Logique & Analyse **135-136** (1991), 319-331

A GENERALISATION OF THE TARSKI-HERBRAND DEDUCTION THEOREM

S.J. SURMA

The paper provides a condition which characterises the implicational fragment of classical sentential logic in the same way as the Tarski-Herbrand deduction theorem together with its conversion characterises the implicational fragment of intuitionistic logic.

The deduction theorem, Tarski-Herbrand-style, can be looked at as providing a general framework for the formalisation and justification of a wide class of intuitively sound, direct proofs. Typical logical theorems provable in this way may be called ascending conditionals or ascending implications because they fall under the schema

$$(\uparrow) \qquad C\alpha_1 C\alpha_2 C\alpha_3 \ldots C\alpha_{n-1}\alpha_n$$

where n is a positive integer number and $C\alpha\beta$ is a conditional or implication with the antecedent α and the consequent β. Clearly, not all provable conditionals are ascending. For instance, Peirce's law $CCC\alpha\beta\alpha\alpha$ is not. The widest class of directly provable conditionals must include what is called here descending conditionals, ie. conditionals of the form

$$(\downarrow) \qquad CC\ldots CC\alpha_1\alpha_2\alpha_3\ldots\alpha_{n-1}\alpha_n$$

The purpose of this note is to come to a Generalization of the above mentioned framework which would allow for the formalisation and justification of intuitively sound, direct proofs of all logical theorems which fall under schema ($\uparrow$) as well as under schema ($\downarrow$). Clearly, a direct proof of a ($\downarrow$)-conditional statement involves exclusively properties of the connective of conditional C alone. In practice, however, a typical proof of a ($\downarrow$)-conditional statement is indirect in that it depends also on properties of negation and/or disjunction.

Below we make use of some of standard set-theoretic notation. In particular, symbols $\subseteq$, $\cap$, $\cup$, $\varnothing$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ denote set-inclusion, intersection, union, the empty set and the set containing $\alpha_1, \alpha_2, \ldots, \alpha_n$ as its only members, respectively.

Let S be the least set containing a denumerable stock of sentential variables and closed under the formation of conditionals using the connective of conditional C as the only sentence-forming connective.
Thus, if $\alpha \in S$ and $\beta \in S$, then also $C\alpha\beta \in S$.

Let Cn be a closure operator on the power set 2^S of S. This means that, for any X, Y, Z $\subseteq S$,

(i) $X \subseteq Cn(X)$,
(ii) the fact that $X \subseteq Y$ implies that $Cn(X) \subseteq Cn(Y)$, and
(iii) $Cn(Cn(X)) \subseteq Cn(X)$.

For the sake of simplicity, expressions of the form

$$Cn(X \cup \{\alpha_1, \alpha_2, \ldots, \alpha_n\})$$

are abbreviated hereafter as

$$Cn(X, \alpha_1, \alpha_2, \ldots, \alpha_n)$$

We also assume that lower-case Greek characters, with or without subscripts, belong to the set S while Latin capitals X, Y, Z, ... are subsets of S.

The condition

$$C\alpha\beta \in Cn(X) \text{ iff } \beta \in Cn(X, \alpha)$$

is the classical deduction theorem and is due to A. Tarski [3] and J. Herbrand [1]. This theorem can be generalised to the following one.

$(D^n\uparrow)$ $C\alpha_1C\alpha_2C\alpha_3\ldots C\alpha_{n-1}\alpha_n \in Cn(X)$ iff $\alpha_n \in Cn(X, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1})$

It may be remarked at the outset that by calling $(D^n\uparrow)$ the deduction theorem we depart slightly from the prevailing terminological convention according to which the term "deduction theorem" applies only to the "if" part of the biconditional condition $(D^n\uparrow)$ while the "only if" part of $(D^n\uparrow)$ is (equivalent to) the rule of detachment. To be sure, X is said to be closed under (generalised) detachment iff, for any α and β, the fact that $\alpha \in Cn(X)$ and

$C\alpha\beta \in Cn(X)$ implies that $\beta \in Cn(X)$.

We have the following well-known fact which settles the scope of application of the Tarski-Herbrand deduction theorem.

LEMMA 1

The following conditions are equivalent.

(i) $(D^n\uparrow)$ for any n;

(ii) $(D^2\uparrow)$;

(iii) $Cn(\varnothing)$ includes the set of all theorems of the intuitionistic implicational logic and is closed under detachment.

It may be noticed that condition (iii) of the above Lemma can be improved by making a reference to an explicit axiom system for the logic involved. For instance, we could re-write (iii) as follows.

(iii') $Cn(\varnothing)$ is the least set closed under detachment and containing the formulae $C\alpha C\beta\alpha$ and $CC\alpha C\beta\gamma CC\alpha\beta C\alpha\gamma$.

Clearly, condition $(D^n\uparrow)$ does not cover the case of $(\downarrow)$-conditionals. In other words, we cannot simply place $CC...CC\alpha_1\alpha_2\alpha_3...\alpha_{n-1}\alpha_n$ instead of $C\alpha_1C\alpha_2C\alpha_3...C\alpha_{n-1}\alpha_n$ in $(D^n\uparrow)$.

We may proceed now to the problem of formalisation of intuitive proofs of $(\downarrow)$-conditionals. Consider the following condition.

$(D^n\downarrow)$ If n is even, then

$$CC...CC\alpha_1\alpha_2\alpha_3...\alpha_{n-1}\alpha_n \in Cn(X) \text{ iff, for any } i \in \{2, 4, 6, ..., n\},$$
$$Cn(X, \alpha_i) \cap Cn(X, \alpha_{i+2}) \cap Cn(X, \alpha_{i+4}) \cap ... \cap Cn(X, \alpha_n) \subseteq Cn(X, \alpha_{i-1}).$$

Clearly, the formula on the right of the biconditional formula $(D^n\downarrow)$ above abbreviates the following condition.

$Cn(X, \alpha_2) \cap Cn(X, \alpha_4) \cap \ldots \cap Cn(X, \alpha_n) \subseteq Cn(X, \alpha_1)$,
$Cn(X, \alpha_4) \cap Cn(X, \alpha_6) \cap \ldots \cap Cn(X, \alpha_n) \subseteq Cn(X, \alpha_3)$,
$Cn(X, \alpha_6) \cap Cn(X, \alpha_8) \cap \ldots \cap Cn(X, \alpha_n) \subseteq Cn(X, \alpha_5)$,
. . . .
$Cn(X, \alpha_{n-2}) \cap Cn(X, \alpha_n) \subseteq Cn(X, \alpha_{n-3})$ and
$Cn(X, \alpha_n) \subseteq Cn(X, \alpha_{n-1})$

The substitution of n = 2 in $(D^n\downarrow)$ gives

$$C\alpha_1\alpha_2 \in Cn(X) \text{ iff } Cn(X, \alpha_2) \subseteq Cn(X, \alpha_1)$$

which coincides with $(D^2\uparrow)$ and which, in turn, is equivalent to

$$C\alpha_1\alpha_2 \in Cn(X) \text{ iff } \alpha_2 \in Cn(X, \alpha_1)$$

Thus, $(D^n\downarrow)$ generalises the Tarski-Herbrand deduction theorem. The theorem below establishes the scope of application of $(D^n\downarrow)$.

THEOREM 1
The following conditions are equivalent.

(i) $(D^n\downarrow)$, for any even n;
(ii) $(D^n\downarrow)$, for any even $n \leq 4$;
(iii) $Cn(\varnothing)$ is the set of theorems of the C-fragment (ie. the implicational fragment) of classical sentential logic, closed under detachment.

As it is well-known, condition (iii) of the above Theorem can be rewritten equivalently as follows.

(iii') $Cn(\varnothing)$ is the least set closed under detachment and containing the formulae $C\alpha C\beta\alpha$, $CC\alpha C\beta\gamma CC\alpha\beta C\alpha\gamma$, $CCC\alpha\beta\alpha\alpha$.

To get this theorem proved we must do some preparatory work. First, let us define an auxiliary sentential formula

$$A(\alpha_1\alpha_2\ldots\alpha_n)$$

where n is an arbitrary positive integer number, by making use of the following recursive schema.

(i) $A(\alpha_1) = \alpha_1$,
(ii) $A(\alpha_1\alpha_2 \ldots \alpha_n\alpha_{n+1}) = CC\alpha_{n+1}A(\alpha_1\alpha_2 \ldots \alpha_n)A(\alpha_1\alpha_2 \ldots \alpha_n)$.

The following property of A can be established easily using induction on i.

LEMMA 2
If $Cn(\varnothing)$ contains all theorems of intuitionistic implicational logic and is closed under detachment, then, for any i such that $0 < i \leq n$,

$$C\alpha_iA(\alpha_1\alpha_2 \ldots \alpha_n) \in Cn(\varnothing).$$

LEMMA 3
If $Cn(\varnothing)$ is the set of theorems of the C-fragment of classical sentential logic, closed under detachment, then
$CC\alpha_1\beta CC\alpha_2\beta \ldots CC\alpha_n\beta CA(\alpha_1\alpha_2 \ldots \alpha_n)\beta \in Cn(\varnothing)$.

Proof. The proof is inductive. That line

I. $CC\alpha_1\beta C\alpha_1\alpha_1\beta$

belongs to $Cn(\varnothing)$ follows directly from the assumption about $Cn(\varnothing)$. To show that the fact that Lemma 3 holds true of k implies that it also holds true of k + 1, we assume that

II. $CC\alpha_1\beta CC\alpha_2\beta \ldots CC\alpha_k\beta CA(\alpha_1\alpha_2 \ldots \alpha_k)\beta$

and we need to show that

III. $CC\alpha_1\beta CC\alpha_2\beta \ldots CC\alpha_k\beta CC\alpha_{k+1}\beta CA(\alpha_1\alpha_2 \ldots \alpha_k\alpha_{k+1})\beta$

To do so suppose that

1. $C\alpha_1\beta$, $C\alpha_2\beta$, ..., $C\alpha_k\beta$
2. $C\alpha_{k+1}\beta$

and

3. $A(\alpha_1\alpha_2 \ldots \alpha_k\alpha_{k+1})$

From 3 and the definition of A

4. $CC\alpha_{k+1}A(\alpha_1\alpha_2 \ldots \alpha_k)A(\alpha_1\alpha_2 \ldots \alpha_k)$

From II, 1 and 4

5. $CC\alpha_{k+1}A(\alpha_1\alpha_2\ldots\alpha_k)\beta$
From 2, 5 and the logical theorem $CCC\alpha\beta\gamma CC\alpha\gamma\gamma$
6. β

so line III is proved. From I, II, III and the principle of mathematical induction, Lemma 3 is proved true of any n.

Q.E.D.

LEMMA 4
Let n be even. If $Cn(\emptyset)$ is the set of theorems of the C-fragment of classical sentential logic, closed under detachment, then, for any $i \in \{2, 4, 6, \ldots, n\}$, $CCC\ldots CC\alpha_1\alpha_2\alpha_3\ldots\alpha_{n-1}\alpha_n C\alpha_{i-1}A(\alpha_i\alpha_{i+2}\alpha_{i+4}\ldots\alpha_n) \in Cn(\emptyset)$.

Proof. In case n = 2 Lemma 4 reduces to line

I. $CC\alpha_1\alpha_2C\alpha_1\alpha_2 \in Cn(\emptyset)$

which follows directly from our assumption about the set $Cn(\emptyset)$. To prove the lemma for n > 2 we proceed by induction on n. Accordingly, we show first that Lemma 4 holds true for n = 4, ie. that

II. $CCCC\alpha_1\alpha_2\alpha_3\alpha_4C\alpha_{i-1}A(\alpha_i\alpha_4) \in Cn(\emptyset)$, for any $i \in \{2, 4\}$

To do so we first show that

1. $CCCC\alpha_1\alpha_2\alpha_3\alpha_4C\alpha_1A(\alpha_2\alpha_4) \in Cn(\emptyset)$

Assuming
1.1. $CCC\alpha_1\alpha_2\alpha_3\alpha_4$
and
1.2. α_1
we need to derive $A(\alpha_2\alpha_4)$, ie. $CC\alpha_4\alpha_2\alpha_2$ so let
1.3. $C\alpha_4\alpha_2$
From 1.1, 1.3 and the transitivity property of C
1.4. $CCC\alpha_1\alpha_2\alpha_3\alpha_2$
By another use of the transitivity of C we may infer from 1.4 and the logical theorem $C\alpha C\beta\alpha$ that
1.5. $CCC\alpha_1\alpha_2\alpha_3C\alpha_1\alpha_2$

From 1.5 and the logical theorem $CCC\alpha\beta\alpha\alpha$
1.6. $C\alpha_1\alpha_2$
From 1.2 and 1.6
1.7. α_2

so line 1 is proved. Next we show that

2. $CCCC\alpha_1\alpha_2\alpha_3\alpha_4C\alpha_3\alpha_4 \in Cn(\varnothing)$

Assume that
2.1. $CCC\alpha_1\alpha_2\alpha_3\alpha_4$
and
2.2. α_3
From 2.2 and the logical theorem $C\alpha C\beta\alpha$
2.3. $CC\alpha_1\alpha_2\alpha_3$
From 2.1 and 2.3
2.4. α_4

which proves line 2. Lines 1 and 2 complete the proof of II. As the next thing we need to show that if Lemma 4 holds true of an arbitrary even p, then it also holds true of p + 2. To do so we assume, as an inductive hypothesis, that

III. $CCC\ldots CC\alpha_1\alpha_2\alpha_3\ldots\alpha_{p-1}\alpha_pC\alpha_{i-1}A(\alpha_i\alpha_{i+2}\alpha_{i+4}\ldots\alpha_p) \in Cn(\varnothing)$, for any $i \in \{2, 4, 6, \ldots, p\}$

Now, to prove line

IV. $CCCC\ldots CC\alpha_1\alpha_2\alpha_3\ldots\alpha_p\alpha_{p+1}\alpha_{p+2}C\alpha_{i-1}A(\alpha_i\alpha_{i+2}\alpha_{i+4}\ldots\alpha_p\alpha_{p+2}) \in Cn(\varnothing)$, for any $i \in \{2, 4, 6, \ldots, p, p+2\}$

we assume that

1. $CCC\ldots CC\alpha_1\alpha_2\alpha_3\ldots\alpha_p\alpha_{p+1}\alpha_{p+2}$
and
2. α_{i-1}
we must deduce $A(\alpha_i\alpha_{i+2}\alpha_{i+4}\ldots\alpha_p\alpha_{p+2})$, ie.
$CC\alpha_{p+2}A(\alpha_i\alpha_{i+2}\alpha_{i+4}\ldots\alpha_p)A(\alpha_i\alpha_{i+2}\alpha_{i+4}\ldots\alpha_p)$ so let
3. $C\alpha_{p+2}A(\alpha_i\alpha_{i+2}\alpha_{i+4}\ldots\alpha_p)$

From 1, 3 and the transitivity of C

4. $CCC\ldots CC\alpha_1\alpha_2\alpha_3\ldots\alpha_p\alpha_{p+1}A(\alpha_i\alpha_{i+2}\alpha_{i+4}\ldots\alpha_p)$

On the other hand, from III and 2

5. $CCCC\ldots CC\alpha_1\alpha_2\alpha_3\ldots\alpha_{p-2}\alpha_{p-1}\alpha_pA(\alpha_i\alpha_{i+2}\alpha_{i+4}\ldots\alpha_p)$

From 4, 5 and the logical theorem $CCC\alpha\beta\gamma CC\alpha\gamma\gamma$

6. $A(\alpha_i\alpha_{i+2}\alpha_{i+4}\ldots\alpha_p)$

The case $\alpha_{i-1} = \alpha_{p+1}$ is easily fixed by using the logical theorem $C\alpha C\beta\alpha$. Thus line IV is proved. From II, III, IV and the principle of mathematical induction, Lemma 4 holds true for $n > 2$.

Q.E.D.

LEMMA 5

Let n be an even positive integer number greater than or equal to 2. If $Cn(\varnothing)$ is the set of theorems of the C-fragment of classical sentential logic, closed under detachment, then

$$CC\alpha_1A(\alpha_2\alpha_4\alpha_6\ldots\alpha_n)CC\alpha_3A(\alpha_4\alpha_6\alpha_8\ldots\alpha_n).\ \ldots\ldots\ldots\ldots$$
$$CC\alpha_{n-3}A(\alpha_{n-2}\alpha_n)CC\alpha_{n-1}\alpha_nCC\ldots CC\alpha_1\alpha_2\alpha_3\ldots\alpha_{n-1}\alpha_n \in Cn(\varnothing).$$

Proof. The case of $n = 2$ is obvious. Assume that the Lemma is true of an even p and suppose that

1. $C\alpha_1A(\alpha_2\alpha_4\ldots\alpha_{p+2})$,
$C\alpha_3A(\alpha_4\alpha_6\ldots\alpha_{p+2})$,
. . . .
$C\alpha_{p-1}A(\alpha_p\alpha_{p+2})$

2. $C\alpha_{p+1}\alpha_{p+2}$

and

3. $CC\ldots CC\alpha_1\alpha_2\alpha_3\ldots\alpha_p\alpha_{p+1}$

From 1 and the definition of A

4. $C\alpha_1CC\alpha_{p+2}A(\alpha_2\alpha_4\ldots\alpha_p)A(\alpha_2\alpha_4\ldots\alpha_p)$,
$C\alpha_3CC\alpha_{p+2}A(\alpha_4\alpha_6\ldots\alpha_p)A(\alpha_4\alpha_6\ldots\alpha_p)$,
. . . .
$C\alpha_{p-1}CC\alpha_{p+2}\alpha_p\alpha_p$

From 4

5. $CC\alpha_{p+2}A(\alpha_2\alpha_4\ldots\alpha_p)C\alpha_1A(\alpha_2\alpha_4\ldots\alpha_p)$,
$CC\alpha_{p+2}A(\alpha_4\alpha_6\ldots\alpha_p)C\alpha_3A(\alpha_4\alpha_6\ldots\alpha_p)$,
. . . .
$CC\alpha_{p+2}\alpha_pC\alpha_{p-1}\alpha_p$

On the other hand, from 2 and 3

6. $CC\ldots CC\alpha_1\alpha_2\alpha_3\ldots\alpha_p\alpha_{p+2}$

An application of the inductive assumption to steps 5 and 6 gives

7. $CC\alpha_{p+2}A(\alpha_2\alpha_4\ldots\alpha_p)CC\alpha_{p+2}A(\alpha_4\alpha_6\ldots\alpha_p)\ldots CC\alpha_{p+2}\alpha_p\alpha_{p+2}$

From 7 and the logical theorem $CCC\alpha\beta\alpha\alpha$

8. $CC\alpha_{p+2}A(\alpha_2\alpha_4\ldots\alpha_p)CC\alpha_{p+2}A(\alpha_4\alpha_6\ldots\alpha_p)\ldots CC\alpha_{p+2}\alpha_{p-2}\alpha_{p+2}$

Repetition of an argument similar to that leading to step 8 gives, finally,

9. α_{p+2}

It follows from 1-9 and 1 that the conclusion of our Lemma holds true of $p + 2$.

Q.E.D.

Proof of Theorem 1
The proof that (i) *implies* (ii) is obvious so let us proceed to the proof that (ii) *implies* (iii). Substitution of $n = 2$ in $(D^n\downarrow)$ gives $(D^2\uparrow)$. Hence by Lemma 1

1. $Cn(\varnothing)$ includes the set of theorems of intuitionistic implicational logic

and

2. $Cn(\varnothing)$ is closed under detachment

Clearly,

3. $Cn(\beta) \cap Cn(\alpha) \subseteq Cn(\alpha)$,
$Cn(\alpha) \subseteq Cn(\alpha)$

From 3 by ($D^n\downarrow$), in which we put n = 4,

4. $CCC\alpha\beta\alpha\alpha \in Cn(\varnothing)$

Formula $CCC\alpha\beta\alpha\alpha$ extends the intuitionistic implicational logic to the C-fragment of classical logic. Hence, from 1 and 4

5. $Cn(\varnothing)$ is the set of theorems of the C-fragment of classical sentential logic

This proves that (ii) implies (iii).

Proof that (iii) implies (i). Let n be an even positive integer number. We have to show that, on the assumption of condition (iii) of Theorem 1, conditions (*) and (**) are equivalent where (*) and (**) are as follows.

(*) $CC\ldots CC\alpha_1\alpha_2\alpha_3\ldots\alpha_{n-1}\alpha_n \in Cn(X)$

and

(**) $Cn(X, \alpha_i) \cap Cn(X, \alpha_{i+2}) \cap Cn(X, \alpha_{i+4}) \cap \ldots \cap Cn(X, \alpha_n) \subseteq Cn(X, \alpha_{i-1})$,
for any $i \in \{2, 4, 6, \ldots, n\}$

First, we will show that (*) implies (**). To do so suppose that, for any $i \in \{2, 4, 6, \ldots, n\}$,

1.1. $\beta \in Cn(X, \alpha_i) \cap Cn(X, \alpha_{i+2}) \cap Cn(X, \alpha_{i+4}) \cap \ldots \cap Cn(X, \alpha_n)$
By Lemma 1, condition (iii) of Theorem 1 implies that ($D^2\uparrow$) holds true.
Hence and from 1.1
1.2. $C\alpha_i\beta, C\alpha_{i+2}\beta, C\alpha_{i+4}\beta, \ldots, C\alpha_{n-2}\beta, C\alpha_n\beta \in Cn(X)$

From 1.2 and Lemma 3
1.3. $CA(\alpha_i\alpha_{i+2}\alpha_{i+4}\ldots\alpha_{n-1}\alpha_n)\beta \in Cn(X)$
From 1.3 and the transitivity of C
1.4. $CC\alpha_{i-1}A(\alpha_i\alpha_{i+2}\alpha_{i+4}\ldots\alpha_{n-1}\alpha_n)C\alpha_{i-1}\beta \in Cn(X)$
From 1.4 and Lemma 4
1.5. $CCCC\ldots CC\alpha_1\alpha_2\alpha_3\ldots\alpha_{n-2}\alpha_{n-1}\alpha_n C\alpha_{i-1}\beta \in Cn(X)$
From condition (*) and 1.5
1.6. $C\alpha_{i-1}\beta \in Cn(X)$
From 1.6 and $(D^2\uparrow)$
1.7. $\beta \in Cn(X, \alpha_{i-1})$

Lines 1.1-1.7 prove that

1. $Cn(X, a_i) \cap Cn(X, a_{i+2}) \cap Cn(X, a_{i+4}) \cap \ldots \cap Cn(X, a_n) \subseteq Cn(X, a_{i-1})$

so we may conclude that condition (*) implies condition (**).
To see that (**) implies (*) assume that

2.1. $Cn(X, \alpha_i) \cap Cn(X, \alpha_{i+2}) \cap \ldots \cap Cn(X, \alpha_n) \subseteq Cn(X, \alpha_{i-1})$,
for any $i \in \{2, 4, 6, \ldots, n\}$
From condition (iii) of Theorem 1 and Lemma 2
2.2. $A(\alpha_2\alpha_4\alpha_6\ldots\alpha_n) \in Cn(X, \alpha_j)$, for any $j \in \{2, 4, 6, \ldots, n\}$,
$A(\alpha_4\alpha_6\alpha_8\ldots\alpha_n) \in Cn(X, \alpha_j)$, for any $j \in \{4, 6, 8, \ldots, n\}$,
. . . .
$A(\alpha_{n-2}\alpha_n) \in Cn(X, \alpha_j)$, for any $j \in \{n-2, n\}$
From 2.1 and 2.2, for any $i \in \{2, 4, 6, \ldots, n\}$,
2.3. $A(\alpha_i\alpha_{i+2}\alpha_{i+4}\ldots\alpha_{n-2}\alpha_n) \in Cn(X, \alpha_{i-1})$
Cn is a closure operator so $\alpha_n \in Cn(X, \alpha_n)$. Hence and from 2.1
2.4. $\alpha_n \in Cn(X, \alpha_{n-1})$
From (iii), 2.3 and 2.4, for any $i \in \{2, 4, 6, \ldots, n\}$,
2.5. $C\alpha_{i-1}A(\alpha_i, \alpha_{i+2}, \alpha_{i+4}, \ldots, \alpha_n) \in Cn(X)$
From 2.5, (iii) and Lemma 5
2.6. $CC\ldots CC\alpha_1\alpha_2\alpha_3\ldots\alpha_{n-1}\alpha_n \in Cn(X)$

This proves that (**) implies (*). We may conclude, then, that conditions (*) and (**) are equivalent and so that (iii) implies (i). This proves Theorem 1.

Q.E.D.

As we have seen, condition (**), which appears in the above proof, is both sufficient and necessary for the provability of a conditional referred to in condition (*). It may be noted, in this context, that a related condition, presented in the Polish paper [2], although sufficient was not necessary and, besides, it was built into a less simple metalogical framework.

It follows from Lemma 1 and Theorem 1 that condition ($D^n\downarrow$) stands to the C-fragment of classical sentential logic in the same relation as condition ($D^n\uparrow$) stands to the C-fragment of intuitionistic implicational logic. It is for this reason that ($D^n\downarrow$) may be called the deduction theorem for classical sentential logic.

Clearly, our classification of conditional statements into ($\uparrow$)-conditionals and ($\downarrow$)-conditionals is not disjoint. For instance, condition $C\alpha_1\alpha_2$ is both ascending and descending. To get the classification improved in this respect we may simply treat $C\alpha_1\alpha_2$ as ascending. Also, the difference between ($D^n\uparrow$) and ($D^n\downarrow$) does not consist in that the first condition generates as logical theorems exclusively ascending conditionals because one of the consequences of ($D^n\uparrow$) is that, for any n,

$$CCCC\ldots CCC\alpha_1\alpha_1\alpha_2\alpha_2\ldots\alpha_{n-1}\alpha_{n-1}\alpha_n\alpha_n \in Cn(\varnothing)$$

where, clearly, $CCCC\ldots CCC\alpha_1\alpha_1\alpha_2\alpha_2\ldots\alpha_{n-1}\alpha_{n-1}\alpha_n\alpha_n$ is descending.

However, the point here is that, unlike ($D^n\uparrow$), condition ($D^n\downarrow$) generates as logical theorems also all those ($\downarrow$)-conditionals which are intuitionistically unprovable.

Let us distinguish between even ($\downarrow$)-conditionals and odd ($\downarrow$)-conditionals according to whether n is even or odd. Clearly, Theorem 1 provides characterization of even ($\downarrow$)-conditionals only. However, it may be remarked that no genuine odd ($\downarrow$)-conditionals are provable as theorems of the C-fragment of the classical sentential logic. Indeed, no conditional of the form $CC\ldots CC\alpha_1\alpha_2\alpha_3\ldots\alpha_{n-1}\alpha_n$ is a theorem of the C-fragment of classical sentential logic where n is odd and α_1, α_2, α_3, ..., α_n are sentential variables.

Modifications or specifications of ($D^n\downarrow$) can be used to provide a full characterization of the C-fragments of some non-classical logical systems. One such modification of ($D^n\downarrow$) provides a full characterization of the C-fragment of system S5 of C. I. Lewis.

Let S^c denote the set of all conditionals of S, ie.

$$S^c = \{C\alpha\beta\colon \alpha, \beta \in S\}$$

and let us consider the condition

$(DC^n\downarrow)$ If $X \cup \{\alpha_4, \alpha_6, \alpha_8, \ldots, \alpha_n\} \subseteq S^c$, then $(D^n\downarrow)$.

We have the following theorem.

THEOREM 2
The following conditions are equivalent.

(i) $(DC^n\downarrow)$ for any even positive integer number n;
(ii) $(DC^n\downarrow)$ for any even positive integer number n ≤ 4;
(iii) $Cn(\varnothing)$ is the set of theorems of the C-fragment of Lewis's system S5, closed under detachment.

The proof of this theorem is similar to that of Theorem 1.

It follows from Theorem 2 that the logic, which condition $(DC^n\downarrow)$ determines, is precisely the C-fragment of Lewis's sentential logic S5 based on the rule of detachment as the only inference rule.

University of Auckland

ACKNOWLEDGMENT

I would like to thank the anonymous referee for the insightful comments.

REFERENCES

[1] HERBRAND, J., Recherches sur la théorie de la démonstration. *Prace Tow. Nauk. Warsz., Wydz. III*, Nr 33, 1930, 1-61.
[2] SURMA, S. J., Twierdzenia o dedukcji dla implikacji zstepujacych. *Studia Logica*, vol. 22, 1968, 61-83.
[3] TARSKI, A., Über einige fundamentale Begriffe der Metamathematik. *Sprawozd. Tow. Nauk. Warsz., Wydz. III*, Nr 23, 1930, 22-29.