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B, A BURIDAN-STYLE PROPOSITIONAL LOGIC (*)

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In Hinckfuss [1991], it was claimed that there was no reason to believe that a paradox free Buridan-style(1) of propositional logic, extended to include a truth predicate for locutions, was not possible. In this paper, I present such a system of propositional logic with a metatheoretic model and show that it is sound and complete with respect to that model. In conclusion, a possibility of the system going unsound when extended to handle formulae of its own metatheory is considered and rejected.

Primitive symbols of B

Propositional variables: p_1, p_2, p_3, ...
Locution constants: l_1, l_2, l_3, ...
Propositional operators: $\sim$, &
Predicate: T (to be read 'states truly').

The Set of Well-Formed Formulae for B. (WFF)

Basic WFFs

1_1. Any propositional variable ϵ WFF.
1_2. For any $l \in \{l_1, l_2, l_3, ...\}$, $Tl \in$ WFF.

Complex WFFs

2_1. If $\alpha \in$ WFF, so is $\sim\alpha$.
2_2. If $\{\alpha,\beta\} \subseteq$ WFF, $(\alpha\&\beta) \in$ WFF.

(*) Thanks are due to Graham Priest and Greg Restal for helpful encouragement and discussion.

(1) Hughes [1982] gives a thorough account of Buridan's treatment of the liar paradox.

Terminal Clause

3. WFF is included in any set, which, like WFF, satisfies conditions 1_1, 1_2, 2_1, 2_2.

Definitions

Formulae of the form $(\alpha \supset \beta)$, $(\alpha \vee \beta)$, $(\alpha \equiv \beta)$ and $(\alpha \not\equiv \beta)$ are defined in the usual way.

Deductions in B.

Deductions in B are sequences of *lines* of the form

P l α

where P is a set of wffs called the *premise set*, 'l' is called the *label* and is a member of $\{l_1, l_2, l_3, \ldots\}$, α is called the *content* of the line and is a wff. We shall allow that reference to a line, P l α, may be made using its label, 'l', or by using what we shall call the *full display* of the line, namely, P l α.

A wff, α, may be the content of more than one line, but each line has one and only one content. We shall use [l] to refer to the content of l. Thus if α_n is the content of l_n, $[l_n] = \alpha_n$.

The *premise set*, P, is the set of premises from which the content of l is derived.

As well as stating its content, each line in a B-deduction is deemed to state its *metacontent*. The metacontent of any line, l, is the statement that Tl, that is, that the line states truly. Unless ambiguity would result, we shall use 'Tl' not only as a shorthand for 'Line l states truly', but autonomously also. Similarly for formulae such as (p&q) and formula surrogates such as α.

Some lines within a deduction are distinguished as *assumptions* of the deduction. For any assumption, P l [l], within a deduction, $P = \{[l]\}$. Lines within a deduction which are not assumptions within the deduction are said to be *derived lines* of the deduction.

The New Label Condition.

Any label may or may not appear as a part of the content of a formula, e.g. when [l] is Tl_1 or $(p\&\sim T_2)$, etc., except that, for any two lines, l_m and l_n, within a deduction, if l_m precedes l_n, then l_n will occur neither as label or as part of the content of l_m. Nor will any label of any line within a deduction occur as either the label or as part of the content of any other line external to the deduction whose label is part of the content of any line within the deduction. In short, any new lines are to have new labels.

The description of a line as *new* with respect to *preceding* lines implies an *ordering* of the lines. We shall assume that this ordering is a strict ordering denoted by '*is later than*', or '>' for short, and that the ordering shall include any lines that are mentioned in the deduction but which are not lines of the deduction itself. Any lines within the deduction shall be deemed to be later in the ordering than lines that are mentioned in the deduction but which do not occur in the deduction.

All and only the derived lines within a deduction will be the consequents of the application of one of the following deduction rules.

&I	P_m	l_m	α	
	P_n	l_n	β	
∴	$(P_m \cup P_n)$	l_o	$(\alpha\&\beta)$	where both $l_o > l_m$ and $l_o > l_n$.
&E	P_m	l_m	$(\alpha\&\beta)$	or P_m l_m $(\alpha\&\beta)$
∴	P_m	l_n	α	∴ P_m l_n β
				where $l_n > l_m$.
~&E	P_m	l_m	$\sim(\alpha\&\beta)$	
	P_n	l_n	α	
∴	$(P_m \cup P_n)$	l_o	$\sim\beta$	where both $l_o > l_m$ and $l_o > l_n$.
~~E	P_m	l_m	$\sim\sim\alpha$	
∴	P_m	l_n	α	where $l_n > l_m$.
TI	$\{\alpha\}$	l_m	α	
∴	$\{Tl_m\}$	l_n	Tl_m	where l_m is an assumption and $l_n > l_m$

or

	P_m	l_m	α	
∴	P_m	l_n	Tl_m	where l_m is derived and $l_n > l_m$.

TE	P_m	l_m	Tl_k	
∴	P_m	l_n	α	where $[l_k] = \alpha, l_m > l_k$, and $l_n > l_m$.

~I	$\{\alpha\}$	l_m	α	where l_m is an assumption.
	P_n	l_n	$(\beta \& \sim \beta)$	
∴	$(P_n - \{\alpha\})$	l_o	$\sim \alpha$	where $l_o > l_n$ and $l_n > l_m$.

or

	$\{\alpha\}$	l_m	α	where l_m is an assumption.
	P_n	l_n	$(\beta \& \sim \beta)$	
∴	$(P_n - \{Tl_m, \alpha\})$	l_o	$\sim Tl_m$	where $l_o > l_n$ and $l_n > l_m$.

Metatheoretic Modelling for B

A model, ϕ, is any function whose arguments are all the basic wffs and whose values are taken from the pair $\{1,0\}$.

The value, if any, of a WFF, α, in model ϕ, $\phi\text{Val}`\alpha$, is given by the following recursive definition.

1. If α is a basic WFF, then $\phi\text{Val}`\alpha = \phi`\alpha$.

2_1. If $\alpha = \sim \beta$, then $\phi\text{Val}`\alpha = 1$ iff $\text{Val}`\beta = 0$, and $\phi\text{Val}`\alpha = 0$ iff $\phi\text{Val}`\beta = 1$.

2_2. If $\alpha = (\beta \& \delta)$, then $\phi\text{Val}`\alpha = 1$ iff $\phi\text{Val}`\beta = 1$ and $\phi\text{Val}`\delta = 1$, and $\phi\text{Val}`\alpha = 0$ iff $\phi\text{Val}`\beta = 0$ or $\phi\text{Val}`\delta = 0$.

3. The function ϕVal is included in any function which also satisfies conditions 1, 2_1 and 2_2.

It follows by a simple recursive proof that all wffs have a value in all models.

A model, ϕ, is said to be a *B-model* (short for Buridan model) iff, for any l,

(i) if ϕ'Tl=1, then ϕVal'[l]=1;
(ii) if ϕVal'[l]=0, then ϕ'Tl=0;
(iii) if (a) ϕVal'[l]=1 and,
(b) for some model, δ, δVal'[l]=1 and δ'Tl=1,
then ϕ'Tl=1.

Note that this allows the possibility, in a B-model, that ϕVal'[l]=1 and ϕ'Tl=0. This is a place of departure from standard Tarski modelling.

A B-model, ϕ, is a B-model *for* a premise set, P, iff, for all $\alpha \epsilon$P, ϕVal'α=1.

A model, ϕ, is a *countermodel for* a line, l, with premise set P, iff ϕ is a B-model for P and ϕ'Tl=0.

A line is *B-valid* iff there is no countermodel for the line. It follows that if ϕ is a B-model for the premise set P of any B-valid line, l, then ϕVal'Tl=1.

Note that the B-validity of some line, l, does not entail that l states truly, that is, that Tl.

Note also that not all assumptions are B-valid. For example, any assumption of the form {~Tl} l ~Tl will yield ϕVal'Tl=0 for any model, ϕ, which is a B-model for {~Tl}.

Theorem. The Soundness of B.
There is no B-deduction which contains a derived line which is not B-valid.

Proof.
The proof is by induction on the generation of B-deductions.

Basis.
No assumption is a derived line. However we note that any B-model for the premise set {[l]} of any assumption, l, yields ϕVal'[l]=1; and that, since B-models are functions to the pair {1,0}, in no such B-model is it the case that ϕVal'[l]=0.

Inductive Steps.
For the inductive part of the proof it is necessary to show that any application of any of the deduction rules on lines that are either assumptions or B-valid lines yields a B-valid line.

Proof for &I.
Assume that lines P l_m α and P_n l_n β are both either assumptions or B-valid. Then we are required to prove that the line

$$(P_m \cup P_n) l_o \ (\alpha \& \beta)$$

where l_o is later than both l_m and l_n, is also B-valid. Assume for the purposes of a *reductio* that the line l_o is not B-valid. Then there is a B-model for $(P_m \cup P_n)$ in which Tl_o has value 0. Let that model be ϕ. It follows that ϕ is a B-model for both P_m and P_n, whence, since l_m and l_n are both either assumptions or B-valid, both ϕVal'$\alpha=1$ and ϕVal'$\beta=1$; whence ϕVal'$(\alpha\&\beta)=1$, that is, ϕVal'$[l_o]=1$.
Now l_o is later than both l_m and l_m. It follows, given the new label condition, that 'l_o' and hence Tl_o are not a part of any member of P_m, or P_n or of $[l_m]$ (that is, α) or of $[l_n]$ (that is, β). We can therefore construct a model ϕ_1, which, like ϕ, is a model for P_m and P_n, and in which, therefore, ϕ_1Val'$\alpha=1$, ϕ_1Val'$\beta=1$, and hence ϕ_1Val'$(\alpha\&\beta)=1$, but in which, unlike ϕ, ϕ_1'$Tl_o=1$. Hence there is a model in which both Tl_o and $[l_o]$ evaluate to 1. Whence, since ϕ is a B-model and ϕVal'$[l_o]=1$, by condition (iii) for B-models, ϕ'$Tl_o=1$, contrary to our hypothesis.

Proofs for &E, ~&E, and ~~E
These follow similarly to the proof for &I.

Proof for TI.
We first have to show that the line $\{Tl_m\}$ l_n Tl_m is valid, where l_m is an assumption and $l_n > l_m$.

Now any B-model, ϕ, for $\{Tl_m\}$ will yield ϕVal'$Tl_m=1$, whence ϕVal'$[l_n]=1$, whence, given the new label condition, ϕ'$Tl_n=1$.

Secondly, we have to show that if a derived line, P_m l_m α, is B-valid, then so is line P_m l_n Tl_m, where $l_n > l_m$.

Assume l_n is not B-valid. Then there is a B-model, ϕ, for P_m such that ϕ'$Tl_n=0$. Given the new label condition for derived lines, it follows that ϕVal'$[l_n]=0$, that is ϕVal'$Tl_m=0$. But if l_m is B-valid, there is no B-model for P_m in which Tl_m evaluates to 0. Our assumption is therefore false and the theorem holds for TI.

Proof for TE.
In this case we have to show that if line P_m l_m Tl_k is an assumption or is B-valid, then so is P_m l_n α, where $[l_k]=\alpha$, $l_m > l_k$, and $l_n > l_m$.

Assume l_m is either an assumption or is B-valid, and that l_n is not B-valid.

If l_n is not B-valid, then there is some B-model, ϕ, for P_m, such that ϕVal'$Tl_n=0$.

Now if l_m is an assumption, $P_m=\{[l_m]\}=\{Tl_k\}$. Hence if ϕ is a B-model for P_m, ϕVal'$Tl_k=1$, whence ϕVal'$\alpha=1$, whence, given the new label condition, ϕ'$Tl_n=1$.

Again, if l_m is B-valid and ϕ is a B-model for P_m, then ϕ'$Tl_k=1$; whence ϕVal'$[l_k]=1$, that is, ϕVal'$\alpha=1$, and as before, the new label condition would then ensure that ϕ'$Tl_n=1$, contrary to our hypothesis.

Proof for ~I.

We have to show firstly, that if $\{\alpha\}$ l_m α is an assumption and the line P_n l_n $(\beta\&\sim\beta)$ is either an assumption or is a B-derived line, then the line $(P_n\text{-}\{\alpha\})$ l_o $\sim\alpha$ is B-valid, for $l_o>l_n>l_m$.

Assume first that l_n is an assumption. Then $P_n=\{(\beta\&\sim\beta)\}$ and hence P_n evaluates to 0 in all models.

Assume now that l_n is B-valid. Then since $[l_n]=(\beta\&\sim\beta)$ and since $(\beta\&\sim\beta)$ evaluates to 0 in all models, Tl_n evaluates to 0 in all models. If l_n is B-valid, there is no model for P_n in which Tl_n evaluates to 0. It follows again, therefore, that there is no B-model for P_n.

So whether l_n is an assumption or is derived, there is no B- model for P_n.

Assume then, that ϕ is a B-model for $(P_n\text{-}\{\alpha\})$. Then if there is no B-model for P_n, it must be the case that ϕVal'$\alpha=0$, that is, ϕVal'$\sim\alpha=1$, whence, given the new label condition, ϕVal'$Tl_o=1$.

For the second form of ~I, we have to show that for assumption $\{\alpha\}$ l_m α and line P_n l_n $(\beta\&\sim\beta)$ where l_n is either an assumption or is B-valid, then the line $(P_n\text{-}\{Tl_m,\alpha\})$ l_o $\sim Tl_m$ is B-valid, for $l_o>l_n>l_m$.

As before, it follows that there is no model for P_n. So if ϕ is a B-model for $(P_n\text{-}\{Tl_m,\alpha\})$, either ϕ'$Tl_m=0$ or ϕVal'$\alpha=0$. If ϕVal'$\alpha=0$, ϕ'$Tl_m=0$. So in either case, if ϕ is a B-model for $(P_n\text{-}\{Tl_m,\alpha\})$, ϕ'$Tl_m=0$, whence ϕ'$\sim Tl_m=1$, whence, by the new label condition, ϕ'$Tl_o=1$.

This completes the proof of the soundness of B.

Theorem. Completeness of B

For any B-valid line, P l_n α, given the full displays of any lines whose labels occur in either the premises or the content of P l_n α, we can construct a B-deduction which ends in P l_n α.

Proof

We show that there is a method for such a construction which involves

(a) constructing a Jeffrey-style tree and
(b) transforming the tree to the required B-deduction.

We first examine α to see if it is of the form Tl_m, where 'l_m' does not occur as a part of any member of P or as part of the contents of any line whose label is a part of any member of P, that is, 'l_m' is new. If so, then P l_n Tl_m is B-valid if and only if P l_m $[l_m]$ is B-valid, and we derive l_n by deriving l_m and then using TI. Again, if $[l_m]=Tl_k$, where 'l_k' is new, we derive l_m from l_k, again using TI. Proceed backward in the derivation from the line P l_n α in this way until a line is reached whose contents are either not of the form Tl, or, if they are, then 'l' is not new. The problem then reduces to generating the residual deduction.

A tree, call it a *B-tree*, for the residual deduction is constructed with initial formulae comprising the premises of the line in question and the negation of the contents of the line. The B-tree rules are:

TE	Tl	$\sim\sim$E	$\sim\sim\alpha$	&E	$(\alpha\&\beta)$	$\sim$&E	$\sim(\alpha\&\beta)$
	[l]		α		α		/ \
					β		$\sim\alpha$ $\sim\beta$

Closure rule: A path of the tree is closed when there appears within the path any formula α and its negation, $\sim\alpha$.

The tree is constructed by applying each of the rules in turn to all relevant formulae appearing in any branch of the tree, giving preference to lines closer to the top when dealing with any one rule. After applying $\sim$&E to every possible formula, we go back to rule TE and proceed again until there are no more lines in the tree to which any of the rules are applicable.

The B-tree system is complete, which is to say, that for any initial formulae for which there is no B-model, all paths in the tree will close in a finite number of operations. The proof for this follows the style of the proof for the completeness of trees for classical propositional logic as given in Richard Jeffrey [1967].

Only two new issues are to be addressed and both concern TE. The first is whether the use of TE in the tree could lead to infinite trees. That this is not so is apparent from the fact that there will be only a finite number of lines whose complete description is given and on which, therefore, TE could apply. Since the tree rules do not include TI or any other mechanism for the generation of new formulae of the form Tl to which TE could apply, applications of TE will be limited by the number of occurrences of formulae of the form Tl appearing as parts of the initial formulae or of other formulae whose complete description is given.

The second concern is whether a finished open path would be upwardly B-valid, that is, whether any B-model for the set of formulae which are either basic formulae or the negations of basic formulae which appear in the path is also a B-model for the total set of formulae appearing in the path, including the initial formulae. The proof of this for trees of classical propositional logic involves showing that each of the tree rules is upwardly valid. With the exception of TE, all of the rules for B-trees are rules for classical propositional logic, and all of these are upwardly B-valid. The TE rule is not upwardly B-valid, since it is possible to have a B-model in which [l] evaluates to 1 and in which Tl evaluates to 0. But this does not matter, because any formula of the form Tl is itself basic and so will receive a valuation of 1 should it occur in an open finished path all of whose basic formulae evaluate to 1.

This completes the proof for the completeness of B-trees.

To generate a B-deduction from a B-tree, begin with the assumptions whose contents are the formulae of the initial lines of the tree, taking care to label these assumptions differently from any fully described lines. Then proceed down the left-most path in the following manner.

(i) Where the formula in the tree path is produced by a non- branching tree rule, use the corresponding deduction rule to generate a new line of the proof.

(ii) Where the formula is produced by the rule ~&E, make an assumption whose content is the formula on the left branch.

(iii) When the closure of the path has been reached, apply &I to produce a line with contents of the form (ß&~ß) and apply ~I and also, where applicable, ~~E. Use ~&E to give a line whose content is the right-hand formula of the last application of ~&E in the tree and carry on down this path as before.

(iv) Proceed in this way until the right-most closure is reached. Use &I, and ~I to produce a line whose content is the negation of the content of the last line of the initial formulae.

(v) Use ~~E and TI, if necessary, to produce the required line.

Example:

Given $[l_1]=p$, $[l_2]=q$, derive the line $\{Tl_1, \sim(p\&\sim q)\}\ l_{15}\ Tl_2$

We first proceed backward from l_{15} via TI to produce l_{14}, namely, $\{Tl_1, \sim(p\&\sim q)\}\ l_{14}\ q$. We then produce a B-tree for l_{14} as follows:

(i)	Tl_1	(premise)
(ii)	$\sim(p\&\sim q)$	(premise)
(iii)	$\sim q$	(neg. of concl.)
(iv)	p	(i, TE)
	/ \	
(v)	$\sim p$ $\quad\sim\sim q$	(ii, $\sim$&E)
	X (iv,v) X (iii,v)	

We then convert this tree into a B-deduction as follows:

$\{Tl_1\}$	l_3	Tl_1	(Assumption)
$\{\sim(p\&\sim q)\}$	l_4	$\sim(p\&\sim q)$	(Assumption)
$\{\sim q\}$	l_5	$\sim q$	(Assumption)
$\{Tl_1\}$	l_6	p	(l_3, l_1, TE)
$\{\sim p\}$	l_7	$\sim p$	(Assumption)
$\{Tl_1, \sim p\}$	l_8	$(p\&\sim p)$	(l_6, l_7, &I)
$\{Tl_1\}$	l_9	$\sim\sim p$	(l_7, l_8, $\sim$I)
$\{Tl_1\}$	l_{10}	p	(l_9, $\sim\sim$E)
$\{Tl_1, \sim(p\&\sim q)\}$	l_{11}	$\sim\sim q$	(l_4, l_{10}, $\sim$E)
$\{Tl_1, \sim(p\&\sim q), \sim q\}$	l_{12}	$(\sim q\&\sim\sim q)$	(l_5, l_{11}, &I)
$\{Tl_1, \sim(p\&\sim q)\}$	l_{13}	$\sim\sim q$	(l_5, l_{12}, $\sim$I)
$\{Tl_1, \sim(p\&\sim q)\}$	l_{14}	q	(l_{13}, $\sim\sim$E)
$\{Tl_1, \sim(p\&\sim q)\}$	l_{15}	Tl_2	(l_2, l_{14}, TI)

Extensions of B

Any logic can be rendered unsound with an inappropriate extension. However, it is desirable that any logic should cope with an extension to its own semantic modelling.

The interpretation of ϕVal'$\alpha=1$ as 'α is true in ϕ' and of ϕVal'$\alpha=0$ as 'α is not true in ϕ', and the allowance that one of the ϕ's, ϕ_a say, is the model corresponding to the real world, might make one wonder whether the semantic paradoxes can be reintroduced along the following lines:

	1. $\phi_a Val‘[l_1]=0$	(Assumption)
$\{1\}$	2. $\phi_a Val‘[l_1]=1$	(1, Truth Intro)
$\{1\}$	3. $\sim\phi_a Val‘[l_1]=1$	(1,Metatheory)
$\{1\}$	4. $\phi_a Val‘[l_1]=1\ \&\ \sim\phi_a Val‘[l_1]=1$	(2,3, &I)
$\{\ \}$	5. $\sim\phi_a Val‘[l_1]=0$	(1,4, $\sim$I)
$\{\ \}$	6. $\phi_a Val‘[l_1]=1$	(5, Excl. middle)
$\{\ \}$	7. $\phi_a Val‘[l_1]=0$	(1,6,Metatheory)
$\{\ \}$	8. $\phi_a Val‘[l_1]=0\ \&\ \sim\phi_a Val‘[l_1]=0$	(7, 5, &I)

The wrong step here, from the point of view of B-logic, is step 2. Line 2 does not follow from the contents of 1, but rather from its metacontents. Making this explicit as demanded by B- logic, we have instead:

$\{\phi_a Val‘[l_1]=0\}$	1. $\phi_a Val‘[l_1]=0$	(Assumption)
$\{Tl_1\}$	2. Tl_1	(1, TI)
$\{Tl_1\}$	3. $\phi_a Val‘[l_1]=1$	(2, Meta-theory)
$\{\phi_a Val‘[l_1]=0\}$	4. $\sim\phi_a Val‘[l_1]=1$	(1, Meta-theory)
$\{Tl_1, \phi_a Val‘[l_1]=0\}$	5. $\phi_a Val‘[l_1]=1\ \&\ \sim\phi_a Val‘[l_1]=1$	(3, 4, &I)
$\{Tl_1\}$	6. $\sim\phi_a Val‘[l_1]=0$	(1, 5, $\sim$I)
$\{Tl_1\}$	7. $\phi_a Val‘[l_1]=1$	(6, Excl. middle)
$\{Tl_1\}$	8. $\phi_a Val‘[l_1]=0$	(1,7,Metatheory)
$\{Tl_1\}$	9. $\phi_a Val‘[l_1]=0\ \&\ \sim\phi_a Val‘[l_1]=0$	(8, 6, &I)

Again we have our contradiction, but only with Tl_1 as a premise. Discharging this premise, we obtain,

$\{\ \}$	10. $\sim Tl_1$	(1, 9, $\sim$I)

No paradox there. Line l_1 does not state truly and that is because its metacontent, namely the proposition that it states truly, is inconsistent with its content, namely the proposition that its content is not true.

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