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SEQUENTS FOR DEPENDENCE LOGICS

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Abstract

In this note we introduce several sequent calculi for propositional dependence logics in which a set of topics is attached to each formula. Some connections between these logics and classical logic are established.

1. *Introduction*

Avoiding the paradoxes of material implication has been an aim whose pursuit has led to significant progress in the domain of formal reasoning. The standard methodology for addressing this problem is to reduce the multiplicity of manipulations of material implication by restricting its connections with the other connectives (for example with negation by eliminating contraposition). The objective is to obtain a logic whose theorems correspond to natural argumentation. Our hypothesis is that argumentations are syntactical variations of a given discourse in order to adapt it to the locuter. In other words to argue does not mean to introduce new elements of the discourse but only means to display its structure.

Some time ago R. L. Epstein introduced a family of logics called dependence logics in which an implication must contain only arguments that refer to the same topic. In other words the subject matters of premiss and conclusion are the same. One of these implications corresponds to the analytic implication introduced by W. T. Parry in which the constitive content of the premiss contains that of the conclusion (see also K. Fine). A similar calculus has been introduced by D. Vanderveken in order to formalise speech acts. In the context of deductive data bases S. Cazalens, R. Demolombe and A. Jones introduced a modal implication which expresses the dependency between topics. This is very useful to represent the notion of cooperation between the data base and the user. E. Orlowska and P. Weingartner define relevant logics based on the same principle.

In this note we will present sequent systems for several dependence logics, and we prove formal relations between classical logic and dependence logics.

2. *Dependence logics*

The idea behind dependence logics is to consider that the conditional expression '$A \Rightarrow B$' is true if and only if A is false or B is true', and the subject matters of A and B are related.

We suppose that to every formula there is attached a non empty subset of the set of topics T. Formally let t be a function from the set of propositional variables VP into the power set of T. Then t is extended to formulae and to sets of formulae by the following clauses:

$t(A) = \cup\{t(p)$: p is a propositional variable in the formula $A\}$

$t(\Gamma) = \cup\{t(A)$: A is a formula in the set of formulae $\Gamma\}$

A *valuation, relative to t*, is a function v from VP into the set of truth values $\{T, F\}$. The valuation v of formulae is defined, for every connective, as in classical logic, except for implication, denoted by " $\Rightarrow$ ", and for which several semantics are possible.

1°. $v(A \Rightarrow B) = T$, if and only if i) $v(A) = F$ or $v(B) = T$, and
ii) $t(B) \subseteq t(A)$

This semantics has been introduced by R. L. Epstein. We denote by D_t the logic under the definition 1°.

The following definition corresponds to the logic called dual dependence logic related to t and denoted by DD_t.

2°. $v(A \Rightarrow B) = T$, if and only if i) $v(A) = F$ or $v(B) = T$, and
ii) $t(A) \subseteq t(B)$

The last possibility determinates the logic of topic equality, denoted by Eq_t.

3°. $v(A \Rightarrow B) = T$, if and only if i) $v(A) = F$ or $v(B) = T$, and
ii) $t(A) = t(B)$

For a given function t, a formula is *t-valid* if and only if it is true for each valuation relative to t.

The function t allows us to represent syntactical properties on formulae, for example, if for every pair of different propositional variables p and q $t(p) \cap t(q) = \emptyset$ then $t(A) \subseteq t(B)$ becomes that the propositional variables of A must be contained in the propositional variables of B.

In the remainder we extend the vocabulary: $V = VP \cup \{\bot_p, \top_p\}_{p \in VP}$.

$\bot_p$ is false and $\top_p$ is true in every valuation and $t(\bot_p) = t(\top_p) = t(p)$. ($\top_p$ can be considered to be an abbreviation of $p \vee \neg p$, and $\bot_p$ of $p \wedge \neg p$). (see Epstein(87) for a similar approach) Using these new constants we introduce Gentzen systems for all these logics.

3. *Sequents for dependence*

In this section we present sequent systems for the dependence logics D_t, DD_t and Eq_t. Basically the idea is to modify the classical rules such that we keep track of the variables that have been used.
In the following, Γ, Π, Δ, Λ denote finite (possibly empty) sequences of formulae separated by commas. For arbitrary Π and Δ, $\Pi \vdash \Delta$ is called a sequent. Π and Δ are called the antecedent and succedent, respectively, of the sequent. An inference is an expression of the form:

$$\frac{S_1}{S} \quad \text{or} \quad \frac{S_1 \quad S_2}{S}$$

where S_1, S_2, S are sequents. S_1 and S_2 are called the upper sequents and S is called the lower sequent of the inference. Intuitively this means that when S_1 (S_2 and S_1) is (are) asserted, we can infer S from it (from them).
We consider the following systems :

1°. *System D_t.*

Structural rules

contraction

$$(CL)\ \frac{A,A,\Gamma \vdash \Delta}{A,\Gamma \vdash \Delta} \qquad (CR)\ \frac{\Gamma \vdash \Delta,A,A}{\Gamma \vdash \Delta,A}$$

exchange

$$(EL)\ \frac{\Pi,A,B,\Gamma \vdash \Delta}{\Pi,B,A,\Gamma \vdash \Delta} \qquad (ER)\ \frac{\Gamma \vdash \Delta,A,B,\Lambda}{\Gamma \vdash \Delta,B,A,\Lambda}$$

weakening

$$(WL)\ \frac{\Gamma \vdash \Delta}{A,\Gamma \vdash \Delta,\bot_A} \qquad (WR)\ \frac{\Gamma \vdash \Delta}{\top_A,\Gamma \vdash \Delta,A}$$

where $\bot_A$ denotes the set $\{\bot_p : p$ is in VP in A$\} \cup \{\bot_p : \bot_p$ or $\top_p$ is in A$\}$
and $\top_A$ denotes the set $\{\top_p : p$ is VP and in A$\} \cup \{\top_p : \bot_p$ or $\top_p$ is in A$\}$

cut

$$(cut)\ \frac{\Gamma\vdash\Delta,A \quad A,\Pi\vdash\Lambda}{\Gamma,\Pi\vdash\Delta,\Lambda}$$

Logical Rules

$$(\wedge L)\ \frac{A,B,\Gamma\vdash\Delta}{A\wedge B,\Gamma\vdash\Delta} \qquad (\wedge R)\ \frac{\Gamma\vdash\Delta,A \quad \Gamma\vdash\Delta,B}{\Gamma\vdash\Delta,A\wedge B}$$

$$(\vee L)\ \frac{A,\Gamma\vdash\Delta \quad B,\Gamma\vdash\Delta}{A\vee B,\Gamma\vdash\Delta} \qquad (\vee R)\ \frac{\Gamma\vdash\Delta,A,B}{\Gamma\vdash\Delta,A\vee B}$$

$$(\Rightarrow L)\ \frac{\Gamma\vdash\Delta,A \quad B,\Pi\vdash\Lambda}{A\Rightarrow B,\Gamma,\Pi\vdash\Delta,\Lambda,\bot_A} \qquad (\Rightarrow R)\ \frac{A,\Gamma\vdash\Delta,B}{\top_A,\Gamma\vdash\Delta,d(A,B)\Rightarrow B}$$

where d(A,B) is the conjunction of the elements of the set
$\{A\}\cup\{\top_p : p$ is a propositional variable in B such that $t(p)\not\subseteq t(A)\}$

$$(\neg L)\ \frac{\Gamma\vdash\Delta,A}{\neg A,\Gamma\vdash\Delta,\bot_A} \qquad (\neg R)\ \frac{A,\Gamma\vdash\Delta}{\top_A,\Gamma\vdash\Delta,\neg A}$$

elimination

$$(\bot E)\ \frac{\Gamma\vdash\Delta,\bot_p}{\Gamma\vdash\Delta} \qquad (\top E)\ \frac{\top_p,\Gamma\vdash\Delta}{\Gamma\vdash\Delta}$$

iff $t(p)\subseteq t(\Gamma)$

Axioms

- identity
  (id) $p\vdash p$ $\qquad p\in V$

- (ax) $A\Rightarrow B\vdash\bot_{A\Rightarrow B}$ $\qquad$ iff $t(B)\not\subseteq t(A)$

The systems DD_t and Eq_t are roughly identical with system D_t. There are the following differences:

2°. System DD_t.

$$(\Rightarrow R)\ \frac{A,\Gamma\vdash\Delta,B}{\top_A,\Gamma\vdash\Delta,A\Rightarrow g(B,A)}$$

where g(B,A) is the disjonction of the elements of the set $\{B\}\cup\{\bot_p : p$ is a propositional variable in A such that $t(p)\not\subseteq t(B)\}$

elimination

$$(\bot E)\ \frac{\Gamma\vdash\Delta,\bot_p}{\Gamma\vdash\Delta}\ \text{if } t(p)\subseteq t(\Delta) \qquad (\top E)\ \frac{\top_p,\Gamma\vdash\Delta}{\Gamma\vdash\Delta}$$

- (ax) $A\Rightarrow B\vdash\bot_{A\Rightarrow B}$ iff $t(A)\not\subseteq t(B)$

3°. System Eq_t.

$$(\Rightarrow R)\ \frac{A,\ \Gamma\vdash\Delta,\ B}{\top_A,\Gamma\vdash\Delta,d(A,B)\Rightarrow g(B,A)}$$

elimination

$$(\bot E)\frac{\Gamma\vdash\Delta,\bot_p}{\Gamma\vdash\Delta}\ \text{iff } t(p)\subseteq t(\Delta)\quad (\top E)\frac{\top_p,\Gamma\vdash\Delta}{\Gamma\vdash\Delta}\ \text{iff } t(p)\subseteq t(\Gamma)$$

- (ax) $A\Rightarrow B\vdash\bot_{A\Rightarrow B}$ iff $t(A)\neq t(B)$

Using these rules as usual we define the notion of proof as follows

Definition 1. A *proof* P is a tree of sequents satisfying the conditions that follow:

1) The topmost sequents of P are axioms.
2) Every sequent in P except the lowest one is an upper sequent of an inference whose lower sequent is also in P.

Definition 2. A sequent S is called *provable* if there is a proof ending with S.

The notion of validity for sequents is defined in the following way :

Definition 3. For a given t, $\Gamma \vdash \Delta$ is *t-valid* if and only if for every valuation v, relative to t, there is a formula in Γ which is false or a formula in Δ which is true and the conditions $t(\Delta) \subseteq t(\Gamma)$, $t(\Gamma) \subseteq t(\Delta)$ and $t(\Gamma) = t(\Delta)$ hold for the systems D_t, DD_t and Eq_t respectively.

Example 1. The sequents $p,q \vdash p$, $p \vdash p,q$ and $p,q \vdash p,q$ are t-valid, for each t, in the systems D_t, DD_t and Eq_t respectively.

4. *Reduction trees*

In order to prove the completeness of the Gentzen systems for dependence logics we use an intermediate tree construction, which is defined using a modified version of the Gentzen rules.

We describe rules that allow us to reduce a sequent in a new sequent (or in two new sequents). These *reduction rules* are defined as follows:

$$(\wedge 1)\ \frac{\Pi, A \wedge B, \Gamma \vdash \Delta}{A, B, \Pi, \Gamma \vdash \Delta} \qquad (\wedge 2)\ \frac{\Pi \vdash \Delta, A \wedge B, \Lambda}{\Pi \vdash \Delta, \Lambda, A, \bot_B \quad \Pi \vdash \Delta, \Lambda, B, \bot_A}$$

$$(\vee 1)\ \frac{\Pi, A \vee B, \Gamma \vdash \Delta}{A, \top_B, \Pi, \Gamma \vdash \Delta \quad B, \top_A, \Pi, \Gamma \vdash \Delta} \qquad (\vee 2)\ \frac{\Pi \vdash \Delta, A \vee B, \Lambda}{\Pi \vdash \Delta, \Lambda, A, B}$$

$$(\neg 1)\ \frac{\Pi, \neg A, \Gamma \vdash \Delta}{\top_A, \Pi, \Gamma \vdash \Delta, A} \qquad (\neg 2)\ \frac{\Pi \vdash \Delta, \neg A, \Lambda}{A, \Pi \vdash \Delta, \Lambda, \bot_A}$$

$$(\Rightarrow 1)\ \frac{\Pi, A \Rightarrow B, \Gamma \vdash \Delta}{\top_A, \top_B, \Pi, \Gamma \vdash \Delta, A \quad B, \top_A, \Pi, \Gamma \vdash \Delta}$$

This rule is applied if and only if the conditions $t(B) \subseteq t(A)$, $t(A) \subseteq t(B)$ and $t(B) = t(A)$ hold for the systems D_t, DD_t and Eq_t respectively.

$$(\Rightarrow 2.1)\ \frac{\Pi \vdash \Delta, A \Rightarrow B, \Lambda}{\Pi \vdash \Delta, \Lambda, \bot_A, \bot_B} \qquad (\Rightarrow 2.2)\ \frac{\Pi \vdash \Delta, A \Rightarrow B, \Lambda}{A, \Pi \vdash \Delta, \Lambda, B, \bot_A}$$

($\Rightarrow$2.1) is applied if the conditions $t(B) \not\subseteq t(A)$, $t(A) \not\subseteq t(B)$ and $t(B) = t(A)$ hold for the systems D_t, DD_t and Eq_t respectively, else we apply ($\Rightarrow$2.2).

Definition 4. Let S be a sequent. By a *reduction tree* for S we mean a tree constructed by placing S at the origin, and then applying the previous reduction rules while it is possible.

Example 2. Let us consider the following reduction trees in D_t associated with the sequent $\neg p \vee q \vdash p \Rightarrow q$:

- assume $t(q) \subseteq t(p)$, then we have :

$$\dfrac{\dfrac{\neg p \vee q \vdash p \Rightarrow q}{p, \neg p \vee q \vdash q, \bot_p}}{\dfrac{\neg p, \top_q, p \vdash q, \bot_p}{\top_p, \top_q, p \vdash q, \bot_p, p} \qquad q, \top_p, p \vdash q, \bot_p}$$

- for $t(q) \not\subseteq t(p)$ we have :

$$\dfrac{\dfrac{\neg p \vee q \vdash p \Rightarrow q}{\neg p \vee q \vdash \bot_p, \bot_q}}{\dfrac{\neg p, \top_q \vdash \bot_q, \bot_p}{\top_p, \top_q \vdash \bot_p, \bot_q, p} \qquad q, \top_p \vdash \bot_p, \bot_q}$$

Definition 5. A sequent $\Gamma \vdash \Delta$ is *closed* if the following two clauses are verified :

1. the conditions $t(\Delta) \subseteq t(\Gamma)$, $t(\Gamma) \subseteq t(\Delta)$ and $t(\Gamma) = t(\Delta)$ hold for the systems D_t, DD_t and Eq_t respectively.
2. either a) Γ and Δ have a propositional variable p in common, or
   b) Γ contains $\bot_p$ for a propositional variable p, or
   c) Δ contains $\top_p$ for a propositional variable p, or
   d) Γ contains $A \Rightarrow B$ with the condition $t(B) \not\subseteq t(A)$, $t(A) \not\subseteq t(B)$, $t(A) \neq t(B)$ for the systems D_t, DD_t and Eq_t respectively.

Definition 6. A reduction tree is *closed* if each end node of the tree is a closed sequent.

Then we have the facts:

Fact 1. Every closed sequent is a theorem of the corresponding dependence system.

Proof. We sketch the proofs of the different types of closed sequents. Some of the steps in the proofs are assured thanks to clause 1 of definition 5.

a) A proof of Γ, p, $\Pi \vdash \Delta$, p, Λ is :

$$\begin{array}{cl}
p \vdash p & \text{(id)} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\top_\Delta, \top_\Lambda, p \vdash \Delta, p, \Lambda & \text{(WR,ER)} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\top_\Delta, \top_\Lambda, \Gamma, p, \Pi \vdash \Delta, p, \Lambda, \bot_\Gamma, \bot_\Pi & \text{(WL,EL)} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\Gamma, p, \Pi \vdash \Delta, p, \Lambda & (\bot\text{E}, \top\text{E})
\end{array}$$

where

$$\begin{array}{ll}
A & \\
\cdots\cdots & \\
B \quad (P) &
\end{array}$$

means that we can derived B from A after possibly several sequent-rules P.

b) A proof of Γ, $\bot_p$, $\Pi \vdash \Delta$ is :

$$\begin{array}{cl}
\bot_p \vdash \bot_p & \text{(id)} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\top_\Delta, \bot_p \vdash \bot_p, \Delta & \text{(WR)} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\top_\Delta, \Gamma, \bot_p, \Pi \vdash \bot_p, \Delta, \bot_\Gamma, \bot_\Pi & \text{(WL,EL)} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\Gamma, \bot_p, \Pi \vdash \Delta & (\bot\text{E}, \top\text{E,ER})
\end{array}$$

c) A proof of $\Pi \vdash \Gamma$, $\top_p$, Δ is :

$$\begin{array}{cl}
\top_p \vdash \top_p & \text{(id)} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\top_\Gamma, \top_\Delta, \top_p \vdash \Gamma, \top_p, \Delta & \text{(WR,ER)} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\top_\Gamma, \top_\Delta, \top_p, \Pi \vdash \Gamma, \top_p, \Delta, \bot_\Pi & \text{(WL,EL)} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\Pi \vdash \Gamma, \top_p, \Delta & (\bot\text{E}, \top\text{E,ER})
\end{array}$$

d) A proof of $\Pi, A \Rightarrow B, \Gamma \vdash \Delta$ (where $A \Rightarrow B$ satisfies the condition of the axiom (ax)) is:

$$\dfrac{\dfrac{\dfrac{A \Rightarrow B \vdash \bot_{A \Rightarrow B} \quad (ax)}{\top_\Delta, A \Rightarrow B \vdash \Delta, \bot_{A \Rightarrow B} \quad (WR,ER)}}{\top_\Delta, \Gamma, A \Rightarrow B, \Pi \vdash \Delta, \bot_{A \Rightarrow B}, \bot_\Gamma, \bot_\Pi \quad (WL,EL)}}{\Pi, A \Rightarrow B, \Gamma \vdash \Delta \quad (\bot E, \top E)}$$

Given a reduction rule $\dfrac{S}{S_1 \quad S_2}$ (or $\dfrac{S}{S_1}$), the reversed form of this rule i.e. $\dfrac{S_1 \quad S_2}{S}$ ($\dfrac{S_1}{S}$ respectively) will be called the Gentzen-version of this rule.

Fact 2. The Gentzen versions of the reduction rules are derived rules in the corresponding dependence logic when the sequents of the rules satisfy clause 1 of definition 5.

Proof. We consider only two main examples. The other cases are similar.

A proof of $\dfrac{\top_A, \Pi, \Gamma \vdash \Delta, A}{\Pi, \neg A, \Gamma \vdash \Delta}$ with $t(\neg A) \subseteq t(\Delta)$, rule associated with the reduction rule $(\neg 1)$, is :

$$\dfrac{\dfrac{\dfrac{\top_A, \Pi, \Gamma \vdash \Delta, A}{\neg A, \top_A, \Pi, \Gamma \vdash \Delta, \bot_A \quad (\neg L)}}{\top_A, \Pi, \neg A, \Gamma \vdash \Delta, \bot_A \quad (EL)}}{\Pi, \neg A, \Gamma \vdash \Delta \quad (\bot E, \top E)}$$

A proof of $\dfrac{A, \top_B, \Pi, \Gamma \vdash \Delta \quad B, \top_A, \Pi, \Gamma \vdash \Delta}{\Pi, A \vee B, \Gamma \vdash \Delta}$, rule associated with the reduction rule $(\vee 1)$, is :

$$
\begin{array}{ccc}
A, \top_B, \Pi, \Gamma \vdash \Delta & & B, \top_A, \Pi, \Gamma \vdash \Delta \\
\cdots\cdots\cdots\cdots\cdots\cdots & & \cdots\cdots\cdots\cdots\cdots\cdots \\
A, \top_A, \top_B, \Pi, \Gamma \vdash \Delta, \bot_A & (WL.,EL) & B, \top_A, \top_B, \Pi, \Gamma \vdash \Delta, \bot_B \\
\cdots\cdots\cdots\cdots\cdots\cdots & & \cdots\cdots\cdots\cdots\cdots\cdots \\
A, \top_A, \top_B, \Pi, \Gamma \vdash \Delta & (\bot E) & B, \top_A, \top_B, \Pi, \Gamma \vdash \Delta \\
\hline
\end{array}
$$

$$
\begin{array}{cc}
A \vee B, \top_A, \top_B, \Pi, \Gamma \vdash \Delta & (\top E, EL) \\
\cdots\cdots\cdots\cdots\cdots\cdots & \\
\Pi, A \vee B, \Gamma \vdash \Delta & (\vee L)
\end{array}
$$

As a consequence of the two previous facts we have the following result :

Fact 3. If the tree associated with a sequent S is closed and S satisfies clause 1 of definition 5, then S is a theorem in the corresponding dependence system.

Proof. If S satisfies clause 1 of definition 5 we can verify that this condition is true for every sequent of the reduction tree relative to S. And then, if the tree associated with S is closed, it is a routine task to write a proof ending with S by modifying the reduction tree for S according to Fact 1 and Fact 2.

Example 3. If $t(q) \subseteq t(p)$, the sequent $\neg p \vee q \vdash p \Rightarrow q$ is t-valid. Using the reduction tree we have already given we can construct the following proof :

Left branch	Rule	Right branch	Rule
$p \vdash p$	(id)		
$\top_p, \top_q, p \vdash q, \bot_p, p$	(WR,ER)	$q \vdash q$	(id)
$\top_p, \top_p, \top_q, p \vdash q, \bot_p, \bot_p, p$	(WL,EL)	$q, \top_p, p \vdash q, \bot_p, \bot_p$	(WL,EL)
$\top_p, \top_q, p \vdash q, \bot_p, p$	($\bot$E, $\top$E)	$q, \top_p, p \vdash q, \bot_p$	($\bot$E)
$\neg p, \top_p, \top_q, p \vdash q, \bot_p, \bot_p$	($\neg$L)	$q, \top_p, \top_q, p \vdash q, \bot_p, \bot_q$	(WL,EL)
$\neg p, \top_p, \top_q, p \vdash q, \bot_p$	($\bot$E)	$q, \top_p, \top_q, p \vdash q, \bot_p$	($\bot$E)

$$\neg p \vee q, \top_p, \top_q, p \vdash q, \bot_p \quad (\vee L)$$

$$\neg p \vee q, p \vdash q, \bot_p \quad (\top E)$$

$$p, \neg p \vee q \vdash q, \bot_p \quad (EL)$$

$$\neg p \vee q \vdash p \Rightarrow q, \bot_p \quad (\Rightarrow R)$$

$$\neg p \vee q \vdash p \Rightarrow q \quad (\bot E)$$

Lemma 1. If the tree associated with a sequent S is not closed, then S is not t-valid in the corresponding dependence system.

Proof. Let S_n be the sequent of an node n which is not closed.
Assume that the constraint associated to the topic function t is not verified in the node n. Since the reduction rules preserve the topics inclusion properties, S does not satisfy it too. So S is not t-valid.
Now we can consider that S_n is a sequent which has only atomic formulae, which has no $\bot_p$ in its antecedent and no $\top_p$ in its succedent and whose antecedent and succedent do not contain a formula in common. We can then obtain an interpretation not satisfying S. Consider the branch consisting of the sequents $S \equiv S_0, S_1, \ldots, S_n$. Let S_i be $\Gamma_i \vdash \Delta_i$, $\Gamma = \cup\Gamma_i$ and $\Delta = \cup\Delta_i$. First notice that from the way the branch was chosen Γ and Δ have no atomic formula in common and for every proposition p: $\bot_p \notin \Gamma$, $\top_p \notin \Delta$.
Let v be the valuation defined for each proposional variable, p, by:

if $p \in \Gamma$ then $v(p)=T$ else $v(p)=F$

This valuation satisfies every formula in Γ, but no formula in Δ. We prove this by induction on the number of logical symbols in the formulae. So v falsifies $S_0, S_1, \ldots, S_n$ and therefore $S \equiv S_0$ is not t-valid.

Example 4 : if $t(q) \not\subseteq t(p)$ the sequent $\neg p \vee q \vdash p \Rightarrow q$ is not t-valid in D_t. Consider the following branch of the reduction tree given in example.

Let S_0 be $\neg p \vee q \vdash p \Rightarrow q$

S_1 be $\neg p \vee q \vdash \bot_p, \bot_q$

S_2 be $\neg p, \top_q \vdash \bot_q, \bot_p$

S_3 be $\top_p, \top_q \vdash \bot_p, \bot_q, p$

and hence let v such that v(p) and v(q) are false.

We have, in D_t, $v(\neg p \vee q)$ true and $v(p \Rightarrow q)$ false. So S_0 is not t-valid.

5. *Soundness and completeness*

Theorem 1. In each system derivability and validity coincide.

Proof. - Derivability implies validity is proved by induction on the number of inferences in a proof of a sequent: axioms are t-valid sequents and the inference rules preserve this property.

- Assume now that a Gentzen sequent is valid. Then it satisfies clause 1 of definition 5 and by lemma 1 the associated tree is closed. So by Fact 3 the sequent is a theorem. And the completeness is assured.

Observation. If a sequent is t-valid there is a cut free proof of it. Indeed the reduction tree associated to it, is closed by lemma 1, and we can then construct a proof from the reduction tree in the same way as previously in Fact 3, without using cut.

A consequence of cut elimination is the subformula property i. e. if S is provable there is a proof of S such that all formulae occuring in this proof are subformulae of those formulae which occur in S.

6. *Connections between dependence logics and classical logic*

In this part, we establish a link between theorems of classical logic and a fragment of dependence logics.

Let's remark first that classical calculus (denoted by PC) may be defined like system D_t, DD_t or Eq_t with the only following differences :

$$(\Rightarrow R)\ \frac{A, \Gamma \vdash \Delta, B}{\top_A, \Gamma \vdash \Delta, A \Rightarrow B}$$

elimination

$$(E\bot)\ \frac{\Gamma\vdash\Delta,\bot_p}{\Gamma\vdash\Delta} \qquad (E\top)\ \frac{\top_p,\Gamma\vdash\Delta}{\Gamma\vdash\Delta}$$

and with no axiom (ax).

For this version of classical logic cut elimination holds. Let us consider the translations from classical formulae into dependence formulae defined by :

$T^{Dt}(A\Rightarrow B) = [d(T^{Dt}(A),B)\Rightarrow T^{Dt}(B)]$
$T^{DDt}(A\Rightarrow B) = [T^{DDt}(A)\Rightarrow g(T^{DDt}(B),A)]$
$T^{Eqt}(A\Rightarrow B) = [d(T^{Eqt}(A),B)\Rightarrow g(T^{Eqt}(B),A)]$
and which are homophonic elsewhere. We gave the definitions of d and g in the third part of this note.

From these translations we have the following theorem:

Theorem 2. $\Pi \vdash_{PC} \Delta$
if and only if $d(T^{Dt}(\Pi),\Delta)\vdash_{Dt}T^{Dt}(\Delta)$
if and only if $T^{DDt}(\Pi) \vdash_{DDt}g(T^{DDt}(\Delta),\Pi)$
if and only if $d(T^{Eqt}(\Pi),\Delta)\vdash_{Eqt} g(T^{Eqt}(\Delta),\Pi)$
where $T^i(\Delta) = T^i(A_1),\ldots , T^i(A_n)$ when $\Delta = A_1,\ldots , A_n$
Proof. The proof is made by induction on the lenght of the proof. Each inference corresponds roughly to the inference whith the same name in the other system, and each axiom to its equivalent. We note that because of the subformula property of D_t, DD_t, Eq_t if $d(T^i(\Pi),\Delta)\vdash_i g(T^i(\Delta),\Pi)$ is provable there is a proof of this sequent which doesn't use (ax). From right to left subformula property of P. C. warrants the valadity of ($\bot$E) and ($\top$E) in the other systems.

7. *Epstein dependence logics*

We can now ask which formulae are t-valid for every function t. The corresponding systems will be denoted by D, DD and Eq. These systems have been introduced by R. L. Epstein. We find it difficult to extend the previous Gentzen system for capturing these new logics. However we can describe a decision procedure which tells if a formula is t-valid for every

t or not.
Let us consider system D (for the other systems the procedure is similar). Let S be a sequent $\Gamma \vdash \Delta$ such that $Var(\Delta) \subseteq Var(\Gamma)$ where $Var(\Pi)$ denotes the set of propositional variables that appear in Π (else there exists a function t such that $t(\Delta) \not\subseteq t(\Gamma)$ and then S is not t-valid: the procedure is finished). For each expression of the form $A \Rightarrow B$ in $\Gamma \cup \Delta$ we associate either the constraint $f(B) \subseteq f(A)$ or $f(B) \not\subseteq f(A)$ where $f(A) = \cup \{f(p):p$ is a propositional variable in $A\}$. Each set of coherent constraints define a set of topic functions which have the same reduction tree for S. So we have only to verify if S is t-valid in D_t for every type of topic function t.

Example. To know wether $\neg p \vee q \vdash p \Rightarrow q$ is valid in D we have to study the following cases :

-$t(q) \subseteq t(p)$

-$t(q) \not\subseteq t(p)$

we saw that for every function t such that $t(q) \subseteq t(p)$, $\neg p \vee q \vdash p \Rightarrow q$ is t-valid in D_t and that for every function t such that $t(q) \not\subseteq t(p)$ it is not t-valid in D_t. So $\neg p \vee q \vdash p \Rightarrow q$ is not valid in D.

8. *Conclusion*

In this note we have presented some connections between propositional dependence logics we have defined and classical logic. In particular we have presented sequent systems which are very close to the classical one. The extension of this kind of work to the first order case is under development (see Krajewski (86) for a first attempt). Concerning applications we think that dependence logics are a comprehensive frame to represent notions like topic which are very important for domains like nonmonotonic inferences or cooperative answers.

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