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NEW RESULTS IN THE ANALYSIS OF SOME CONDITIONAL QUANTIFIERS AND THEIR LOGICS

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0. *Introduction*

Binary generalized quantifiers are binary relations Q_E between subsets of a given non-empty universe E. Examples are $some_E$ (non-empty intersection) and no_E (empty intersection). It has been pointed out in van Benthem [1984] that there is a striking analogy between the analysis of binary quantifiers and conditional sentences of the form if *X*, (*then*) *Y*, considered as expressing relations between antecedent sets and consequent sets of situations. More precisely, given a relevant non-empty universe E of situations, the functor *if* may be analysed as a determiner denoting a binary quantifier if_E, so that given any two terms *X*, *Y* having the extensions [[X]], [[Y]] $\subseteq$ E, if *X*, (*then*) *Y* is true if and only if $<[[X]], [[Y]]> \in if_E$.

The above analogy can be exploited from two opposite but complementary perspectives. First, there is the "inverse logic" perspective, which consists of bringing out some general constraints of conditionality and finding which quantifiers satisfy those constraints. The opposite view —the "direct logic" perspective— consists of considering some of the selected quantifiers and studying the conditional logics determined by them. Of course, neither perspective presupposes that there is a "correct conditional logic", but rather that there are many intuitions of conditionality which can be formulated in many ways.

From the inverse logic perspective, several properties, expressing more or less a priori intuitions of conditionality, have already been suggested. The central ones may be formulated by the following patterns of inference (*X*, *Y*, *Z* are set variables or combinations of set variables with the operations "$\neg$", "$\wedge$" and "$\vee$", which we interprete as complementation, intersection and union respectively) :

CONS	$ifXY \Leftrightarrow ifX(Y \wedge X)$	*conservativity*
C1	$ifXY \Rightarrow ifX(Y \vee Z)$	*confirmation*
C2	$ifX(Y \wedge Z) \Rightarrow if(X \wedge Y)Z$	
C3	$ifXY \Rightarrow if(X \vee Z)(Y \vee Z)$	

R $\Rightarrow$ *ifXX* *reflexivity*
BE *Replacement of Boolean Equivalents*

These principles characterize the *minimal conditional logic M*. They have been extensively explained in van Benthem [1984]. For our purpose, it is sufficient to note that most current accounts of conditionals obey them.

Let us call every quantifier satisfying the minimal conditional logic *M* a *conditional quantifier*. Though uncountably many quantifiers are conditional in this sense, few of them have been studied in detail. Among these latter the best known ones are (where A, B $\subseteq$ E and we write "Q_EAB" instead of "$\langle A, B\rangle \in Q_E$") :

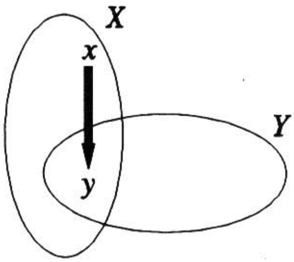
(i) *all*$_E$AB *iff* $A \subseteq B$;
(ii) *all or some*$_E$AB *iff* $A \subseteq B$ or $A \cap B \neq \varnothing$;
(iii) *at least half*$_E$AB *iff* $|A \cap B| \geq |A - B|$;
(iv) *all but finitely many*$_E$AB *iff* $\begin{cases} A \subseteq B, \text{if } A \text{ is finite} \\ A - B \text{ is finite, if } A \text{ is infinite.} \end{cases}$
(v) *all preferred*$_{\langle E,R\rangle}$ AB *iff* all R-maximal situations in A are in B.

It is easy to verify that the first fourth quantifiers validate all basic *M*-principles and satisfy in addition the following two constraints :

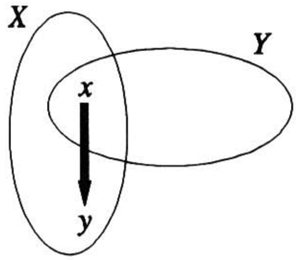
Consistency $Q_E A\varnothing$ only if $A = \varnothing$.
Quantity Whether Q_EAB or not depends only on the cardinalities $|A \cap B|$ and $|A - B|$.

Being quantitative, these quantifiers are *context-neutral* in the sense that whether Q_EAB or not depends only on the sets A and B. This allows us to omit to specify the parameter E whenever convenient.

The last quantifier operates on *structured universes* $\langle E, R\rangle$, R being a binary relation between situations in E. Restricted to *finite* structured universes $\langle E, R\rangle$ where R is *irreflexive* and *transitive*, it is a consistent quantifier which validates all basic *M*-principles. It is not quantitative however; witness the following counter-example (the extensions of *X*, *Y* are as indicated) :



ifXY



not *ifXY*

From now on, when speaking about this quantifier, we will assume that it is restricted to finite structured universes ⟨E, R⟩ with irreflexive and transitive R.

Though these quantifiers are well known, many issues concerning their inferential behaviours are still to be settled. The purpose of this paper is precisely to give the answers to some of these questions. So our contribution belongs essentially to the direct logic, since it is a matter of studying the conditional logics determined by a certain class of conditional quantifiers.

Our study presupposes some prior choice of formalism, expressing more or less the inferential properties of our quantifiers. Here we will stick with the minimal formalism consisting only of elementary conditional formulae *ifXY*, where *X*, *Y* may be complex Boolean terms. Most fundamental patterns of conditional inference are expressible in this restricted formalism.

Beside a prior choice of formalism, another important decision concerns the cardinality of the universes. Must we admit finite universes only, or infinite universes as well? *All but finitely many* requires universes which are at least denumerable, otherwise it reduces to *all*. On the other hand, *all preferred* behaves well on finite universes, but on infinite universes it does not satisfy the minimal logic *M* ; for instance, it does not validate C2. (To see this, consider an infinite increasing R-chain in some infinite set [[*X*]] which includes a non-empty and finite set [[*Y*]] and let [[*Z*]] = ∅ : so *ifX*(*Y* ∧ *Z*), since there is no R-maximal element in [[*X*]], yet not if(*X* ∧ *Y*)*Z*, since the irreflexivity of R ensures that there is at least one R-maximal element in [[*Y*]].) Moreover, when considering infinite universes, we may decide to be careful not to become entangled in higher infinite cardinalites, and so to restrict the range of admissible universes to denumerable ones. In this (cautious) perspective, here is a list of the logics we may consider :

Quantifiers	*Logics*	
Finite universes		
all	C	*Classical conditional logic*
all or some	E	*Exemplary conditional logic*
at least half	QD	*Quasi democratic conditional logic*
all preferred	S	*Subjunctive conditional logic*
Denumerable universes		
all but finitely many	N	
all	C_ω	
all or some	E_ω	
at least half	QD_ω	

For the most part, this paper is about these logics, although other possibilities, such as the inferential behaviours of the quantifiers on *at most* denumerable universes, or on infinite universes of higher cardinalities, will be considered when they seem relevant.

The logics C, E and QD and their mutual relationships have been studied in detail in Lapierre [1991] ; so in the next section we will only summarize the main known facts about them. In Section 2 we will complete our analysis by providing some answers concerning the relationships between these and other logics.

1. *Summary of the main known facts about C, E and QD.*

The main known facts about the mutual relationships between the logics C, E and QD are summarized in Figure 1 above. We will give a quick proof of each indicated relation and specifications about each logic involved in the relation.

Figure 1

$$E$$

$$\subset \qquad \supset$$

$$M \subset E \cap QD \qquad E \cup QD \subset C$$

$$\supset \qquad \subset$$

$$QD$$

The logic C, the one of *all*, is the most inclusive logic in this figure. Obviously, this logic corresponds, in our restricted formalism, to the logic of (S5)-strict implication. So it contains, beside all M-principles, the following additional ones :

CNT	$ifXY \Rightarrow if\neg Y \neg X$	*contraposition*
LM	$ifXY \Rightarrow if(X \wedge Z)Y$	*left-monotonicity*
TRN	$ifXY,\ ifYZ \Rightarrow ifXZ$	*transitivity*
CNJ	$ifXY,\ ifXZ \Rightarrow ifX(Y \wedge Z)$	*conjunction*
DSJ	$ifXZ,\ ifYZ \Rightarrow if(X \vee Y)Z$	*disjunction*

However, we know that all basic principles of M together with TRN axiomatize C completely.(1) Moreover, there is another important and general result about the inferential behaviour of *all*, due to van Benthem : on finite universes, *all* is the only quantitatve and consistent quantifier which satisfies CNJ.(2) But we note that the situation is different for denumerable universes or if we consider non-quantitative quantifiers. For instance, both the quantitative *all but finitely many* and the non-quantitative *all preferred*

(1) We give here the sketch proof. Suppose that a given conditional formula $ifXY$ is not derivable from M+TRN. Then there must be a finite Boolean algebra with an additional binary relation Q interpreting *if* which refutes $ifXY$. This Boolean algebra may be represented isomorphically as a power set algebra and under this representation, Q becomes a binary relation between sets. Then one shows that under a homomorphic restriction of this power set algebra, Q becomes inclusion.

(2) The proof runs as follows. Suppose that *If* satisfies CNJ, that *If*AB and $A \not\subseteq B$. Then$(A \cap B) \subset A$ and so *If*$A(A \cap B)$, by Conservativity (which follows from Quantity). Thus by Quantity, for every $C \subseteq E$ such that $|A \cap C| = |A \cap B|$, *If*$A(A \cap C)$, and so, *If*$A((A \cap B) \cap C)$ (by CNJ). Applying the same reasoning, we obtain in the end that *If*$A\varnothing$. Therefore $A = \varnothing$, by Consistency. So $A \subseteq B$: a contradiction.

validate CNJ. But we will return to these quantifiers in the next section.

The assertion that $E \cup QD \subset C$ contains the following non-immediate propositions.

$E \cup QD \neq C$: this follows from the fact that neither *all or some* nor *at least half* validates CNJ.

$E \subseteq C$: let $ifX_1Y_1,\ldots, ifX_nY_n/ifXY$ be any inference refuted by inclusion in some model. Already, every or at least one X_i-situation is a Y_i-situation ($1 \leq i \leq n$), because every X_i-situation is a Y_i-situation. On the other hand, there is an X-situation which is not a Y-situation, and thus, if there is no $X \wedge Y$-situation, we have a C-counter-example which is also an E-counter-example. Otherwise, consider the "homomorphic sub-model" consisting only of all non-$X \wedge Y$-situations, behaving in exactly the same way with regard to (non-)membership of the extensions of the set variables. All ifX_iY_i are still true according to inclusion, because every former inclusion must still hold in this new model. On the other hand, $ifXY$ is still false according to inclusion, but now there is no $X \wedge Y$-situation — and thus we have a C-counter-example which is also an E-counter-example.

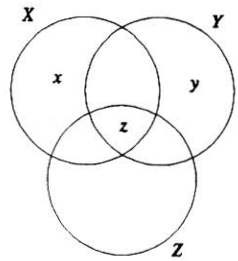
$QD \subseteq C$: let $ifX_1Y_1,\ldots, ifX_nY_n/ifXY$ be any inference refuted by inclusion in some model. Convert this model into a homomorphic sub-model which is both a C-counter-example and an E-counter-example (as above). Then all ifX_iY_i are true according to at least half, because they are true according to inclusion, while $ifXY$ is false according to *at least half*, because it is false according to *all or some* — and thus we have a C-counter-example which is also a QD-counter-example.

In order to give a more precise idea of the logic E, note first that DSJ belongs to this logic, as well than the following principles ($\top =_{def} X \vee \neg X$, X being any term) :

CCNJ	$ifXY, if(X \wedge Y)Z \Rightarrow ifX(Y \wedge Z)$	*cautious conjunction*
CSYM	$if\top X, ifXY \Rightarrow ifYX$	*conditional symmetry*
TN	$if\top X, ifXY \Rightarrow ifYX$	*transmissibility of necessity*
CWA	$if\top X, ifXY \Rightarrow if(X \vee Z)Y$	*conditional weakening of the antecedent*

However, it is easy to verify that all these principles, including DSJ, are derivable from all basic M-principles plus CCNJ. Though that does not mean that this set of principles axiomatize E completely, it is a very likely conjecture at this stage.

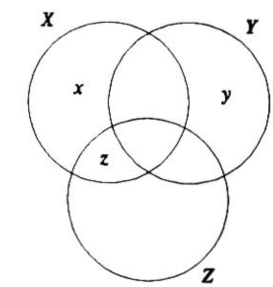
The assertion that $E \not\subseteq QD$ and $QD \not\subseteq E$ is established by the following two facts. First, as we pointed out, DSJ is valid according to *all or some*. However, DSJ is not valid according to *at least half*, witness the following *QD*-counter-example :



Secondly, it has been established in van Benthem [1986] that *at least half* validates the following principle (where X Δ Y abbreviates the symmetric difference $(X \wedge \neg Y) \vee (Y \wedge \neg X)$) :

PΔ $\quad if(X \Delta Y)Y,\ if(Y \Delta Z)Z \Rightarrow if(X \Delta Z)Z$

(To see this, note that a few calculations show that $|(A \Delta B) \cap B| \geq |(A \Delta B) - B|$ *iff* $|B - A| \geq |A - B|$ *iff* $|B| \geq |A|$. So PΔ follows from the transitivity of "$\geq$". Many other *QD*-principles can be found using this equivalence.) But PΔ is not valid according to *all or some*, as the following *E*-counter-example indicates :



In order to establish that $M \subset E \cap QD$, it is sufficient to show that $M \neq E \cap QD$, since both E and QD include M. This is quite simple since we easily verify that the following principle is valid according to both *all or some* and *at least half* without being deriviable in M alone ($\bot =_{def} X \wedge \neg X$, X being any term) :

CDSJ	$\mathrm{if}(X \wedge Y)\bot$, $ifXZ$, $ifYZ \Rightarrow if(X \vee Y)Z$	*cond. disjunction.*

Another way to see this is to consider the class of the quantifiers *all but at most n* ($n \geq 1$) : all of them validate all basic *M*-principles, but none validates CDSJ.

We may now turn our attention to the other logics.

2. *Some new facts about the other conditional logics*

Continuing with finite universes, we have to determine the location of *S* —the logic of *all preferred*— in the Figure 1 of Section 1. We noted above that CNJ is an *S*-principle. In fact, we know more than this. We know that *S* is precisely, in our restricted formalism, the counterfactual logic of Burgess [1981], which is completely axiomatized by R, CNJ, DSJ and the following two additional principles :

SIMP	$ifX(Y \wedge Z) \Rightarrow ifXY$	*simplification*
CLM	$ifXY$, $ifXZ \Rightarrow if(X \wedge Y)Z$	*cautious left-monotonicity*

But, as van Benthem pointed out, this axiomatization is equivalent to all basic *M*-principles plus CNJ, as we may easily verify.

The fact that *all preferred* validates the counterfactual logic of Burgess in our restricted formalism can be established as follows. Burgess' logic has a finite semantics which consists of "similarity models" ⟨W, C, V⟩, where W is a finite and non-empty set of worlds, V is a function assigning to each sentence *X* a subset of W (the worlds where *X* is true) and C is a function assigning to each world *w* a binary relation C_w between worlds in W (see also Lewis [1981], or van Benthem [1991], chap. 7, sec. 7.4, for the presentation of this semantics). Intuitively, $C_w yx$ means that *y* is closer to *w* than *x* is. So the minimal requirement is that C_w be a *strict partial order*, which amounts to transitivity and irreflexivity. The truth condition for a conditional *ifXY* is then as follows: *ifXY* is true in a similarity model ⟨W, C, V⟩ at some world *w* if and only if every C_w-closest *X*-world is a *Y*-world. The semantics for the general case is more complicated, but for our purpose the important fact to note is this : for every finite similarity model M = ⟨W, C, V⟩ and for every world $w \in$ W, there exists an *S*-model ⟨E, R, [[ ]]⟩ such that, given any conditional formula *ifXY* of our formalism, *ifXY* is true in M at *w* if and only if $\textit{all preferred}_{\langle E, R\rangle}[[X]][[Y]]$ (just set E = W, let *Rxy* if and only if $C_w yx$ and let [[ ]] be the restriction of V to Boolean terms). Conversely, for every S-model ⟨E, R, [[ ]]⟩ (which is finite by definition), there exists a finite similarity model M = ⟨W, C, V⟩ and a world $w \in W$ such that, given any conditional formula *ifXY* of our formalism, *all preferred*$_{\langle E, R\rangle}$[[X]][[Y]] if and only if *ifXY* is true in M at w (just set $W = E \cup \{w\}$, let $C_w xy$ if and only if Ryx and let V be any assignment which agrees with [[ ]] which respect to Boolean terms). The identity between S and the counterfactual logic of Burgess is established in this way.(3)

Given the identity between S and Burgess' logic, it is easy to analyse the mutual relationships between our logics. To begin with, S is obviously properly included in C : the C-principles TRN and LM are not valid according to all preferred, as we can easily verify, but all S-principles are valid according to inclusion. (Incidentally, *all preferred* reduces to inclusion when R is the empty relation, which establishes independently the inclusion of S in C.) Secondly, van Benthem [1986] showed that QD is properly included in S, and it was conjectured in Lapierre [1991] that the same relation holds between E and S. But a proof of this conjecture can in fact be extracted from van Benthem's proof, which is partially based on the above equivalence between finite similarity models and S-models. However, van Benthem's original proof was rather elleptical in many parts, so it is worth reformulating it, and in order to do this we will prove our former conjecture first.

THEOREM 1. $E \subset S$.

Proof. The S-principle CNJ is not valid according to *all or some*, and thus $E \neq S$. But every valid inference according to *all or some* is an S-principle too. Indeed, let $ifX_1Y_1,\ldots, ifX_nY_n/ifXY$ be any inference refuted in some S-model, and thus, in some finite similarity model at some world w. Already we have that every or at least one X_i-world is a Y_i-world ($1 \leq i \leq n$), since every C_w-closest X_i-world is a Y_i-world. On the other hand, there is a C_w-closest X-world where Y fails, say x. Consider now the sub-model consisting of w, x and all worlds on C_w-paths between these two. Then

(i) *ifXY* is still false at w, but now there is no $X \wedge Y$-world (note that x is the only X-world in this new model) ;

(3) There is another way to see this. First, all Burgess axioms are S-valid. Conversely, to every Burgess-counter-example, which can always be considered as finite, corresponds an S-counter-model, as we just showed.

(ii) all ifX_iY_i are still true at w, and so every or at least one X_i-world is a Y_i-world. Indeed, suppose that ifX_iY_i is false at w, i.e., there is a C_w-closest X_i-world where Y_i fails, say y. Then y was a $\neg Y_i$-world in the former model, and thus it was not a C_w-closest X_i-world, i.e., there was another X_i-world, say z, such that C_wzy . But either z was an X-world, or it was a $\neg X$-world. In the first case, z was an X-world closer to w than x was: a contradiction. In the second case, it must still be the case that C_wzy , and so y is not a C_w-closest X_i-world after all : another contradiction. Thus we have an E-counter-model for the same inference. ∎

THEOREM 2 (van Benthem [1986]). $QD \subset S$.
Proof. The S-principle CNJ is not valid according to *at least half*, and thus $QD \neq S$. Now, condider any inference which is refuted in some S-model, and thus, in some finite similarity model at some world w. Convert this model into the sub-model consisting of w, only one of the C_w-closest X-worlds where Y fails, and all worlds on C_w-paths between these two. As we showed above (proof of Theorem 1), all truth-values are preserved. Now, transform this sub-model into a system of concentric spheres *à la* Lewis [1973], according to the function $Distance(w, y)$ = the maximum lenght of a C_w-path going from w to y. It is easy to verify that the truth-values of the premises and of the conclusion are preserved in this Lewis model. (Now the key truth condition is this : $ifXY$ is true at w if and only if in the smallest sphere containing an X-world, every X-world is a Y-world.) Now, add in the biggest sphere (say the sphere 1) a copy of each world in this sphere. Then, going progressively to the smallest sphere (so to the center), add in the sphere $i+1$ as many copies of each world in this sphere as the sum of all copies already created in the previous spheres $1,\ldots,i$. In the end, one obtains a model where the condition "in the smallest sphere containing an X-world, every X-world is a Y-world" is equivalent to "there are no more $X \wedge \neg Y$-worlds than $X \wedge Y$-worlds", so a model which refutes the inference according to *at least half*.

The fact that neither *all or some* nor *at least half* validates CNJ implies that $E \cup QD \neq S$. So from the previous remarks and theorems, we infer this

THEOREM 3. $E \cup QD \subset S \subset C$.

Now let us consider denumerable universes. It is not very surprising that the logic of *all* as well than the logic of *all or some* do not change on these universes. We give here the proofs of these identities, which will be useful in some forthcoming demonstrations.

THEOREM 4. $C_\omega = C$ and $E_\omega = E$.
Proof. The non-immediate propositions are the followings.

$C_\omega \subseteq C$ and $E_\omega \subseteq E$: let ifX_1Y_1,..., ifX_nY_n/*if*XY be any inference refuted according to *all* (resp. *all or some*) in some finite model. Select one situation in the universe of this model and add countably many copies of this situation, behaving in exactly the same way with regard to (non-)membership of the extensions of the set variables. This procedure preserves inclusion as well than overlaping, and thus we have a C_ω-counter-model (resp. an E_ω-counter-model) for the same inference.

$C \subseteq C_\omega$: let *if*X_1Y_1,..., *if*X_nY_n/*if*XY be any inference refuted by inclusion in some denumerable model. So there is at least one $X \wedge \neg Y$-situation in this model, say x. Consider the model consisting of x alone, behaving in exactly the same way with regard to (non-)membership of the extensions of the set variables. The previous truth-values are still the same in this new model, and so we have a C-counter-model for the same inference.

$E \subseteq E_\omega$: let *if*X_1Y_1,..., *if*X_nY_n/*if*XY be any inference refuted according to *all or some* in some denumerable model. For each $1 \leq i \leq n$, select exactly one $X_i \wedge Y_i$-situation (if there is any) and select exactly one $X \wedge \neg Y$-situation. Consider the homomorphic sub-model consisting only of these selected situations, behaving in exactly the same way with regard to (non-)membership of the extensions of the set variables. Clearly, this model is finite (note that there is only a finite number of premises). Moreover, it is still the case that every or at least one X_i-situation is a Y_i-situation ($1 \leq i \leq n$), that there is one $X \wedge \neg Y$-situation, but that there is no $X \wedge Y$-situation. Thus we have an E-counter-model for the same inference. ■

Incidentally, this latter result about the inferential behaviours of *all* and *all or some* can be generalized to *at most* denumerable universes as well as to infinite universes of higher cardinalities.

With *at least half*, matters change, as the following theorem indicates.

THEOREM 5. $QD_\omega \subset E$.
Proof. DSJ is an E-principle which is not QD_ω-valid, and thus $QD_\omega \neq E$. (To see that DSJ is not QD_ω-valid, consider the QD-counter-example of

Section 1, and add countably many new situations outside the three relevant sets. The result, though trivial, is obviously a QD_ω-counter-example.) However, every QD_ω-valid inference is E-valid too. Indeed, let $ifX_1Y_1,\ldots,$ $ifX_nY_n/ifXY$ be any inference refuted in some E-model. For each $1 \leq i \leq n$, select exactly one $X_i \wedge Y_i$ -situation (if there is any) and add countably many copies of this situation, behaving in exactly the same way with regard to (non-)membership of the extensions of the set variables. Clearly this new model is denumerable. Moreover, for every $1 \leq i \leq n$, either there is no $X_i \wedge \neg Y_i$-situation, or there are countably many $X_i \wedge Y_i$-situations, which means in both cases that there are no more $X_i \wedge \neg Y_i$-situations than $X_i \wedge Y_i$-situations. On the other hand, there are more $X \wedge \neg Y$-situations than $X \wedge Y$-situations, since there was no $X \wedge Y$-situation at all, but at least one X-situation in the former model, which is still the case — and thus we have a QD_ω-counter-model for the same inference. ■

One notes again that this latter result about the inferential behaviour of *at least half* can be generalized to at most denumerable universes as well as to infinite universes of higher cardinalities.

Given what we know so far, Theorem 5 gives us two by-products, the first one concerning the mutual relationships between M, QD_ω, E, S and C.

THEOREM 6. $M \subset QD_\omega \subset E \subset S \subset C$.

Proof. The non-immediate assertions are the followings.

$QD_\omega \subset E \subset S \subset C$: from Theorem 3 and Theorem 5.

$M \subset QD_\omega$: every M-principle is a QD_ω-principle, of course ; but CDSJ (from if$(X \wedge Y)\bot$, $ifXZ$, $ifYZ$ to if$(X \vee Y)Z$) is QD_ω-valid, as we may easily verify, though it is not an M-principle, as we pointed out in Section 1. ■

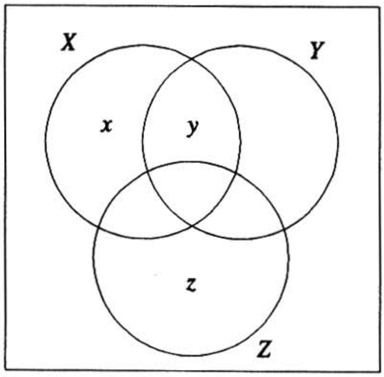
THEOREM 7. $QD \not\subseteq QD_\omega$.

Proof. From Theorem 5 and the fact that QD $\not\subseteq$ E (Section 1). ■

There is also another interesting fact about the inferential behaviour of *at least half* which has nothing to do with the previous ones.

THEOREM 8. $QD_\omega \not\subseteq QD$

Proof. The idea is this. The principle CWA (from $if\top X$, $ifXY$ to $if(X \vee Z)Y$) is not QD-valid, as this QD-counter-example shows :



However, it is a QD_{ω}-valid principle. For, suppose that $if \top X$ and $ifXY$ are both true according to *at least half* in some denumerable model. Then there are countably many X-situations, since there are no more $\neg X$-situations than X-situations. Therefore there are also countably many $X \wedge Y$-situations, since there are no more $X \wedge \neg Y$-situations than $X \wedge Y$-situations. So a fortiori there are countably many $(X \vee Z) \wedge Y$-situations, and this is sufficient for verifying the conclusion. ∎

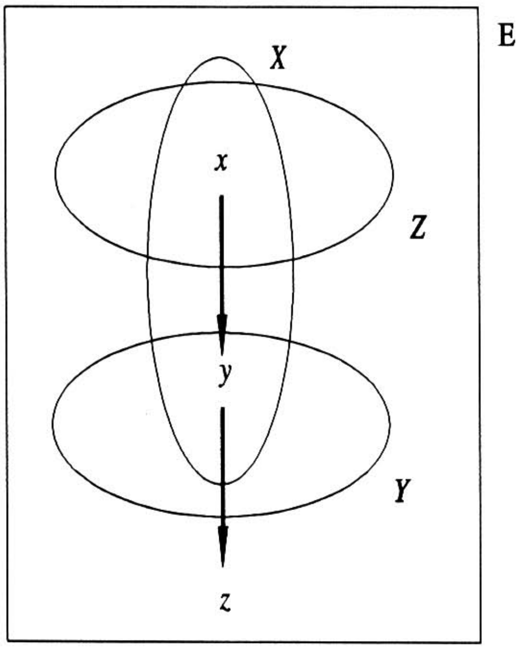
Now, note that the situation is different for at *most* denumerable universes, since every QD-invalid inference is also an invalid inference according to *at least half* on finite or denumerable universes.

It remains now to establish the mutual relationships between S, N and C.

THEOREM 9. $N \subset C$.

Proof. LM is a C-principle but not an N-principle, and thus $N \neq S$. (Too see this, consider the numerical model where $[[Y]]$ = the set of even numbers, $[[X]] = [[Y]] \cup \{1\}$ and $[[Z]] = \{1\}$; clearly, we have that $ifXY$ and not $if(X \wedge Z)Y$ according to *all but finitely many*.) But every inference which is C-invalid is N-invalid too. Indeed, let $ifX_1Y_1, \ldots, ifX_nY_n/ifXY$ be any inference refuted by inclusion in some finite model, and thus (Theorem 4) in some denumerable model. Then in this model all premises are verified according to *all but finitely many*, because they are verified by inclusion. On the other hand, if there are countably many $X \wedge \neg Y$-situations, then there are also countably many X-situations, and so the conclusion is already refuted according to *all but finitely many*. If there are only finitely many $X \wedge \neg Y$-situations and that the set of X-situations is denumerable, add countably many copies of any $X \wedge \neg Y$-situation, behaving in exactly the same way with regard to (non-)membership of the extensions of the set variables. Again this procedure does not disturb any of the previous thruth-values, but now there are countably many $X \wedge \neg Y$-situations — and thus we have an N-counter-model for the same inference. ∎

THEOREM 10. $S \subset N$.
Proof. All but finitely many validates all basic M-principles as well as CNJ, and so every S-principle is an N-principle (since $S = M +$ CNJ). However, there are some N-principles which are outside S. For instance, consider the inference from $if\top \neg X$, $ifXY$ to $if(X \wedge Z)Y$. It is not S-valid, as this S-counter-example shows :



However, it is clearly N-valid, since $if\top \neg X$ means here that there are only finitely many X-situations, and in this case the inference from $ifXY$ to if$(X \wedge Z)Y$ is validated by inclusion. ∎

Given the theorems 3, 9 and 10, we can complete Figure 1 of Section 1 as follows :

Figure 2

$$
\begin{array}{ccc}
 & E & \\
\rotatebox{45}{$\subset$} & & \rotatebox{-45}{$\supset$} \\
M \subset E \cap QD & & E \cup QD \subset S \subset N \subset C \\
\rotatebox{-45}{$\supset$} & & \rotatebox{45}{$\subset$} \\
 & QD &
\end{array}
$$

Now, the location of QD_ω in this figure is not established yet. Theorem 5 gives us a partial answer. This motivates us to ask this

QUESTION. Does $E \cap QD \subset QD_\omega$?

3. *Concluding remarks*

We cannot claim that we have exhausted the subject. The analysis of conditionals from the perspective of generalized quantifiers theory raises many other issues. Nevertheless, even from the point of view of our restricted analysis, some questions are still unanswered. Here are some of these questions.

Among the logics we have just considered, the most interesting one is *N*. Like the subjunctive logic *S*, *N* contains CNJ but neither LM nor TRN, satisfying in this way the minimal desiderata of every counterfactual logic. But unlike *S*, *N* has a purely quantitative semantics which requires denumerable universes only. It contains some additional principles however, but it does not seem that these extra principles may lead us to reject *N* as a plausible candidate for a counterfactual logic. In any case, a natural question at this stage is how to axiomatize *N* completely.

Another important question is whether there are, on denumerable universes, some other conjunctive and non-(left-)monotonic quantitative conditional quantifiers. On universes of higher infinite cardinalities, the answer is *yes*. It has been pointed out in van Benthem [1986] that the following condition, due to Frank Veltman, yields precisely the subjunctive logic *S* :

$$
If\,AB \; iff \; \begin{cases} A \subseteq B, \text{if } A \text{ is finite} \\ |A \cap B| > |A - B|, \text{if } A \text{ is infinite.} \end{cases}
$$

(The idea is this. All *S*-principles are valid according to this condition. Conversely, take any *S*-counter-model for an inference, convert it into a finite Lewis model (as in the proof of Theorem 2) and then add infinitely many copies of worlds, starting from the biggest sphere with stepwise increasing infinite cardinalities. This procedure has the effect of simulating "in all closest *X*-worlds" by "for most *X*-worlds".) But, as we can see, on denumerable universes this condition reduces to *all but finitely many*. Does this mean that *S* has a purely quantitative semantics only in the realm of the non-denumerable? This question is still open.

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