Logique & Analyse **133-134** (1991), 251-270

GRADED MODALITIES IN EPISTEMIC LOGIC*

W. VAN DER HOEK
J.-J. Ch. MEYER**

Abstract

We propose an epistemic logic with so-called graded modalities, in which certain types of knowledge are expressible that are less absolute than in traditional epistemic logic. Beside 'absolute knowledge' (which does not allow for any exception), we are also able to express 'accepting φ if there are at most n exceptions to φ'. This logic may be employed in decision support systems where there are different sources to judge the same proposition. We argue that the logic also provides a link between epistemic logic and the more quantitative (even probabilistic) methods used in AI systems. In this paper we investigate some properties of the logic as well as some applications.

1. *Introduction*

'Infallible' computers are computers that have multiple processors (usually each from a different company and programmed in a different way using different programming languages) to check and double-check on the results. Decisions are taken on a kind of democratic basis: the results that come up most often as the results of a certain calculation, are the ones that matter and are used to make a decision. The idea is just based on statistics: the chance that n independent processors are faulty at the same time is p^n for an already very small probability p. Typically, infallible computers are used in situations where the failure of a computer would have disastrous conseq-

* This work is partially supported by ESPRIT III Basic Research Action No. 6156 (DRUMSII).

** Also at the University of Nijmegen, Dept. of Math & Comp. Sc., Toernooiveld, 6525 ED Nijmegen.

uences, such as the stock exchange, certain security situations, and the so-called flying-by-wire (i.e. using a steering computer) of an airplane like the Airbus A320.

Decision Support Systems working on infallible computers, and devices with several input sensors in general, may have knowledge that is source-dependent. In this paper we will propose an epistemic logic that can deal with knowledge (or some may prefer the term *belief* here) that is not absolutely true in all worlds, but may have exceptions in the sense that there are worlds in which the assertion (that was believed) is nevertheless not true, such as in the case of a faulty processor or sensor in the situation described above.

Consider an agent getting input from three different sources w_1, w_2 and w_3. Suppose further-more, that two types of information are relevant for this agent, say p and q. All the sources agree on p: they mark p as true. Finally, in w_1 and w_2, q is true, whereas in w_3, it is false. When using 'standard' modal logic to model epistemic notions (cf. [MHV91] for an overview), one would consider the resources w_i ($i \leq 3$) to be worlds in an S5-Kripke model (cf. [HC68] for an introduction to modal logic), and observe that the agent knows p, i.e. Kp holds, but does not know q or $\neg q$, since he considers both alternatives to be possible: $Mq \wedge M\neg q$ holds.

This is about the limit of the expressibility of standard modal epistemic logic (we will formalize this claim in the next Section), where the only operators that are available are K and M, to express 'truth in all accessible worlds' and 'truth in some accessible world', respectively. Since the favourite system for knowledge (S5) may be interpreted on Kripke models in which the accessibility relation is universal (cf. [HM85, MHV91]), we may leave out the reference to this relation, leaving one with a system in which one can associate 'K' with 'all (worlds)' and M with 'some (world)' (cf. [GP90]).

However, in the above example, it might be desirable to be able to express that the agent has more confidence in q than in $\neg q$. (For instance, a robot which (who?) is searching for block A, may choose to first look for it on block B, if two of its sensors tell him it is there —while the third sensor tells him it is on block C.) One way to achieve this is to add a qualitative modality '$>$', enabling the agent to judge ($q > \neg q$), as is done in [Ho91a] (and, in a specifically epistemic context, see also [Le80]). Here, we will take an alternative approach, in which we add quantitative modalities to the language (M_n, $n \in \mathbb{N}$), with the intended meaning that more than n successors verify φ. Then, in the above example, we can describe the agent's point of view in a more precise manner, like for instance the fact that he considers exactly two q-alternatives possible ('input-sources'), and exactly one $\neg$q-alternative.

Actually, adding such 'graded modalities' to the modal language is not new. We refer to [Ho92a] for some history, and a first investigation into the expressibility, decidability and definability of this graded language. An application of those graded modalities, especially of the graded analogue of S5, has been studied in the area of *Generalized Quantifiers*, (cf. [HoR91]). In this paper, we try to explain how the greater expressive power of graded modalities may be used in *epistemic logic*. We already showed in [HoM92] how these new modalities may help to make some issues in the field of *implicit knowledge* explicit. However, there, the graded modalities are motivated to establish some properties on a 'meta-level'; adding them to the language enables one to more precisely define accurate models for implicit knowledge; in particular, we showed how one can employ graded modalities to define the intersection of accessibility relations. Here, though, we try to use the new operators directly in the object language in order to obtain a more fine-tuned epistemic logic. We think that, using the enriched language, one has an appropriate tool to deal with notions like 'uncertain', or 'almost certain' knowledge (or belief). The new operators may then be helpful to reason with *degrees of acceptance*. In fact, one may distinguish as many degrees of belief as there are graded modalities. To support our claim, we will sketch some directions in which such modalities might be employed.

The rest of this paper is organised as follows. In the following Section, we will introduce our main system, together with its natural semantics. In Section 3, we investigate how this system of Section 2 can be interpreted epistemically. Then we give some examples in Section 4 and conclude by indicating some further directions of research in Section 5.

2. *The system Gr(S5)*

Before plunging into the definitions of the graded language and the formal system, it may be useful to keep in mind how standard modal logic (together with its semantics) is used to model knowledge. There, Kφ (φ is known) is defined to be true in a Kripke model (M,w) iff in all worlds v accessible from w, (M,v) is a model for φ. Also, Mφ is defined to be $\neg$K$\neg\varphi$, which will be true in w iff φ is true in some accessible world v.

Now, consider the following Kripke model $M_k = \langle W, R, \pi \rangle$, where W

$= \{w_1, w_2, w_3, \ldots w_k\}$ $(k \geq 4)$, $R = W \times W$ and $\pi(w_i)(p) =$ true for all $i \leq k$; $\pi(w_i)(q) =$ true iff $i \notin \{2, 3\}$; $\pi(w_i)(r) =$ true iff $i \in \{1,3\}$ (cf. Figure 1 below).

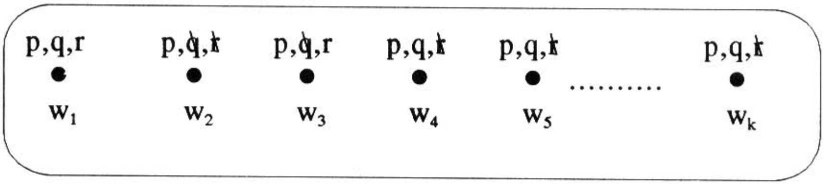


Figure 1

At the end of this Section, we will demonstrate, in a precise way, that in all *purely modal* formulas that are true in this model, one may freely interchange the role of q and r. In other words, despite the fact that q is true 'almost everywhere' in the model, and r is false 'almost everywhere', the modal language is too weak to express this difference between q and r. We claim that, both in the cases where worlds are interpreted in one-one correspondence to counterparts in the physical world (e.g. like sensors —Section 1) and where worlds correspond to possible (but made up) situations for some agent, a tool to distinguish 'q-statements' from 'r-statements' in the above model is highly desirable.

We provide such a tool by adding *graded modalities* $M_0, M_1, \ldots$ to the modal language. The intended interpretation of $M_n\varphi$, $(n \in \mathbb{N})$ will be that there are more than n accessible worlds verifying φ. By defining $K_n\varphi \equiv \neg M\neg_n\varphi$, $K_n\varphi$ is true iff at most n accessible worlds refute φ. In terms of epistemic operators, note that $K_0\varphi$ boils down to $K\varphi$, so that we may interpret K_0 as our (certain) knowledge operator. Generally, $K_n\varphi$ means that the agent reckons with at most n exceptions for φ. Dually, $M_n\varphi$ then means that the agent considers more than n alternatives possible, in which φ is true. Now, what would be the appropriate properties of these 'defeasible' necessity operators? For instance, what kind of introspective properties are desirable? Many possibilities present themselves at this point, but for the time being we will remain on solid ground by considering the graded analogue of S5.

Our language L is built, in the usual way, from propositional atoms p, q, ..., $\in$ P, using the standard connectives $\bot$, $\top$, $\wedge$, $\vee$, $\neg$, $\rightarrow$ and $\leftrightarrow$. Moreover, if $\varphi \in L$, then so is $M_n\varphi$ $(n \in \mathbb{N})$. From now on, we will assume that n, m, k $\in \mathbb{N}$. We use K_n as an abbreviation for $\neg M_n \neg$. Finally we introduce the abbreviation $M!_n\varphi$, where $M!_0\varphi \equiv K_0\neg\varphi$, $M!_n\varphi \equiv (M_{n-1}\varphi \wedge \neg M_n\varphi)$, if $n > 0$. From the definitions above, it is clear that $M!_n$ means 'exactly n'.

2.1 *Definition*. The system Gr(S5) is defined as follows (cf. [HoR91]). It has inference rules Modus Ponens and Necessitation:

R0 $\vdash \varphi, \vdash \varphi \rightarrow \psi \Rightarrow \vdash \psi$
R1 $\vdash \varphi \Rightarrow \vdash K_0\varphi$

It has also the following axioms (for each $n \in \mathbb{N}$):

A0 all propositional tautologies
A1 $K_0(\varphi \rightarrow \psi) \rightarrow (K_n\varphi \rightarrow K_n\psi)$
A2 $K_n\varphi \rightarrow K_{n+1}\varphi$
A3 $K_0\neg(\varphi \wedge \psi) \rightarrow ((M!_n\varphi \wedge M!_m\psi) \rightarrow M!_{n+m}(\varphi \vee \psi))$
A4 $\neg K_n\varphi \rightarrow K_0\neg K_n\varphi$
A5 $K_0\varphi \rightarrow \varphi$

Before elaborating on the impact of the axioms on our intended epistemic reading of the operators, which we will do in the following Section, let us pause for a moment to sharpen our understanding of the postulates as such. The system with rules R0 and R1, axioms A0 - A3 is the graded modal analogue of K, the basic modal system —so let us refer to it with Gr(K). In Gr(K), A1 is a kind of 'generalized K-axiom' (cf. 2.3), A2 is a way to 'decrease grades' in the possibility operator (A2 is equivalent to $M_{n+1}\varphi \rightarrow M_n\varphi$) and using A3, one can go to 'higher grades'. To ensure that the definitions work out rightly, we take proposition 2.10 from [Ho92a]:

2.2 *Proposition*. The following are derivable in Gr(K) (and hence in Gr(S5)):

(i) $M_n(\varphi \wedge \psi) \rightarrow (M_n\varphi \wedge M_n\psi)$
(ii) $M!_n\varphi \wedge M!_m\varphi \rightarrow \bot$ $(n \neq m)$
(iii) $K_n\neg\varphi \leftrightarrow (M!_0\varphi \,\nabla\, M!_1\varphi \,\nabla \wedge \nabla\, M!_n\varphi)$ (∇ denotes 'exclusive or')
(iv) $\neg M_n(\varphi \vee \psi) \rightarrow \neg M_n\varphi$
(v) $M_{n+m}(\varphi \vee \psi) \rightarrow (M_n\varphi \vee M_m\psi)$
(vi) $M!_n\varphi \wedge M_m\varphi \rightarrow \bot$ $(m \geq n)$
(vii) $M_n(\varphi \wedge \psi) \wedge M_m(\varphi \wedge \neg\psi) \rightarrow M_{n+m+1}\varphi$

(viii) $(K_0\neg(\varphi \wedge \psi) \wedge (M_n\varphi \wedge M_m\psi)) \rightarrow M_{n+m+1}(\varphi \vee \psi)$

To see the system in action, we will give a derivation of a theorem which is a generalisation of the K-axiom from standard modal logic.

2.3 *Proposition.* The following is derivable in Gr(K) (and in Gr(S5)).

$$K_n(\varphi \rightarrow \psi) \rightarrow (K_m\varphi \rightarrow K_{n+m}\psi).$$

Proof. We implicitly use the (Gr(K)-derivable) rule of substitution: $\vdash \alpha \leftrightarrow \beta \Rightarrow \vdash \varphi \leftrightarrow \varphi[^{\alpha}/_{\beta}]$. Then, observe that $\vdash$ A1 $\leftrightarrow (K_0(\varphi \rightarrow \psi) \rightarrow (M_n\varphi \rightarrow M_n\psi))$ (*). To see this, note that $\vdash K_0(\varphi \rightarrow \psi) \leftrightarrow K_0(\neg\psi \rightarrow \neg\varphi)$, and $\vdash (M_n\varphi \rightarrow M_n\psi) \leftrightarrow (K_n\neg\psi \rightarrow K_n\neg\varphi)$.

1	$\vdash$	$\neg\psi \rightarrow (\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \neg\psi)$	A0
2	$\vdash$	$K_0(\neg\psi \rightarrow (\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \neg\psi))$	R1,1
3	$\vdash$	$M_{n+m}\neg\psi \rightarrow M_{n+m}((\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \neg\psi))$	(*),2
4	$\vdash$	$M_{n+m}((\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \neg\psi)) \rightarrow M_n(\varphi \wedge \neg\psi) \vee M_m(\neg\varphi \wedge \neg\psi)$	2.2(v)
5	$\vdash$	$((\varphi \wedge \neg\psi) \rightarrow \neg(\varphi \rightarrow \psi)) \wedge ((\neg\varphi \wedge \neg\psi) \rightarrow \neg\varphi)$	A0
6	$\vdash$	$(M_n(\varphi \wedge \neg\psi) \rightarrow M_n\neg(\varphi \rightarrow \psi)) \wedge (M_m(\neg\varphi \wedge \neg\psi) \rightarrow M_m\neg\varphi)$	R1,1, A1
7	$\vdash$	$M_{n+m}\neg\psi \rightarrow (M_n\neg(\varphi \rightarrow \psi) \vee M_m\neg\varphi)$	A1, 3,4,6
8	$\vdash$	$\neg M_m\neg(\varphi \rightarrow \psi) \rightarrow (\neg M_n\neg\varphi \rightarrow \neg M_{n+m}\neg\psi)$	A0,7
9	$\vdash$	$K_n(\varphi \rightarrow \psi) \rightarrow (K_m\varphi \rightarrow K_{n+m}\psi)$	Def K_k,8

Note that, by taking n = m = 0 in 2.3, we get the K-axiom in Gr(S5). In the presence of the necessitation rule, this means that K_0 is a 'normal' modal operator. In fact, the axioms A4 and A5 are graded versions of Euclidicity and reflexivity, respectively. Before making this explicit, we give the definition of the models on which we want to interpret formulas of L.

2.4 *Definition.* A *Kripke structure M* is a tuple $\langle W,\pi,R\rangle$, where W is a set (of 'worlds' or 'states'), π a truth assignment for each $w \in W$ and R a binary relation on W. If R is both reflexive and Euclidean (i.e. $\forall xyz((Rxy \wedge Rxz) \rightarrow Ryz))$, we say that $M \in$ S5. It is easily verified that the accessibility relations R of $M \in$ S5 are equivalence relations. A model $M \in$ S5 is known to be a model of (standard) S5 (cf. [MHV91]).

2.5 *Definition*. For a Kripke structure M we define the *truth of* φ at w inductively:

(i) $(M,w) \models p$ iff $\pi(s)(p) = \textit{true}$, for all $p \in P$.
(ii) $(M,w) \models \neg\varphi$ iff not $(M,w) \models \varphi$.
(iii) $(M,w) \models \varphi \vee \psi$ iff $(M,w) \models \varphi$ or $(M,w) \models \psi$.
(iv) $(M,w) \models M_n\varphi$ iff $|\{w' \in W \mid Rww' \text{ and } (M,w') \models \varphi\}| > n, n \in N$.

2.6 *Remark*. Note that $(M,w) \models K_n\varphi$ iff $|\{w' \in W \mid Rww' \text{ and } (M,w') \models \neg\varphi\}| \leq n$. Also, note that the modal operators M and K (in the literature also written as M and L, or $\Diamond$ and $\Box$) are special cases of our indexed operators: $M\varphi \equiv M_0\varphi$ and $K\varphi \equiv K_0\varphi$.

2.7 *Definition*. We say that φ is *true* in *M* at w if $(M,w) \models \varphi$. If such an *M* and w exist for φ, we say that φ is *satisfiable*. Formula φ is *true* in *M* ($M \models \varphi$) if $(M,w) \models \varphi$ for all $w \in W$, and φ is called *valid* ($\models \varphi$) if $M \models \varphi$ for all *M*. If C is a class of models (like S5), $C \models \varphi$ means that for all $M \in C$, $M \models \varphi$.

With these semantic definitions, we can formalize our claim about the model of Figure 1. For two propositional variables x and y, let $[x \leftrightarrow y]\varphi$ be the formula obtained from φ by interchanging the x and y in φ. (This can be defined in terms of $[^u/_v]\varphi$, substitution of u for v in φ, as follows: $[x \leftrightarrow y]\varphi \equiv [^y/_z][^x/_y][^z/_x]\varphi$, where z is some atom not occurring in φ.) We suppose that the accessibility relation in the model of Figure 1 is *universal*, i.e. for all w and v we have Rwv.

2.8 *Theorem*. Let φ be a (non-graded) modal formula, and *M* the model of Figure 1. With the definition of $[x \leftrightarrow y]\varphi$ given above, we claim that $M \models \varphi \Leftrightarrow M \models [q \leftrightarrow r]\varphi$.

Proof. The theorem follows from the following observation. Let $f: W \to W$ be the following function: $f(w_1) = w_1$; $f(w_2) = w_2$; $f(w_3) = w_4$, $f(x) = w_3$ for all $x \in W \setminus \{w_1, w_2, w_3\}$. Then, we claim, that for all $w \in W$ and all non-graded modal formulas φ: $(M,w) \models \varphi \Leftrightarrow (M,f(w)) \models [q \leftrightarrow r]\varphi$. This claim is established using a simple induction, of which we demonstrate the modal case $\varphi = M\psi$: suppose that $(M,w) \models M\psi$. By the truth-definition of M, there must be some $v \in W$ such that $(M,v) \models \psi$. By the induction hypothesis, we obtain $(M,f(v)) \models [q \leftrightarrow r]\psi$. Since R is universal, we have

$(M,\mathrm{f(w)}) \models \mathrm{M}[\mathrm{q}\underline{\leftrightarrow}\mathrm{r}]\psi$ which is of course equivalent to $(M,\mathrm{f(w)}) \models [\mathrm{q}\underline{\leftrightarrow}\mathrm{r}]\mathrm{M}\psi$, i.e. $(M,\mathrm{f(w)}) \models \varphi$. The converse of this claim is proven similarly. The proof of the theorem then proceeds as follows: $\mathrm{M} \not\models \varphi \Leftrightarrow$ there is some w such that $(M,\mathrm{w}) \models \neg\varphi \Leftrightarrow$ there is some w such that $(M,\mathrm{f(w)}) \models [\mathrm{q}\underline{\leftrightarrow}\mathrm{r}]\neg\varphi \Leftrightarrow$ there is some w such that $(M,\mathrm{f(w)}) \models \neg[\mathrm{q}\underline{\leftrightarrow}\mathrm{r}]\varphi \Leftrightarrow \mathrm{M} \not\models [\mathrm{q}\underline{\leftrightarrow}\mathrm{r}]\varphi$.

We end this introduction to Gr(S5) by recalling the following results:

2.9 *Theorem.* (Completeness: [Fi72], [FC88]). For all $\varphi \in \mathrm{L}$, $\mathrm{Gr(S5)} \vdash \varphi$ iff $\mathrm{S5} \models \varphi$.
Thus S5 is also a class of models characterizing Gr(S5).

2.10 *Theorem.* (Finite models: [Ho92a]). Any $\varphi \in \mathrm{L}$ is satisfiable iff it is so on a finite model.

2.11 *Theorem.* (Freedom of nestings: cf. [HoR91]). In Gr(S5), each formula is equivalent to a formula in which no nestings of (graded) modal operators occur.

Related to the last theorem, a popular slogan in modal logic is that in S5, 'the inner modality always wins', we have e.g. $\mathrm{KM}\varphi \equiv \mathrm{M}\varphi$, $\mathrm{MK}\varphi \equiv \mathrm{K}\varphi$ and $\mathrm{MM}\varphi \equiv \mathrm{M}\varphi$ in S5. However, in the case of Gr(S5) this is not always sufficient: we *do* have $\mathrm{M}_3\mathrm{M}_5\varphi \equiv \mathrm{M}_5\varphi$, but instead of $\mathrm{M}_5\mathrm{M}_3\varphi \equiv \mathrm{M}_3\varphi$ we now have $\mathrm{M}_5\mathrm{M}_3\varphi \equiv \mathrm{M}_5\top \wedge \mathrm{M}_3\varphi$, accounting for the fact that $\mathrm{M}_5\mathrm{M}_3\varphi$ implies that, so to speak, 5 worlds are around.

3. *Epistemic Reading*

Returning to the main point of this paper: how can Gr(S5) serve as an appropriate starting point to study epistemic phenomena? To start with, R0 and A0 express that we are dealing with an (extension of) classical propositional logic: we may use Modus Ponens and reason 'classically' (A0). By taking S5 as a 'standard' system for knowledge, the observations in the preceding Section suggest that we interpret $\mathrm{K}_0\varphi$ as 'φ is known' (by the agent: for the moment, we focus on one-agent systems, although graded modalities do not prevent us from studying multi-agent systems —on the contrary, cf. [HoM92]).

Then, R0, R1, A0 and A5 find their motivation in the same fashion as the corresponding properties in S5, i.e., we may use Modus Ponens, the agent knows all (Gr(S5))-derivable facts, we are dealing with an extension of propositional logic (A0) and moreover the agent cannot know facts that are not true (A5).

In order to interpret the other axioms, we need to have some intuition about the meaning of $K_n\varphi$. The semantics suggest, that it should be something like 'the agent reckons with at most n exceptional situations for φ', or 'the agent "knows-modulo-n-exceptions" φ'. Thus, the greater n is in $K_n\varphi$, the less confidence in φ is uttered by that sentence. The latter observation immediately hints at A2, $K_n\varphi \rightarrow K_{n+1}\varphi$: if the agent foresees at most n exceptions to φ, he also does so with at most n+1 exceptions. Of course, the generalisation of A3, for $n > 0$; $K_n\varphi \rightarrow \varphi$ is not valid: if the agent does not know φ for sure, i.e., if he allows for exceptions regarding φ, he cannot conclude that φ is the case. Thus $K_n\varphi$ expresses a form of "uncertain knowledge".

In standard S5, we have the axiom $\neg K\varphi \rightarrow K\neg K\varphi$, expressing the agent's negative introspection: if he does not know a given fact, he knows that he does not (this is of course an 'over-idealised' property of knowledge, especially if we have in mind capturing human knowledge; see [MHV91] for a short discussion and further references). We may write this introspection axiom equivalently as

(1) $M\varphi \rightarrow KM\varphi$,

saying that the agent has awareness (see [FH88] or [HoM89] for a discussion on this 'awareness' —defined in a technical sense) of what he considers to be possible. Now that we have at hand a more fine-tuned mechanism to distinguish between 'grades' of possibility, it seems straightforward to strengthen the bare introspection formula (1) to

(2) $M_n\varphi \rightarrow K_0M_n\varphi$,

saying that the agent is aware of the fact that he considers more than n φ-situations possible. (2) is equivalent with our axiom A4. Note that (2) is at the same time the 'most general' way to generalise (1): it implies, (using A2 m-1 times) for instance $M_n\varphi \rightarrow K_mM_n\varphi$.

In the same spirit, we can interpret A1: if the agent knows that φ implies ψ, then, if he believes that there can be at most n exceptions to φ, he will not imagine more than n exceptions to ψ, since every exception to ψ will be an exception to φ as well, i.e. $K_0(\varphi \rightarrow \psi) \rightarrow (K_n\varphi \rightarrow K_n\psi)$, or equivalently (cf. 2.3), $K_0(\varphi \rightarrow \psi) \rightarrow (M_n\varphi \rightarrow M_n\psi)$ (A1'). In epistemic logic, the K-axiom, $K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$, has been considered a source of *logical omniscience* ([FH88] or [Ho92c]), which yields too idealistic a notion of knowledge (and certainly of belief). It would mean that the agent is capable to close his knowledge (belief) under logical implication. However, now that we allow for weaker notions of knowledge, it appears that the K-axiom is only valid for K_0, which we may consider as a kind of 'ideal' knowledge. Instead of a K-axiom for each K_n, we have the much more realistic (cf. 2.3)

(3) $K_n(\varphi \rightarrow \psi) \rightarrow (K_m\varphi \rightarrow K_{n+m}\psi)$.

This seems very reasonable (suppose n, m > 0): if the agent has some confidence that φ implies ψ, and also has some confidence in φ, his conclusion that ψ holds should be stated with even less certitude than that of the two assertions separately. This is reminiscent of *plausible* ([Re76]) or *defeasible reasoning*, where reasoning under uncertainties is also the topic of investigation. Note that (3) guarantees that, the longer the chain of reasoning with uncertain arguments, the less certain the conclusion can be stated by the agent. Moreover, note that, although (3) holds, if n > 0 we do not have $K_n(\varphi \rightarrow \psi) \rightarrow (K_n\varphi \rightarrow K_n\psi)$: this makes it questionable to call K_n a *modal operator* (if n > 0). However, here we do so because of the interpretation of such operators in Kripke models.

Finally, to understand A3, we must recall that $M!_n\varphi$ means that the agent is aware of exactly n possible situations in which φ is true. But then, A3 simply states this property of additivity: if the agent knows that φ and ψ are mutual exclusive events, and he is thinking of exactly n situations in which φ is true and, at the same time, m situations in which ψ is true, altogether he has to reckon with (n + m) situations in which one of these two alternatives is the case.

Up to now, we have been deliberately slightly vague about what $M_n\varphi$ and $M!_n\varphi$ exactly should mean. For instance, is this index n within the scope of the agent's knowledge? That is, does the agent know himself of (exactly) n concrete situations in which φ holds, and if so, is it possible that there are still other situations he does not know about where φ holds as well? This makes sense in situations in which the agent has to make decisions that depend on rules that allow for *exceptions*. The alternative interpretation is, that these n situations are only known to the reasoner *using* the system at a meta-level, interpreting K_n as some abstract n-degree of knowledge (or perhaps belief, if n is greater than some threshold)? We believe that the logic can be used in both these cases, and will not fix the interpretation in this paper.

It is argued (cf. [HM85]) that the axiom which distinguishes knowledge (K) from belief (B) is $(K\varphi \rightarrow \varphi)$. Instead of $(B\varphi \rightarrow \varphi)$, for belief, the weaker axiom $\neg B\bot$ is added. Now that we have (infinitely) many operators around, we might see how they behave in this respect. In Gr(S5), $\neg K_n\bot$ (meaning that more than n possibilities are reckoned with) is derivable only for $n = 0$. If we would have (add) $\neg K_n\bot$, implying that he allows for more than n possibilities, he 'does not know too much' (if n is big). And indeed, as long as the agent considers at least one possible world, it means that he does not know contradictions ($\neg K\bot$). In case he has no epistemic alternative left his knowledge is all encompassing but inconsistent ($K\varphi$, for any φ). This is of course excluded in S5 (and hence in Gr(S5)), but so far, there was no way to exclude the extreme case of an 'omniscient knower', i.e., one for which $(K\varphi \leftrightarrow \varphi)$ holds. Semantically speaking, there was no way to define the class of Kripke models in which each world had more than one successor. Using graded modalities this can be enforced by adding $M_1\top$ to any system.

4. *Examples*

When interpreting K_n as an 'n-degree of knowledge', we recall that the higher the degree, the less certain the knowledge. The picture is denoted in the following chain:

$$K_0\varphi \rightarrow K_1\varphi \rightarrow \ldots K_n\varphi \rightarrow K_{n+1}\varphi \ldots \Rightarrow \ldots M_{n+1}\varphi \rightarrow M_n\varphi \rightarrow \ldots \rightarrow M_1\varphi \rightarrow M_0\varphi.$$

Here, the '$\rightarrow$' denotes logical implication. If, semantically speaking, the number of alternatives is infinite, the sequence is an infinite one, and '$\Rightarrow$' denotes implication, in the sense that all M_i-formulas are logically weaker than all the K_φ-formulas. We could, as argued above, interpret the strongest formula in this chain ('$K_0\varphi$') as "φ is known", and the weakest ('$M_0\varphi$') as "φ is not impossible" —but even as "φ is believed", cf. [HoM89].

If, however, the number of alternatives is *finite*, say N, we get the sequence

$$K_0\varphi\ (\equiv M_{N-1}\varphi) \rightarrow K_1\varphi\ (\equiv M_{N-2}\varphi) \rightarrow \dots K_n\varphi\ (\equiv M_{N-n-1}\varphi) \rightarrow \dots$$
$$K_{N-1}\varphi\ (\equiv M_0\varphi) \rightarrow K_N\varphi\ (\equiv \top)$$

In fact, this is the case in the situation of the introduction, where the agent is capable to sum up a complete description of the model by listing a (finite) number of possible situations determined by some finite set of propositional atoms.

The property that each formula of L is equivalent to one in which no nestings of the operator occur (2.11), supports to consider an S5-model to be a collection of 'points' (worlds) that can have certain properties (summarized by the atomic formulas that are true in each world), the language L being sufficiently expressive to sum up the quantitative distribution of those properties over the model. Alternatively, identifying worlds with truth assignments to primitive propositions, as is usual in standard S5-models, we can view a Gr(S5)-model as a multi-set of truth assignments rather than a set of these as in standard, ungraded modal logic. A special case, of course, is that situations (= truth assignments) occur only once in a description. We shall refer to these models as *simple* (referring to the original Latin meaning of this word). Note that in simple models it is still sensible to use graded modalities, since an assertion (even a primitive proposition) may nevertheless hold in more than one situation, as e.g. p in the situations {p is true, q is false} and {p is true, q is true}.

To be more specific, let us consider a simple example. Suppose we are given that the agent knows $(p \vee q)$ and also $(p \vee r)$. Since q and r are 'independent' propositional atoms, we try to formalise our intuition that the agent has more confidence in p than in q (or r). Given the three propositional atoms, the agent will consider five of the eight (a priori) possible worlds: the worlds in which $(\neg p \wedge (\neg q \vee \neg r))$ is true, left out. Thus, assuming that we have a simple model in the sense above, we get $(M!_5\top \wedge M!_4p \wedge M!_3q \wedge M!_3r)$, indicating that indeed, p is the 'most frequent' atom. (This is perhaps more appealing when interpreting the premises as $(\neg q \rightarrow p)$ and $(\neg r \rightarrow p)$, expressing that there are two (independent) reasons for p.)

Michael Freund has proposed a formal system for defaults, in which the number of worlds refuting some default is important when imposing an order on such defaults. In Freund's words: "... if we have to choose between two assertions of Δ that are in conflict, our natural move is to drop the one that is violated by the greatest number of worlds..." (Cf. Fr93]). Graded modalities provide a tool to explicitly reason with such numerical values. However, in Freund's general approach, the worlds themselves may have attached weights to them, so that a full treatment seems to be out of scope here; in our set up, all worlds would have the same weight (although generalizing this to arbitrary weights seems to be feasible).

The following example is well known in the literature on probabilistic reasoning ([Pe88]) and on non-monotonic reasoning ([Gi87]) where it is called the lottery paradox. It deals with the situation of a lottery with n tickets, numbered 1 ... n. Let w_i denote 'ticket i will be the winning ticket' ($1 \leq i \leq n$). Many default theories (cf. [Gi87]) allow one to obtain the defeasible conclusion $\neg w_i$ for each $i \leq n$, using a default rule expressing "if you can assume that $\neg w_i$, conclude $\neg w_i$". In particular, one derives $(\neg w_1 \wedge \ldots \neg w_n)$, raising the question why we call the happening a lottery, if we can derive on forehand that no ticket will (probably) win.

In our graded language, we would model the situation as follows, using the premises P1-P2:

P1 $K_0\neg(w_i \wedge w_j)$ $(i \neq j)$	no two tickets will win simultaneously
P2 $M!_n T \wedge M!_1 w_i$ $(i \leq n)$	of all n possibilities, there is one in which ticket i wins

From these premises, one safely deduces that $K_0(w_1 \vee w_2 \vee \ldots \vee w_n)$, and even $K_0(w_1 \nabla w_2 \nabla \ldots \nabla w_n)$ (with ∇ standing for exclusive or) expressing that exactly one of the tickets will win. Moreover, one deduces $K_1\neg w_i$, expressing, that, except for (at most one) possibility, ticket i will not win. One should compare this with ungraded modal approaches, in which it is possible to express P1 together with the fact that there are *at least* n possibilities (the latter is done by adding $[M(w_1 \wedge \neg w_2 \wedge \neg w_3 \wedge \ldots \wedge \neg w_n) \wedge M(\neg w_1 \wedge w_2 \wedge \neg w_3 \wedge \ldots \wedge \neg w_n) \wedge \ldots \wedge M(\neg w_1 \wedge \neg w_2 \wedge \neg w_3 \wedge \ldots \wedge w_n)]$ but in which there is no way to guarantee that there are *at most* n of such worlds (adding copies to the 'intended model' is never excluded).

In the example of the introduction, the number of worlds (sources) was fixed. This gives rise to considering $Gr_k(S5)$, with fixed $k \in \mathbb{N}$, which is obtained from Gr(S5) by adding $M!_k T$ to it. Let $k^\wedge = \min\{m \in \mathbb{N} \mid m > \frac{1}{2}k\}$. Using a preference modality (*use belief* in the sense of Perlis [Pe86]) expressed by operator P as in [MH91], we may express the democratic principle of infallible computers in $Gr_k(S5)$, with k denoting the number of computers, as $P\varphi \leftrightarrow K_{k^\wedge}\varphi$, that is, φ is preferred (is a practical/working/use belief) iff it is true in more than the half of all sources.

Note that there is no logical omniscience in this respect, in a way resembling the local reasoning approach of [FH88].

However, note that here, P is not a normal modality as it *is* in [MH91], since, as follows from our discussion about the K-axiom, $P(\varphi \rightarrow \psi) \rightarrow (P\varphi \rightarrow P\psi)$ is not valid. To illustrate this, consider the case of an airplane with three sensors w_1, w_2 and w_3 in which "it is foggy" (φ) is true according to w_1 and w_2 (and not according to w_3), and "permission to take off" (ψ) according to sensor w_1 only. Then we have that both $P(\varphi \rightarrow \psi)$ (since $\varphi \rightarrow \psi$ is true in w_1 and w_3) and $P\varphi$ (since φ is true in w_1 and w_2), thus both φ and $(\varphi \rightarrow \psi)$ are working beliefs, without the conclusion "permission to take off" (ψ) being one.

One might contrast this with the situation where *rules* are added to the system (in the form of (certain) knowledge: cf. [MH91]). For instance, in the above example, $K_0(\varphi \rightarrow \neg\psi)$ might be a rule (it is *known* by the decision support system, independently of the information supplied by the sources, that fog is sufficient to deny permission to leave). If in addition, $P\varphi$ would be the case (the systems supposes φ based on the information of its sources), it would take as a working belief $\neg\psi$, i.e. there is no permission to fly! (This follows directly from axiom A1: $K_0(\varphi \rightarrow \neg\psi) \rightarrow (K_{k^\wedge}\varphi \rightarrow K_{k^\wedge}\neg\psi)$, i.e., $K_0(\varphi \rightarrow \neg\psi) \rightarrow (P\varphi \rightarrow P\neg\psi)$.)

Recall that $Gr_k(S5) = Gr(S5) + M!_k\top$. Using Proposition 2.2 we see that for any φ, $M!_0\varphi \nabla M!_1\varphi \nabla \ldots \nabla M!_k\varphi$ is derivable in $Gr_k(S5)$. Here, a formula of the form $M!_m\varphi$ is rather informative, since we know the *relative number* of occurrences of φ. This is close to adding *probabilities* to (modal) logic. In the literature, there have been several attempts to do so (Cf. [FH89, HR87]). In order to avoid the problem of losing *compactness* (cf. [Ho92b]) it was suggested to allow only for a *finite* number of probabilities ([FA89, Ho92b]). We will give the main idea now.

Let us denote the language with graded modalities with L_c (a language for counting). The language L_p (a language for probabilities) is like L_c, but instead the operators M_n, we now add operators $P^{>}{}_r$, for each $r \in [0,1]$. The intended meaning of $P^{>}{}_r\varphi$ is, that the probability of φ is greater than r. Two interpret this language, we assume to have a finite set F, such that $\{0, 1\} \subseteq F \subseteq [0,1]$ and $\forall r,s(r \in F \wedge s \in F \wedge r + s \leq 1) \Rightarrow (r + s \in F)$. Now, a Probability Kripke model M over F is a tuple $\langle W, \pi, R, P_F\rangle$, where W is a set of worlds, π as before, R is a serial relation on W ($\forall w \exists v Rwv$) and P_F: $W \times P(W) \rightarrow F$ is a function from the powerset of W to F, for each $w \in W$, satisfying:

- $X \cap Y = \emptyset \Rightarrow P_F(w, X \cup Y) = P_F(w,X) + P_F(w,Y) \quad X,Y \in \mathcal{P}(W)$
- $P_F(w,\{v \mid Rwv\}) = 1$

The truth definition for L_P formulas is obtained straightforwardly, with the modal case

$$(M,w) \models P^{>}{}_r\varphi \text{ iff } P_F(w,\{v \mid Rwv \text{ and } (M,v) \models \varphi\}) > r$$

We denote the class of all these models by PK_F. In [Ho92b] a logic P_FD is given, such that we have $P_FD \vdash \varphi \Leftrightarrow PK_F \models \varphi$.

Let $Gr_k(K)$ be $Gr(K) + M!_k\top$. This class is obviously sound and complete with respect to K_k, the class of Kripke models in which each world has exactly k successors. Moreover, let

$$Fk = \{0 = \frac{0}{k}, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k}{k} = 1\}$$

Given a model $M_c = \langle W, \pi, R\rangle \in K_k$, we straightforwardly associate a model $M_p = \langle W, \pi, R, P_{Fk}\rangle \in PK_{Fk}$ with it by stipulating that $P_{Fk}(w,v) = \frac{1}{k}$ if Rwv holds, and $P_{Fk}(w,v) = 0$ otherwise. The relation between the classes of valid formulas of those two models is as follows. Let $\tau: L_c \to L_p$ be a translation from graded to probabilistic formulas, distributing over the logical connectives and such that $\tau(M_n\varphi) = \bot$ if $n \geq k$ and $\tau(M_n\varphi) = P^{>}{}_r t(\varphi)$, with $r = \frac{n}{k}$, if $n < k$. We claim that for all $\varphi \in L_c$, $(M_c,w) \models \varphi$ iff $(M_p,w) \models \tau(\varphi)$. Conversely, we define $\sigma: L_p \to L_c$ as a translation that distributes over the connectives and for which moreover $\sigma(P^{>}{}_r\varphi) = M_n\sigma(\varphi)$, where $n = \max\{m \mid \frac{m}{k} \leq r\}$. In this case, we have for all $\varphi \in L_p$, $(M_p,w) \models \varphi$ iff $(M_c,w) \models \sigma(\varphi)$. Note that, although in general $\tau(\sigma(\varphi)) \neq \varphi$ and $\sigma(\pi(\varphi)) \neq \varphi$, we do have $M_p \models \varphi \leftrightarrow \tau(\sigma(\varphi))$ and $M_c \models \varphi \leftrightarrow \sigma(\tau(\varphi))$.

The above observation immediately ties up the ways to reason about relative occurrences with ways to reason about probabilities, at least for the minimal graded modal logic $Gr_k(K)$. However, we argued that the natural graded system to reason epistemically with (or about) numbers, is Gr(S5). So what is the counterpart, in the sense of the paragraph above, of $S5_k$, the semantic class of $Gr_k(S5)$? Models for $Gr_k(S5)$ are $M = \langle W, \pi, R\rangle$ in which R is an equivalence relation, and for which $\forall w(|\{v \mid Rwv\}| = k)$. But, by an argument using generated models (which still holds for the graded cases, cf. [Ho92a]), we can also conceive them as models in which R is *universal* ($\forall wvRwv$) and for which $\forall w(|\{v \mid Rwv\}| = k)$. In such a model, there is no need to explicitly refer to the relation R. So let $U_k = \{M = <W,\pi> \mid |W| = k\}$, where $(M,w) \models M_n\varphi$ iff $|\{v \in W \mid (M,v) \models \varphi\}| > n$. Then we have that $Gr_k(S5) \vdash \varphi \Leftrightarrow U_k \models \varphi$.(1)

On the side of probabilistic models , we add the following constraint in order to compare them with U_k. The class PKU_k is a subclass of probabilistic PK-models $M = \langle W, \pi, R, P_{Fk}\rangle$ in which P_{Fk} is such that $\forall wv(P_{Fk}(w,v) = \frac{1}{k})$. In particular, we have a kind of reflexivity: $P_{Fk}(w,w) = \frac{1}{k}$. We claim that we have, for all $\varphi \in L_c$: $U_k \models \varphi \Leftrightarrow PK_k \models \tau(\varphi)$ and for all $\varphi \in L_p$: $U_k \models \sigma(\varphi) \Leftrightarrow PK_k \models \varphi$.

We now proceed by mentioning the use of graded operators to express the 'numerical syllogisms' as introduced in [AP88]. In the following, the left hand side is our translation of the numerical syllogisms on the right hand side.

1	$M!_7 d$	exactly 7 days of the week are known
2	$M!_5(w \wedge d)$	I know 5 of them to be working days
3	$M_3(s \wedge d)$	at least 4 days are shopping days
4	$\therefore M_1(w \wedge s)$	∴ I know at least 2 days to go working and shopping.

To prove such a conclusion formally, it turns out to be worthwhile to split up the set of formulas (the formulas in d, w and s), in to a set of partitions (cf. Figure 2).

The formal derivation now reads as follows (note $\Gamma = \{\alpha, \beta, \gamma, \delta, \epsilon, \phi, \pi\}$):

(1)	$M!_7(\gamma \vee \delta \vee \phi \vee \pi)$	translation of 1 (cf. Figure 2)
(2)	$M!_5(\delta \vee \pi)$	translation of 2
(3)	$M_3(\phi \vee \pi)$	translation of 3
(4)	$K_0\neg((\phi \vee \pi) \wedge (\gamma \vee \delta))$	definition of G
(5)	$\neg M_7(\gamma \vee \delta \vee \phi \vee \pi)$	(1), def. M!
(6)	$\neg M_3(\gamma \vee \delta)$	2.2.(viii), (3), (4), (5)
(7)	$(K_0(\delta \rightarrow (\gamma \vee \delta)) \rightarrow (M_3\delta \rightarrow M_3(\gamma \vee \delta))) \wedge K_0(\delta \rightarrow (\gamma \vee d))$	A1' and A0, R1
(8)	$\neg M_3\delta$	A0, (6), (7)
(9)	$M!_0\delta \vee M!_1\delta \vee M!_2\delta$	(8), 2.2.(iii)

(1) We will not prove this here, but the result is easily obtained by combining results of [Fi72] and [HR91].

(10) $\neg M_1\pi \rightarrow (M!_0\pi \vee M!_1\pi)$ — 2.2.(iii)

(11) $K_0\neg(\delta \wedge \pi) \rightarrow [((M!_0\delta \vee M!_1\delta \vee M!_2\delta) \wedge (M!_0\pi \vee M!_1\pi)) \rightarrow (M!_0(\delta \vee \pi) \vee M!_1(\delta \vee \pi) \vee M!_2(\delta \vee \pi)]$ — A0, A3

(12) $\neg M_1\pi \rightarrow (M!_0(\delta \vee \pi) \vee M!_1(\delta \vee \pi) \vee M!_2(\delta \vee \pi)$ — (9)(10)(11), $\vdash K_0\neg(\delta \wedge \pi)$

(13) $[(M!_0(\delta \vee \pi) \vee M!_1(\delta \vee \pi) \vee M!_2(\delta \vee \pi)] \rightarrow \neg M_2(\delta \vee \pi)$ — A0,def. M!

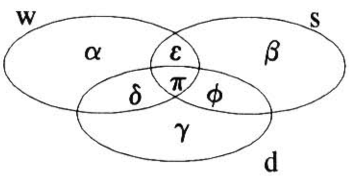
(14) $(M!_5(\delta \vee \pi) \rightarrow M_4(\delta \vee \pi)) \wedge (M_4(\delta \vee \pi) \rightarrow M_2(\delta \vee \pi))$ — def. M!,A2 twice

(15) $(\neg M_2(\delta \vee \pi) \wedge M!_5(\delta \vee \pi)) \rightarrow \bot$ — A0, (14)

(16) $(\neg M_1\pi \wedge M!_5(\delta \vee \pi)) \rightarrow \bot$ — A0, (12), (13), (15)

(17) $M_1\pi(2)$, — (16)

(18) $M_1(\pi \vee \epsilon)$ — (17), A1'



For example, it is understood that $w \leftrightarrow (\alpha \vee \beta \vee \delta \vee \varepsilon \vee \pi)$. If $\Gamma = \{\alpha, \beta, \gamma, \delta, \varepsilon, \phi, \pi\}$, then for all different x1, ... xk $\in \Gamma$ $(2 \leq k \leq 7)$:
(*)$\vdash \neg(x1 \wedge \ldots \wedge xk)$

Figure 2

We round off this Section by mentioning a link between the graded formalisms presented here and so called *terminological* or *concept languages*, used for knowledge representation (cf. [DLN91] for these languages, and [HoR92] for a deeper analysis of the connection with graded modal systems). Such languages provide a means for expressing knowledge about hierarchies of sets of objects with common properties. Expressions in such languages are built up using *concepts* and *roles*. Compound expressions are then made using a number of constructs.

Typical examples of such constructs are intersection, complement and restricted quantification, yielding examples like the concept 'mathematicians whose pupils are all clever' (m ⊓ ALL p c, with the modal counterpart (m $\wedge$ $[R_p]$c), where $[R_p]$ is the necessity operator for the relation R_p). Many of such concept languages also allow for number restriction, as in the concept 'mathematicians who have at least 4 clever pupils (m ⊓ $\geq$ 4 p c). Obviously, here the graded modalities come into play: the latter concept would translate into the formula (m $\wedge$ $\langle R_p\rangle_3$c), where $\langle R_p\rangle$ is the dual of $[R_p]$. In [HoR92], one can find several results on relating several concept languages in the hierarchy of concept languages to (some fragment of) some graded modal logic.

5. *Conclusion*

We have argued that extending the modal language with graded modalities (taking into account the number of accessible worlds) gives some interesting options for epistemic logic. We provided some examples of how this new language can be used in an epistemic context. Particularly, we indicated how these operators can be used in the context of a fixed number of sources. It thus provides us with a framework for *reasoning with exceptions.*

We think the graded modalities are especially useful in 'laboratory-like situations', where explicit bounds are prescribed. Areas of application that may be worthwhile may therefore typically be found in situations where numbers of counter-examples have a clear evidence and meaning. Typical examples (that have not been worked out by us, yet) may thus be found in 'laboratory situations' like (reasoning about) *a voting* or in a *legal context* (where for instance a petition is granted when at least n requirements are met) or more generally, intelligent databases of which the quantities of the data matters (cf. [Ho92c], for several examples).

We see two lines of future work. Firstly, we may transfer some standard questions from 'standard' epistemic logic to the graded language. For instance, it might be interesting to study the introspection properties more systematically, like was done e.g. in [Ho91b]. Secondly, we think that several of our proposals have natural generalisations. For instance, where the P-operator models the notion of 'more than-a-half', we could have such operators P_n for 'more than-an-n-th'.

Free University, Amsterdam

REFERENCES

[AP88] P. Atzeni and D.S. Parker. Set containment inference and syllogisms. *Theoretical Computer Science,* 62:39-65, 1988.

[DLN91] F. Donini, M. Lenzerini, D. Nardi & W. Nutt. The complexity of concept languages, in *Proc. 2nd. Conf. on Knowledge Representation*, Morgan Kaufman, San Mateo, 151-162, 1991.

[FA89] M. Fattorosi-Barnaba & G. Amati, *Modal operators With Probabilistic Interpretations, I,* in Studia Logica 4 (1989) 383-393.

[FC88] M. Fattorosi-Barnaba and C. Cerrato. Graded modalities III.

Studia Logica, 47:99-110, 1988.

[FH88] R.F. Fagin and J.Y. Halpern. Belief, awareness, and limited reasoning. *Artificial Intelligence,* 34:39-76, 1988.

[FH89] R.F. Fagin & J.Y. Halpern, *Unertainty, Belief and Probability,* in: Proc. of the International Joint Conference on Artificial Intelligence (1989) 1161 - 1167. Extended version: IBM Research Report RJ 6191 (1991).

[Fi72] K. Fine. In so many possible worlds. *Notre Dame Journal of Formal Logic,* 13:516-520, 1972.

[Fr93] M. Freund, Default extensions: an alternative to the probabilistic approach. Unpublished manuscript, University of Orleans, France.

[Gi87] M.L. Ginsberg (ed.), *Readings in Nonmonotonic Reasoning,* Morgan Kaufmann, Los Altos, 1987.

[GP90] V. Goranko and S. Passy. Using the universal modality: Gains and questions. Preprint, Sofia University, 1990.

[HC68] G.E. Hughes and M.J. Cresswell. *Introduction to Modal Logic.* Methuen, London, 1968.

[HM85] J.Y. Halpern and Y.O. Moses. A guide to the modal logics of knowledge and belief. *Proceedings IJCAI-85.* Los Angeles, CA, 1985, pages 480-490.

[Ho91a] W. van der Hoek, Qualitative Modalities, *proceedings of the Scandinavian Conference on Artificial Intelligence -91,* B. Mayoh (ed.) IOS Press, Amsterdam (1991), 322 - 327.

[Ho91b] W. van der Hoek, Systems for Knowledge and Beliefs, in: In J. van Eijck, editor, *Logics in AI-JELIA'90,* Lecture Notes in Artificial Intelligence 478, Springer, Berlin, 1991, pp. 267 - 281. Extended version to appear in *Journal of Logic and Computation.*

[Ho92a] W. van der Hoek. On the semantics of graded modalities. *Journal of Applied Non-Classical Logics,* Vol. 2, number 1 (1992) pp. 81-123.

[Ho92b] W. van der Hoek, Some Considerations on the Logic P_FD, (a Logic Combining modalities and Probabilities), in: *Logic Programming,* A. Voronkov (ed.), LNCS 592, Springer, Berlin (1992), pp. 474-485. Extended version to appear in *Journal of Applied Non-Classical Logics.*

[Ho92c] W. van der Hoek, *Modalities for Reasoning about Knowledge and Quantities,* Ph.D. thesis, Amsterdam, 1992.

[HoM89] W. van der Hoek and J.-J.Ch. Meyer. Possible logics for belief. *Logique et Analyse*, 127-128, (1989), 177-194.

[HoM92] W. van der Hoek & J.-J.Ch. Meyer, Making Some Issues of Implicit Knowledge Explicit. *International Journal of Foundations of Computer Science*, Vol. 3, number 2 (1992) pp. 193-224.

[HoR91] W. van der Hoek & M. de RiJke, Generalized Quantifiers amd Modal Logic, in: *Generalized Quantifiers Theory and Applications*, J. van der Does and J. van Eijck (eds), Dutch Network for Logic, Language and Information (1991), pp. 115-142. To appear in *Journal of Logic, Language and Information*.

[HoR92] 'Counting Objects in Generalized Quantifier Theory, Modal Logic, and Knowledge Representation' Report Free University, IR-307, Amsterdam (1992).

[HR87] J.Y. Halpern and Rabin, *A Logic to Reason about Likelihood*, in: Artificial Intelligence 32:3 (1987) 379 - 405.

[Le80] W. Lenzen. *Glauben, Wissen und Warscheinlichkeit*. Springer Verlag, Wien, 1980.

[MH91] J.-J.Ch. Meyer and W. van der Hoek. Non-monotonic reasoning by monotonic means. In J. van Eijck, editor, *Logics in AI-JELIA'90*, Lecture Notes in Artificial Intelligence 478, Springer, Berlin, 1991, pages 399-411.

[MHV91] J.-J.Ch. Meyer, W. van der Hoek, and G.A.W. Vreeswijk. Epistemic logic for computer science: A tutorial. *EATCS bulletin*, 44:242-270, 1991. (Part I), and *EATCS bulletin*, 45:256-287, 1991. (Part II).

[Pe86] D. Perlis. On the consistency of commonsense reasoning. *Computational Intelligence*, 2:180-190, 1986.

[Pe88] J. Pearl. *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*. Morgan Kaufmann, San Mateo, California (1988).

[Re76] N. Rescher. *Plausible Reasoning, an Introduction to the Theory and Practice of Plausibilistic Inference*. Van Gorcum, Assen, 1976.