lemma 6.1) |

where 4 and 6 contradict the consistency of x. Hence, $w_0$ (= $\{A: ow^m A \in x\}$) is consistent.

*Ad* (b). Assume that $\{A: ow^m A \in x\}$ is not negation-complete, i.e. that

| | | |
|---|---|---|
| 1. | $\left\{\begin{array}{l} ow^m B \notin x [B \notin w_0] \\ ow^m \neg B \notin x [\neg B \notin w_0] \end{array}\right\}$ for some wff B | counterassumption |
| 2. | $\neg ow^m B \in x$ and $\neg ow^m \neg B \in x$ | from 1 by the maximal consistency of x |
| 3. | $ow^m B \in x$ | from the second conjunct in 2 by T4 in Lemma 6.4 |

where 3 and the first conjunct in 2 contradict the consistency of x. Hence, $w_0$ is maximal.

*Ad* (c). We want to show that if $\boxminus A \in x$, then $A \in w_0$, i.e. $ow^m A \in x$.

| | | |
|---|---|---|
| 1. | $\boxminus A \in x$ | assumption |
| 2. | $\boxminus A \rightarrow ow^m A \in x$ | T3 in Lemma 6.4, x max cons |
| 3. | $ow^m A \in x$ | from 1, 2 by the fact that x is closed under R0, x being max cons |

where 3 is our desired conclusion.

This completes the proof that $w_0$ belongs to U. The case of the "eastern limit" $e_0$ is handled similarly —just appeal to the Corollary of Lemma 6.5 and appropriate items in Lemma 6.4.

For the remainder of our proof of Theorem 7.2 (as well as for that of the Coincidence Lemma to be dealt with in the next Section) we need the following

7.3. LEMMA. Let u,v be any members of U, let i be any member of the closed interval of integers [-m,n]. Then:

(i) For each wff A, $\boxminus A \in u$ iff $\boxminus A \in v$.

(ii) $a_i \in u$ iff $\begin{cases} u = E^i u_0, & \text{if } 0 \le i \le n \\ u = W^{-i} u_0, & \text{if } 0 > i \ge -m \end{cases}$

(iii) $a_n \in u$ iff $u = e_0$; $\neg a_n \in u$ iff $u \neq e_0$.
(iv) $a_{-m} \in u$ iff $u = w_0$; $\neg a_{-m} \in u$ iff $u \neq w_0$.

*Proof.*
*Ad* (i). Beginning with the left-to-right direction, we have:

| | | |
|---|---|---|
| 1. | $\boxminus A \in u$ | assumption |
| 2. | $\diamondminus \boxminus A \in x$ | from 1 by an equivalent formulation of the condition of x-adequacy |
| 3. | $\boxminus\boxminus A \in x$ | from 2 by T21 in Lemma 6.4 |
| 4. | $\boxminus A \in v$ | from 3 by the x-adequacy of $v \in U$ |

where 4 is our desired result. The converse direction is handled similarly.
*Ad* (ii). Starting with the left-to-right direction, suppose that some i with $0 \le i \le n$ and some $u \in U$ are such that

| | | |
|---|---|---|
| 1. | $a_i \in u$, whereas | counterassumption |
| 2. | $u \neq E^i u_0$ | |
| 3. | $A \in u$ and $oe^i A \notin u$ (for some A) | from 2 by the definition of $E^i u_0$ [see the Remark under Definition 7.1], T1 in Lemma 6.4 and clause (i) of the present Lemma; $x, u \in U$ |

| | |
|---|---|
| 4. $oe^iA \in u$ | from 1 and 3 [1st conjunct] by A1(e), since $u \in U$ so that u max cons |

where 4 contradicts the second conjunct in 3. Suppose next that, for some i with $0 > i \geq -m$ and some $u \in U$, we have

| | |
|---|---|
| 1. $a_i \in u$, whereas<br>5. $u \neq W^{-i}u_0$ | counterassumption |
| 6. $A \in u$ and $ow^{-i}A \notin u$ (some A) | from 5 by the definition of $W^{-i}u_0$ etc. |
| 7. $ow^{-i}A \in u$ | from 1 and 6 [1st conjunct] by A1(f), $u \in U$ |

where 7 contradicts the second conjunct in 6.

To handle the right-to-left direction, suppose that $0 \leq i \leq n$ and that

| | |
|---|---|
| 1. $u = E^iu_0$ | assumption |

Then (with a view to showing that $a_i \in u$):

| | |
|---|---|
| 2. $u = \{A: oe^iA \in x\}$ | from 1 by definition [Remark under Definition 7.1 *supra*] |
| 3. $u = \{A: oe^iA \in u\}$ | from 2 by T1 [Lemma 6.4] and clause (i) of the present Lemma; u, $x \in U$ |
| 4. For all A, $A \in u$ iff $oe^iA \in u$ | immediate from 3 |
| 5. $a_i \in u$ iff $oe^ia_i \in u$ | from 4 by instantiation |
| 6. $oe^ia_i \in u$ | by axiom schema A1(a), u max cons |
| 7. $a_i \in u$ | from 5 and 6 |

where 7 is our desired result.

The remaining case where $0 > i \geq -m$ goes through by analogous reasoning: just appeal to axiom schema A1(b) in the crucial step 6!

*Ad* (iii) and (iv). These clauses are easily obtained as special cases of (ii): just use the definitions of $e_0$ and $w_0$, while bearing in mind our Remark under Definition 7.1 above.

The proof of Lemma 7.3 is complete. Armed with this Lemma, we now deal with our remaining task, viz. to establish the following fact:

(II) As defined in clause (iv) of Definition 7.1 *supra*, E and W are *partial functions* on U *satisfying conditions* (C1E), (C1W) through (C8E), (C8W) on a $\mathcal{T}$m,n-frame (§3 *supra*).

*Ad* (C1E). We are to show that, for each u $\in$ U with u $\neq e_0$, {A: eA $\in$ u} (= Eu) is consistent, maximal and x-adequate. Furthermore, we must show that {A: $oe^{n+1}$A $\in$ x} (= $Ee_0$) does not belong to U. We deal with the first task here as follows, beginning with consistency. Suppose that u $\neq e_0$ and

| | |
|---|---|
| 1. {A: eA $\in$ u} $\vdash \bot$ | counterassumption |
| 2. {eA: eA $\in$ u} $\vdash$ e$\bot$ | from 1 by Lemma 6.5 with § = e |
| 3. {eA: eA $\in$ u} $\subseteq$ u | immediate |
| 4. u $\vdash$ e$\bot$ | from 2,3 by (P5) in Lemma 6.1 |

On the other hand, we have:

| | |
|---|---|
| 5. $\vdash$e$\top$ and e$\top$ $\in$ u | R2, A3(a); u max cons |
| 6. $\neg a_n \to (e\top \to \neg e\bot)$ | by T6 in Lemma 6.4 |
| 7. $\neg a_n \in$ u | by clause (iii) of Lemma 7.3, since u $\neq e_0$ *ex hypothesi* |
| 8. $(e\top \to \neg e\bot) \in$ u | from 6,7 since u is closed under R0 |
| 9. $\neg$ e$\bot$ $\in$ u | from the second conjunct in 5 and 8, u closed under R0 |
| 10. u $\vdash \neg$e$\bot$ | from 9 by property (P4) [Lemma 6.1] |

where 4 and 10 contradict the consistency of u (u $\in$ U). Hence, the counterassumption 1 is reduced *ad absurdum* and Eu is seen to be consistent.

As for maximality, assume that {A: eA $\in$ u} is not negation-complete, where still u $\neq e_0$. Then, for some wff B we have

| | |
|---|---|
| 1. eB $\notin$ u [B $\notin$ Eu] | counterassumption |
| 2. e$\neg$B $\notin$ u [$\neg$B $\notin$ Eu] | (counterassumption, with 1) |
| 3. $\neg$eB $\in$ u | by 1 and u max cons |
| 4. $\neg$e$\neg$B $\in$ u | by 2 and u max cons |
| 5. eB $\in$ u | from 4 by A3(b) [first item] |
| 6. u $\vdash \bot$ | from 3,5 by a tautology under A0; etc. |

where 6 contradicts the consistency of u. Hence, Eu is maximal.
Finally, as to x-adequacy, assume that for any wff A:

| | |
|---|---|
| 1. $\boxminus A \in x$ | assumption |
| 2. $\boxminus A \in u$ | from 1 by clause (i) of Lemma 7.3; x, u $\in$ U |
| 3. $eA \in u$ | from 2 by A3(a) [first item], u max cons |

so that $A \in Eu$, as desired.

It remains to show that $Ee_0$ (= $\{A: oe^{n+1}A \in x\}$) is *not* a member of U. Assume otherwise, i.e. that it is. Then:

| | |
|---|---|
| 1. $Ee_0 \nvdash \bot$ (i.e., is consistent) | by the conditions of membership in U |
| 2. $a_n \in e_0$ | by Lemma 7.3, clause (iii) |
| 3. $\vdash a_n \rightarrow e\bot$, $a_n \rightarrow e\bot \in e_0$ | by A3(d) [first item], $e_0 \in U$ |
| 4. $e\bot \in e_0$ | from 2,3 since $e_0 \in U$ and is closed under R0 |
| 5. $\bot \in Ee_0$, $Ee_0 \vdash \bot$ | from 4 by the definition of $Ee_0$ |

where 5 contradicts 1. Hence, our counterassumption is reduced *ad absurdum* and $Ee_0$ is seen *not* to belong to U.

This completes the verification that, as defined, E satisfies our condition (C1E).
*Ad* (C1W). In the light of the proof just given, this case presents no novelties and can be left to the reader: just make use of the right items in Lemmata 6.4, 6.5 and 7.3!

*Ad* (C2E). Suppose, contrary to (C2E), that some u, v $\in$ U are such that
1. $u \neq v$
2. $u \neq e_0$ and $v \neq e_0$, whilst
3. $Eu = Ev$

} counterassumption

Then:

| | |
|---|---|
| 4. For all wffs A, $eA \in u$ iff $eA \in v$ | from 3 by definition: $Eu = \{A: eA \in u\} = \{A: eA \in v\} = Ev$ |

| | |
|---|---|
| 5. $\neg a_n \in u$ and $\neg a_n \in v$ | from 2 by Lemma 7.3, clause (iii) |
| 6. $B \in u$ and $B \notin v$, for some wff B | immediate from 1 |
| 7. $ewB \in u$ iff $ewB \in v$ | from 4 by instantiation [$A = wB$] |
| 8. $ewB \in u$ | from 5 [1st conjunct] and 6 [1st conjunct] by axiom schema A3(f); $u \in U$ |
| 9. $ewB \in v$ | immediate from 7 and 8 |
| 10. $B \in v$ | from 5 [2nd conjunct] and 9 by A3(f), since $v \in U$ so that v max cons |

where 10 contradicts the second conjunct in 6. Hence Q.E.D.

*Ad* (C2W). The argument is a "mirror-image" of the preceding one: use the second schema in A3(f) in the crucial steps 8 and 10!

*Ad* (C3). Consider any $u, v \in U$ with $u \neq e_0$ and $v \neq w_0$: we are to show that $Eu = v$ iff $Wv = u$. Starting with the left-to-right direction, we assume for *reductio ad absurdum* that there exist $u,v \in U$ with

1. $\{B: eB \in u\} = v$ [$Eu = v$]
2. $\neg a_n \in u$ and $\neg a_{-m} \in v$ [$u \neq e_0$, $v \neq w_0$]

whilst, for some wff A:

3. $A \in Wv$, $A \notin u$ [$Wv \neq u$]

(1–3: counterassumption)

Then:

| | |
|---|---|
| 4. $wA \in v$ [$A \in \{B: wB \in v\}$] | from 3 [1st conjunct] by the definition of Wv |
| 5. $ewA \in u$ | from 1 and 4 |
| 6. $A \in u$ | from 2,5 by A3(f) [1st item] |

where 6 contradicts the second conjunct in 3. Hence Q.E.D.

To do the opposite direction, suppose there are $u, v \in U$ with

1. $\{B: wB \in v\} = u$ [$Wv = u$]

whilst, still assuming 2 above, we have for some wff A:

| | |
|---|---|
| 3. $A \in Eu$, $A \notin v$ [$Eu \neq v$] | counterassumption |

Then:

| | |
|---|---|
| 4. $eA \in u$ [$A \in \{B: eB \in u\}$] | from 3 [1st conjunct] by the definition of Eu |
| 5. $weA \in v$ | from 1 and 4 |
| 6. $A \in v$ | from 2,5 by A3(f) [2nd item] |

where 6 contradicts the second conjunct in 3. Hence the desired result.

*Ad* (C4) and (C5). The verification of these redundant conditions is left to the reader.

*Ad* (C6). The verification is immediate by the definition of $w_0$ and $e_0$ [clause (ii) in Definition 7.1 *supra*] and by the Remark under that definition [$k=m$ and $k=n$, as the case may be].
Finally, we deal with the somewhat more complicated remaining conditions (C7), (C8E) and (C8W).

*Ad* (C7) [finite accessibility from $u_0$]. Suppose, contrary to (C7), that some $u \in U$ is such that

| | |
|---|---|
| 1. $u \neq u_0$, and<br>2. $u \neq E^k u_0$, for all k with $0 < k \leq n$, and<br>3. $u \neq W^k u_0$, for all k with $0 < k \leq m$. | counterassumption |
| 4. Either $a_{-m} \in u$ or $a_{-m+1} \in u$ or...or $a_0 \in u$ or...or $a_{n-1} \in u$ or $a_n \in u$ | from axiom schema A1(g) by the maximal consistency of u ($\in$ U) |

We must now consider these $m+1+n$ cases in turn:

| | |
|---|---|
| 5. $a_{-m} \in u$ | assumption |
| 6. $A \in u$ and $A \notin W^m u_0$ (for some wff A) | from 3 with $k = m > 0$ |
| 7. $ow^m A \in u$ | from 5,6 [1st conjunct] by A1(f) [u max cons] |
| 8. $\neg A \in W^m u_0$ | from 6 [2nd conjunct] by the maximality of $W^m u_0$ ($= w_0 \in U$) |

| | |
|---|---|
| 9. $ow^m \neg A \in u$ | from 8 by the definition of $W^m u_0$, T1 [Lemma 6.4], clause (i) [Lemma 7.3] |
| 10. $\bot \in u$ | from 7,9 by T4.1 in Lemma 6.4 |

where 10 contradicts the consistency of $u \in U$.

Similar arguments take care of the cases where $a_{-m+1} \in u$, ..., $a_{-1} \in u$. We then proceed to the case:

| | |
|---|---|
| 11. $a_0 \in u$ | assumption |
| 12. $A \in u$ and $A \notin u_0$ (for some wff A) | from 1 |
| 13. $oA \in u$ | from 11, 12 [1st conjunct] by A1(e) [$i = 0$, u max cons] |
| 14. $\neg A \in u_0$ | from 12 [2nd conjunct] by the maximality of $u_0$ |
| 15. $o\neg A \in u$ | from 14 by the definition of $u_0$, T1 and clause (i) in Lemma 7.3 |
| 16. $\bot \in u$ | from 13 and 15 by T2 [Lemma 6.4] |

where 16 contradicts the consistency of $u \in U$.
Analogous arguments take care of the remaining cases; consider e.g.:

| | |
|---|---|
| 17. $a_n \in u$ | assumption |
| 18. $A \in u$ and $A \notin E^n u_0$ (for some A) | from 2 with $k = n > 0$ |
| 19. $oe^n A \in u$ | from 17, 18 [1st conjunct] by axiom schema A1(e) [u max cons] |
| 20. $\neg A \in E^n u_0$ | from 18 [2nd conjunct] by the maximality of $E^n u_0$ ($= e_0 \in U$) |
| 21. $oe^n \neg A \in u$ | from 20 by the definition of $E^n u_0$ and Lemma 7.3, clause (i); T1 [Lemma 6.4] |
| 22. $\bot \in u$ | from 19, 21 by T4.1 in Lemma 6.4 |

Thus, one and the same conclusion to the effect that $\bot \in u$ can be derived, whether we assume that $a_{-m} \in u$ or $a_{-m+1} \in u$ or... or $a_0 \in u$ or...

or $a_{n-1} \in u$ or $a_n \in u$. Hence, by a step of "disjunction elimination", the dependency of that conclusion on any of those m+1+n assumptions is eliminated and replaced by dependency on their disjunction, i.e. line 4, which in turn just depends on the initial counterassumption that some u ∈ U satisfies lines 1, 2 and 3. Hence, the conclusion that ⊥ ∈ u just so depends and, as already noticed, contradicts the consistency of u (∈ U). Therefore, this counterassumption to (C7) is reduced *ad absurdum*. This completes our verification that, as defined, the functions E and W satisfy condition (C7).

*Remark.* In the above proof we assumed, in lines 2 and 3 and in lines depending on them, that n,m > 0. In the three cases where n or m or both = 0, our proof can obviously be simplified — the reader should tell us how!

*Ad* (C8E) [E-analogue of Peano's fourth axiom]. (C8E), as formulated in §3 above, is clearly equivalent to the following statement:
for each $u \in U$ with $u \neq e_0$,

| | | |
|---|---|---|
| if $u=W^m u_0$ $(=w_0)$, | then $Eu \neq u$ and $E^2u \neq u$ and...and $E^{n+m}u \neq u$, | *and* |
| if $u=W^{m-1}u_0$, | then $Eu \neq u$ and $E^2u \neq u$ and...and $E^{n+m-1}u \neq u$, | *and* |
| . | | |
| . | | |
| . | | |
| if $u=u_0$, | then $Eu \neq u$ and $E^2u \neq u$ and...and $E^n u \neq u$, | *and* |
| if $u=Eu_0$, | then $Eu \neq u$ and $E^2u \neq u$ and...and $E^{n-1}u \neq u$, | *and* |
| . | | |
| . | | |
| . | | |
| if $u=E^{n-1}u_0$, | then $Eu \neq u$. | |

Thus, in the scope of the universal quantifier, we have a conjunction of n+m implications, the succedents of which form a decreasing series of conjunctions (the first having n+m conjuncts, the last just one).

Having premised this observation, we can now easily formulate our counterassumption to (C8E): it asserts that some $u \in U$ with $u \neq e_0$ is such that *either*

$$
1.\ \left\{
\begin{array}{l}
\quad u = W^{m}u_0 \text{ and } (Eu = u \text{ or } E^2u = u \text{ or} \ldots \text{ or } E^{n+m}u = u) \\
\text{or } u = W^{m-1}u_0 \text{ and } (Eu = u \text{ or } E^2u = u \text{ or} \ldots \text{ or } E^{n+m-1}u = u) \\
\text{or } . \\
\quad . \\
\quad . \\
\quad . \\
\text{or } u = u_0 \text{ and } (Eu = u \text{ or } E^2u = u \text{ or} \ldots \text{ or } E^{n}u = u) \\
\text{or } u = Eu_0 \text{ and } (Eu = u \text{ or } E^2u = u \text{ or} \ldots \text{ or } E^{n-1}u = u) \\
\text{or } . \\
\quad . \\
\quad . \\
\quad . \\
\text{or } u = E^{n-1}u_0 \text{ and } Eu = u.
\end{array}
\right\} \text{ counterassumption}
$$

We must then consider each of these n+m disjuncts in turn (with a view to showing its absurdity), and begin with the first one:

| | | |
|---|---|---|
| 2. | $u = W^m u_0$ and $(Eu = u$ or $E^2u = u$ or...or $E^{n+m}u = u)$ | assumption |
| 3. | $u = \{A: e^kA \in u\}$, for some k with $0 < k \leq n+m$ | from 2 [2nd conjunct] by the definition of $E^ku$ |
| 4. | For all wffs A, $A \in u$ iff $e^kA \in u$ $(0 < k \leq n+m)$ | immediate by 3 |
| 5. | For all wffs A, $A \in u$ iff $ow^mA \in u$ | from 2 [1st conjunct] by the definition of $W^mu_0$; Lemma 7.3, (i); T1 |
| 6. | $(H\neg a_{-m} \wedge a_{-m} \wedge G\neg a_{-m}) \in u$ iff $ow^m(H\neg a_{-m} \wedge a_{-m} \wedge G\neg a_{-m}) \in u$ | from 5 by instantiation |
| 7. | $ow^m(H\neg a_{-m} \wedge a_{-m} \wedge G\neg a_{-m}) \in u$ | by A1(b) with i = -m, u max cons |
| 8. | $a_{-m} \in u$ and $G\neg a_{-m} \in u$ | from 6,7 etc. |
| 9. | $a_{-m} \in u$ iff $e^ka_{-m} \in u$ $(0 < k \leq n+m)$ | from 4 by instantiation |
| 10. | $e^ka_{-m} \in u$ | from 9 and the first conjunct in 8 |
| 11. | $e^k\neg a_{-m} \in u$ | from 8 by the first axiom schema in A4(c) $[A = \neg a_{-m}]$! |
| 12. | $\perp \in u$ | from 8 [1st conjunct], 11 and 10 by T7 [1st item] in Lemma 6.4 |

where 12 contradicts the consistency of $u \in U$. Note the crucial step yielding line 11.

Analogous arguments take care of the disjuncts in 1 where $u = W^{m-1}u_0$, $u = W^{m-2}u_0$... down to, say, $u = Wu_0$ $(= W^{m-(m-1)}u_0)$. We then proceed to the following disjunct in 1:

| | |
|---|---|
| 13. $u=u_0$ and ($Eu=u$ or $E^2u=u$ or...or $E^nu=u$) | assumption |
| 14. $u=\{A: e^kA\in u\}$, for some k: $0<k\leq n$ | from 13 [2 nd conjunct] by the definition of $E^ku$ |
| 15. For all wffs A, $A\in u$ iff $e^kA\in u$ $(0<k\leq n)$ | immediate by 14 |
| 16. For all wffs A, $A\in u$ iff $oA\in u$ | from 13 [1st conjunct] by the definition of $u_0$; Lemma 7.3 clause (i); T1 in Lemma 6.4 |
| 17. $(H\neg a_0\wedge a_0\wedge G\neg a_0)\in u$ iff $o(H\neg a_0\wedge a_0\wedge G\neg a_0)\in u$ | from 16 by instantiation |
| 18. $o(H\neg a_0\wedge a_0\wedge G\neg a_0)\in u$ | by A1(a) with $i = 0$, u max cons |
| 19. $a_0\in u$ and $G\neg a_0\in u$ | from 17, 18 etc. |
| 20. $a_0\in u$ iff $e^ka_0\in u$ $(0<k\leq n)$ | from 15 by instantiation |
| 21. $e^ka_0\in u$ | from 20 and 19 [1st conjunct] |
| 22. $e^k\neg a_0\in u$ | from 19 by the (m+1)st axiom schema in A4(c) [$A = \neg a_0$]! |
| 23. $\perp\in u$ | from 19 [1st conjunct], 22 and 21 by T7 [(m+1)st item] in Lemma 6.4 |

where 23 contradicts the consistency of $u \in U$.

Again, similar arguments work for the remaining n-1 disjuncts in our counterassumption 1. Considering the last one among them, call it

| | |
|---|---|
| 24. $u=E^{n-1}u_0$ and $Eu=u$ | assumption |

we proceed as above and end up with

.

.

| | |
|---|---|
| 30. $a_{n-1}\in u$ and $G\neg a_{n-1}\in u$ | justified as usual, using A1(a) |

.

.

.

| | |
|---|---|
| 32. $ea_{n-1}\in u$ | by 24 [2nd conjunct] and 30 [1st one] |
| 33. $e\neg a_{n-1}\in u$ | from 30 by the (m+n)th axiom schema in A4(c) with $A=\neg a_{n-1}$ |

34. $\bot \in u$ from 30 [1st conjunct], 33, 32 by T7 [last item]

contrary to the consistency of $u \in U$.

Thus, we have shown that each of the $n+m$ disjuncts in our counterassumption 1 to (C8E) implies the conclusion that $\bot \in u$. Hence, that counterassumption is reduced *ad absurdum*, and the proof that, as defined, the function E satisfies (C8E) is complete.

*Remark.* The present proof, like that of (C7), can obviously be simplified in certain special cases. For instance, if the sum $n+m = 0$, so that $n = m = 0$, $e_0 = w_0 = u_0$ and $U = \{u_0\}$, the condition (C8E) will be "vacuously" satisfied and the need for a proof vanishes almost altogether. If $n = 0$, whereas $m > 0$, the last n disjuncts disappear from the counterassumption 1 and don't have to be considered in the proof. Again, if $m = 0$, wheras $n > 0$, the first m disjuncts disappear from the counterassumption 1, and the proof starts with line 13 *supra*. Finally, if $n = 1$, so that $E^{n-1}u_0 = u_0$, the part of our argument consisting of lines 24-34 can be dropped, since they simply reduce to lines 13-23.

*Ad* (C8W) [W-analogue of Peano's fourth axiom]. In the light of the preceding proof, the validation of this condition can safely be left to the reader as a tedious exercise. Note the usefulness of the axiom schemata in A4(d) in the present context!

The proof of Theorem 7.2 is hereby complete.

## 8. *Coincidence lemma for canonical $\mathcal{F}m,n$-structures*

In this section we show that, as applied to any wffs, the notions of *truth* and *membership* coincide (are co-extensive) with respect to the points in generated canonical $\mathcal{F}m,n$-structures. More precisely, we have the following

8.1. COINCIDENCE LEMMA. Let x be any fixed maximal consistent (in $\mathcal{F}m,n$) set of wffs, and let $\mathcal{M}^x = ((U, (w_0, u_0, e_0), (m,n), E, W), V)$ be the canonical $\mathcal{F}m,n$-structure generated by x. Then, for each wff A and each u in U,

$\mathcal{M}^x, u \models A$ iff $A \in u$.

*Proof.* By induction on the length of A.

*Basis.* A is either (a) some propositional variable $p_i$, or (b) $\top$, or (c) $\bot$, or (d) some systematic frame constant $a_i$ with $i \in [-m,n]$. The three cases (a)-(c) are immediate. We then deal with case (d) as follows.

Suppose first $0 \leq i \leq n$. Then we have the following chain of equivalences:

$$\mathcal{M}^*,u \models a_i \text{ iff } u = E^i u_0 \text{ iff } a_i \in u.$$

Here, the first "iff" holds by virtue of the truth condition for $a_i$ and the second by clause (ii) of Lemma 7.3 *supra* [when $0 \leq i \leq n$]. Hence the desired result in this case.

Suppose next that $0 > i \geq -m$. Now, we have these equivalences:

$$\mathcal{M}^*,u \models a_i \text{ iff } u = W^{-i} u_0 \text{ iff } a_i \in u.$$

Again, the first "iff" holds by the truth condition for $a_i$ and the second by clause (ii) of Lemma 7.3 [when $0 > i \geq -m$].

This settles case (d) as a whole.

*Induction Step.* The cases where A is a Boolean compound are left to the reader. Consider next

*Case* A = oB (for some wff B). The case will be clinched if we establish the following chain of equivalences:

$$\mathcal{M}^*,u \models oB \text{ iff } \mathcal{M}^*,u_0 \models B \text{ iff } B \in u_0 \text{ iff } oB \in x \text{ iff } oB \in u.$$

Here, the first "iff" is guaranteed by the truth condition for oB, the second by the inductive hypothesis [$u_0 \in U$], the third by the definition of $u_0$ in canonical $\mathcal{F}$m,n-structures, and the fourth "iff" is provided by Lemma 7.3, clause (i) together with theorem schema T1 of Lemma 6.4 [$x,u \in U$].

*Case* A = eB. Suppose first that $u \neq e_0$. Hence $Eu \in U$ by the fact that, as defined, E satisfies (C1E). We then argue as follows:

$$\mathcal{M}^*,u \models eB \text{ iff } \mathcal{M}^*,Eu \models B \text{ iff } B \in Eu \text{ iff } eB \in u,$$

where the first "iff" is given by the truth condition for eB, the second by the hypothesis of induction [$Eu \in U$], and the third by the definition of Eu in canonical $\mathcal{F}$m,n-structures.

Suppose next that $u = e_0$. Then, the above argument does not work, because $Ee_0$ (= Eu) $\notin$ U and the inductive hypothesis is simply not applicable. Instead, we argue as follows: since there is no v $\in$ U with v = $Ee_0$ (= Eu) [by the second item of (C1E)], we have $\mathcal{M}^*$,u $\models$ eB for *any* wff B *whatsoever* (including $\perp$), the truth condition for eB being vacuously satisfied on the present assumption that u = $e_0$. In like manner, we have $e\perp$ $\in$ u and eB $\in$ u, for any wff B whatsoever [cf. the validation of the second item in (C1E) in the preceding Section]. Hence our desired result that for each wff B, $\mathcal{M}^*$,u $\models$ eB iff eB $\in$ u, also in the present case where u = $e_0$.

*Case* A = wB. The treatment is perfectly similar to the one just given: according to the truth condition for wB, two subcases will be considered and handled in the spirit just indicated.

*Case* A = GB (for some wff B). Inspecting the truth condition for (wffs of the form) GB, we see that m+n+1 subcases have to be dealt with, viz. according to whether u = $W^m u_0$, u = $W^{m-1}u_0$, ..., u = $u_0$, u = $Eu_0$, ..., or u = $E^{n-1}u_0$ [in which m+n subcases the inductive hypothesis is clearly applicable] or, finally, u = $e_0$ = $E^n u_0$ [in which subcase the inductive hypothesis is simply not applicable, because $Ee_0$ $\notin$ U]. These subcases are now considered in turn.

*Subcase* u = $w_0$ = $W^m u_0$. By Lemma 7.3 above, clause (iv), we get $a_{-m}$ $\in$ u. We then have the following chain of equivalences:

| | | |
|---|---|---|
| $\mathcal{M}^*$,u $\models$ GB | iff $\mathcal{M}^*$,$E^k u$ $\models$ B | for all natural numbers k with $0<k\leq n+m$ |
| | iff B $\in$ $E^k u$ | for all k with $0 < k \leq n+m$ |
| | iff $e^k B$ $\in$ u | for all k with $0 < k \leq n+m$ |
| | iff $\bigwedge_{k=1}^{n+m} e^k B \in u$ | (for this notation, see the Remark relating to A4 in Section 5 *supra*) |
| | iff GB $\in$ u | |

From this chain of "iff"s, the desired result is of course immediate. So, just observe that the first "iff" is supplied by the truth condition for GB, the second by the inductive hypothesis [all the $E^k u$ are in U by virtue of (C1E)], the third by the definition of $E^k u$ in canonical $\mathcal{T}$m,n-structures, the fourth by the fact that u is maximal consistent and hence closed under finite conjunctions, and that the fifth, indeed "characteristic", "iff" is provided by the first axiom schema in A4(c) by virtue of the fact that $a_{-m}$ $\in$ u, u being maximal consistent since belonging to U.

*Subcases* $u = W^{m-1}u_0, \ldots, u = u_0, u = Eu_0, \ldots, u = E^{n-1}u_0$. By Lemma 7.3, clause (ii), we obtain in these respective cases that $a_{-m+1} \in u, \ldots, a_0 \in u, a_1 \in u, \ldots a_{n-1} \in u$. We then argue just as in Subcase $u = W^m u_0$, keeping track of the "right indices", when applying the relevant axiom schemata in A4(c).

*Subcase* $u = e_0 = E^n u_0$. Here, as in *Case* A = eB, previous reasoning does not work, because $Ee_0$ (= Eu) $\notin$ U and the inductive hypothesis does not apply. Instead, we argue as follows: the truth condition for GB being vacuously satisfied in the present subcase, we have $\mathcal{M}^*, u \models$ GB for any wff B whatsoever (including $\bot$); similarly, since $a_n \in u$ here, we obtain GB $\in$ u for any wff B whatsoever, by the last axiom schema in A4(c). Hence our desired result.

This completes the treatment of *Case* A = GB as a whole.

*Case* A = HB. The proof is a mirror image of the preceding one; details are left to the reader. The relevant axiom schemata are those in A4(d), of course.

*Cases* A = FB and A = PB. These are also taken care of in a straightforward way: just argue "dually"! Don't forget the schemata in A3 and in T11-T12 of Lemma 6.4!

*Case* A = ⊟ B. The case will be settled, if the following equivalences go through:

$\mathcal{M}^*, u \models$ ⊟ B iff for all v in U: $\mathcal{M}^*, v \models$ B
iff for all v in U: B $\in$ v
iff ⊟ B $\in$ u.

Here, the first two "iff"s are verified as usual. And the "if" half of the third "iff" is obvious in view of Lemma 7.3, clause (i) together with the x-adequacy of any v $\in$ U. To establish the crucial "only if" half, then, we argue as follows. Suppose that, for any u $\in$ U, we have

| | |
|---|---|
| 1. ⊟ B $\notin$ u | assumption |
| 2. ⟐ ¬B $\in$ u | from 1 by the maximal consistency of u $\in$ U |
| 3. (P¬B ∨ ¬B ∨ F ¬B) $\in$ u | from 2 by T17 of Lemma 6.4 |
| 4. Either P¬B $\in$ u or ¬B $\in$ u or F¬B $\in$ u | from 3 by u max cons |

We consider these three cases in turn:

| | |
|---|---|
| 5. $P\neg B \in u$ | assumption |
| 6. Either $a_n \in u$ or $a_{n-1} \in u$ or...or $a_0 \in u$ or...or $a_{-m+1} \in u$ or $a_{-m} \in u$ | from axiom schema A1(g) by u max cons |

Again, we must consider these n+1+m subcases in turn:

| | |
|---|---|
| 7. $a_n \in u$ $(=e_0)$ | assumption |
| 8. $w^k\neg B \in u$, for some k: $0<k\leq m+n$ | from 7,5 by the first item in T12 of Lemma 6.4, $\bigvee_{k=1}^{m+n} w^k\neg B \in u$, u max cons |
| 9. $\neg B \in W^k u$ | from 8 by the definition of $W^k u$ |
| 10. For some v in U, $B \notin v$ | from 9 by the fact that $W^k u \in U$ |

We then go on to establish the same conclusion 10 in the remaining n+m subcases. Only the last one calls for special treatment:

| | |
|---|---|
| 11. $a_{-m} \in u$ $(=w_0)$ | assumption |
| 12. $\neg P\neg B \in u$ | from 11 by the last item in T12 |
| 13. $\bot \in u$ | from 5, 12 by u max cons and A0 |
| 14. $\neg B \in u$ | from 13, u max cons, A0 |
| 15. For some v in U, $B \notin v$ | from 14 by the fact that u $(=w_0) \in U$ |

Hence, whichever $a_i$, $i \in [n,-m]$, is assumed to be in u, we obtain the conclusion that B is a non-member of some v in U. By disjunction elimination, that conclusion will depend just on the assumption 5. Consider next the case:

| | |
|---|---|
| 16. $\neg B \in u$ | assumption |
| 17. For some v in U, $B \notin v$ | from 16 since $u \in U$ |

Consider then the third and last case:

| | |
|---|---|
| 18. $F\neg B \in u$ | assumption |

Bearing in mind that u ($\in$ U) still satisfies line 6 *supra* (by A1(g)), we must consider the subcases in 6, say, in reverse order:

| | |
|---|---|
| 19. $a_{-m} \in u$ ($= w_0$) | assumption |
| 20. $e^k \neg B \in u$, for some k: $0 < k \leq n+m$ | from 19, 18 by the first item in T11 of Lemma 6.4, $\bigvee_{k=1}^{n+m} e^k \neg B \in u \in U$ |
| 21. $\neg B \in E^k u$ | from 20 by the definition of $E^k u$ |
| 22. For some v in U, $B \notin v$ | from 21 by the fact that $E^k u \in U$ |

We then go on to establish the same conclusion 22 in the remaining m+n subcases. Again, only the last one calls for special treatment:

| | |
|---|---|
| 23. $a_n \in u$ ($= e_0$) | assumption |
| 24. $\neg F \neg B \in u$ | from 23 by the last item in T11 |
| 25. $\bot \in u$ | from 18, 24 by A0, u max cons |
| 26. $\neg B \in u$ | from 25 by A0, u max cons |
| 27. For some v in U, $B \notin v$ | from 26 by the fact that u ($= e_0$) $\in$ U |

Once again, whichever $a_i$, $i \in [-m,n]$, is assumed to be in u, we can derive the conclusion that B is a non-member of some v in U, so, by disjunction elimination, that conclusion will depend just on the assumption 18.

Hence, that very conclusion is derivable from any of the three assumptions 5, 16 and 18. Therefore, by another step of disjunction elimination, that conclusion follows from 4 and, ultimately, from the initial assumption 1. Clearly, this result is equivalent to the desired "only if" half in our third, crucial "iff".

The induction is complete (leaving *Case* A = $\diamond$ B to the reader).

## 9. *Semantic soundness and completeness of the axiomatic systems* $\mathscr{T}m,n$

### 9.1. *Soundness theorem.*

*Weak version*: Every $\mathscr{T}m,n$-provable wff is $\mathscr{T}m,n$-valid.
*Strong version*: Every $\mathscr{T}m,n$-satisfiable set of wffs is $\mathscr{T}m,n$-consistent.
*Proof.* As usual, the weak version is established by showing (i) that every instance of the axiom schemata A0, A1(a)-(h), ..., A5(a)-(c) is $\mathscr{T}m,n$-valid, and (ii) that the rules R0 and R2 preserve $\mathscr{T}m,n$-validity. This is tedious, but entirely routine.
As to the strong version, it is easily obtained as a corollary of the weak one; see e.g. Åqvist (1987) §10. (The situation is different as regards the infinitary system $\mathscr{T}$ dealt with in my Åqvist (1992); see Theorem 5.3 of that essay.)

### 9.2. *Completeness theorem.*

*Weak version*: Every $\mathscr{T}m,n$-valid wff is $\mathscr{T}m,n$-provable.
*Strong version*: Every $\mathscr{T}m,n$-consistent set of wffs is $\mathscr{T}m,n$-satisfiable.
*Proof.* As the weak version is immediate from the strong one, let us concentrate on the latter.
Let $\Gamma$ be any $\mathscr{T}m,n$-consistent set of wffs. Form the Lindenbaum extension $\Gamma_\omega$ of $\Gamma$ as defined in Definition 6.7: by Theorem 6.8, $\Gamma_\omega$ is maximal consistent (in $\mathscr{T}m,n$). Again, form the canonical $\mathscr{T}m,n$-structure generated by $\Gamma_\omega$, i.e. the structure $(\mathscr{M})^{\Gamma_\omega}$ as defined in Definition 7.1 *supra*: by the fundamental Theorem 7.2, then, $(\mathscr{M})^{\Gamma_\omega}$ is a Tm,n-model. By the Coincidence Lemma 8.1 for (generated) canonical $\mathscr{T}m,n$-structures, we obtain in particular that for each wff A:

$$(\mathscr{M})^{\Gamma_\omega}, \Gamma_\omega \models A \text{ iff } A \in \Gamma_\omega$$

since $\Gamma_\omega$ is known to belong to the "universe" U of $(\mathscr{M})^{\Gamma_\omega}$ [appeal to axiom schema A5(a), if necessary]. Hence, since $\Gamma \subseteq \Gamma_\omega$, we have $(\mathscr{M})^{\Gamma_\omega}, \Gamma_\omega \models$ A for every $A \in \Gamma$. In other words, assuming $\Gamma$ to be any $\mathscr{T}m,n$-consistent set of wffs, we have constructed a $\mathscr{T}m,n$-model, viz. $(\mathscr{M})^{\Gamma_\omega}$, such that for some u in its universe U, viz. $\Gamma_\omega$, $(\mathscr{M})^{\Gamma_\omega}, u \models A$ for each A in $\Gamma$; i.e. we have shown $\Gamma$ to be $\mathscr{T}m,n$-satisfiable. Q.E.D.

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