Logique & Analyse **131-132** (1990), 287-293

ON MODELS OF QUINE'S NF.

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NF is the well-known axiomatic system introduced by Quine [1]. Its specific axioms are the extensionality axiom and all the instances of the comprehension schema $\forall\exists y\forall x[x \in y \equiv A]$, where $\forall$ means the universal closure, A is a stratified formula [1] and y is a variable having no free occurrences in A.

This article is concerned with the following question: how does a model of *NF* look like? First of all the following remark is the starting point of the results of Grishin [4] and Boffa and Crabbé [2]: each model of *NF* is an infinite atomic Boolean algebra. To obtain more detailed answer one ought to continue as follows.

All models of *NF* are partitioned into standard and non-standard ones. According to Rosser and Hao Wang [3] a model of a system of axioms is standard iff it satisfies the following conditions:

a) the relation of the model which represents the equality relation in the formal logic is the equality relation for objects of the model;
b) that portion of the model which is supposed to represent the non-negative integers is well-ordered by the relation $\leq$;
c) that portion of the model which is supposed to represent the ordinal numbers of the formal logic is well-ordered by the relation $\leq$.

Rosser and Hao Wang [3] proved that no model of *NF* is standard in this sense because of violation of the condition c) (even in the case when both a) and b) are satisfied). So it makes sense to subdivide further the class of non-standard models of *NF* into two non-overlapping subclasses. Let all models of NF, which satisfy both conditions a) and b) belong to the first subclass, while those ones, for which either a) or b) does not hold belong to the second subclass. The models of *NF* belonging to the first subclass are called in this article standard in the broad sense. The models of *NF* belonging to the second subclass are called non-standard in the narrow sense. The words 'in the broad sense' as well as 'in the narrow sense' are consequently omitted below for the sake of brevity. One can say informally that a model of *NF* is standard iff its set of all natural numbers contains only 'standard' natural numbers 0, 1, 2, ... One ought to say that this article deals only with those non-standard models of NF, in which the relation = is regular equality. All models of *NF* being considered in this text are countable. Since any such a model is isomorphic to some structure of the form $\langle \omega, =, \in_M \rangle$, where ω is the set of all standard natural numbers 0, 1, 2, ... while $\in_M \subseteq \omega \times \omega$, one can (without loss of generality) consider that a countable non-standard model is a structure having this form.

The following lemma proved by Grishin [4] is essentially used hereafter. (The wording is slightly changed.)

Lemma 1. Let R and Q be two families of subsets of the sets X and Y correspondingly. Let both R and Q satisfy the following conditions:

1. $|R| = |Q| = \aleph_0$, where $|R|$ and $|Q|$ are the cardinalities of R and Q respectively;
2. R and Q are closed under the operations of intersection and complementation with respect to X and Y correspondingly;
3. R and Q contain all singleton subsets of X and Y respectively;
4. For every infinite $A \in R$ ($B \in Q$) there exists an $A_1 \in R$ ($B_1 \in Q$) such that $A_1 \subseteq A$ and $|A_1| = |A - A_1|$ ($B_1 \subseteq B$ and $|B_1| = |B - B_1|$)

Then the families R and Q are isomorphic with respect to inclusion, i.e. there exists a one-to-one correspondence between R and Q such that $\alpha \subseteq \beta \equiv \varphi(\alpha) \subseteq \varphi(\beta)$ for all $\alpha, \beta \in R$.

The families of sets satisfying all conditions 1-4 are (for the sake of brevity) called countably saturated below. (The terminology is induced by Crabbé[5]. One can easily see that a family of sets satisfying the conditions 1-4 is an infinite atomic Boolean algebra [2].). A good example of such a family is the set of all recursive sets of natural numbers. This is the gist of the following Lemma (see also [9]).

Lemma 2. The set of all recursive sets of natural numbers is countably saturated.

Proof. Let us recall that a set of natural numbers is called recursive if its characteristic function is recursive. It is a well-known fact that the set of all recursive functions is countable. So, the set of all recursive sets of natural numbers is also countable. It is also well-known that the intersection of two recursive sets of natural numbers is also a recursive one as well as the complement of a recursive set of natural numbers with respect to the set of all natural numbers. It is also obvious that a singleton subset of the set of all natural numbers is recursive. So, it remains only to prove that the set of all recursive sets of natural numbers satisfies the condition 4. This can be done as follows. Let A be an infinite recursive set of natural numbers. One can enumerate its members in their natural order by all natural numbers. Let A_1 be the set of all members of A whose numbers (in this enumeration) are even. A_1 is obviously a countable set as well as $A - A_1$, i.e. its complement with respect to A. It follows from the way of its construction that A_1 is a recursive set of natural numbers. So, the set of all recursive sets of natural numbers satisfies all the conditions 1-4 and is a countably saturated one. Hence, the lemma holds.

Another good example of a countably saturated family of sets can be described as follows. If *NF* is consistent, which is equivalent to the existence of a model (and a countable one) of NF, then *NF* + $\neg$ C, where C is the axiom of counting, is also consistent as it has been proved by Orey [6] and by Henson [7]. In such a case *NF* + $\neg$ C has a countable model, which is obviously a non-standard countable model of NF. That the set N defined for such a model of *NF* as $\{\alpha \mid \alpha = \{z \mid z \in_M x\}$ for some $x \in \omega\}$ is a countably saturated one is implied by the following lemma.

Lemma 3. Let $M = \langle \omega, =, \in_M \rangle$ be a non-standard countable model of NF. Let $N = \{\alpha \mid \alpha = \{z \mid z \in_M x\}$ for some $x \in \omega\}$. Then N satisfies all the conditions 1-4 of Lemma 1.

Proof. The set N is obviously countable. Since *NF* contains the axiom of existence of one-element set and since M is a model of NF, N contains all singleton subsets of ω. Since *NF* contains the axiom of existence of the complement of a set to the universal one and the axiom of existence of intersection of any two sets, N satisfies the condition 3. It remains now only to prove that it satisfies the condition 4. Let $\alpha \in N$ be an infinite set. By assumption there exists $x \in \omega$ such that $\alpha = \{z \mid z \in_M x\}$. Let Nn be the set representing in M the set of all natural numbers. (This denotation as well as some other ones are taken from [8].). There are only two possible cases: either $M \models Nc(x) \in Nn$ or $M \models Nc(x) \notin Nn$. Consider the first of them. Let $M \models Nc(x) \in Nn$. In this case either $Nc(x) = 2m$ or $Nc(x) = 2m + 1$ for some $m \in Nn$. This means that either x itself or x without one of its elements can be partitioned into two sets having the same cardinality in M. This implies that α also can be partitioned into two sets such that there exists a one-to-one correspondence between them. Consider now the second case. One can prove by the induction that the formula

$$\forall x[Nc(x) \notin Nn \supset \forall b[b \in Nn \supset \exists z[z \subseteq x \& Nc(x) = b]]]$$

is a theorem of NF. So, it holds in M. Since M = Nc(x) $\notin$ N, it follows that

$$M = \forall b[b \in Nn \supset \exists z[z \subseteq x \& Nc(x) = b]].$$

As M is a non-standard model of NF, there exists in it a natural number k distinct from all standard numbers 0, 1, 2, ... Without loss of generality one can suppose that k is of the form 2m, where m is also distinct from all standard numbers 0, 1, 2, ... Since $M \models k \in Nn$ and $M \models 2m \in Nn$, one can conclude that $M \models \exists z[z \subseteq x \& Nc(x) = 2m]$. Let p be such that p $\subseteq$ x&Nc(p) = 2m. This p obviously has a subset (say, q), whose cardinality in M equals to m. Taking the sets from N corresponding to q and to its complement to p respectively, one can easily see that their cardinalities are infinite. But at the same time the countability of the universe of M implies that both these cardinalities are no greater than $\aleph_0$. So, they are both equal to $\aleph_0$. This means that the lemma holds.

Combining both Lemma 2 and Lemma 3 one can prove the following.

Theorem 1. Let $M = \langle \omega, =, \in_M \rangle$ be a non-standard countable model of NF. Then there exists an enumeration of the set of all recursive sets of natural numbers such that M is isomorphic to $R = \langle \omega, =, \in_R \rangle$, where for all m, n $\in \omega$ m $\in_R$ n means that m is a member of the recursive set of natural numbers, whose number in this enumeration is n.

Proof. The proof is carried out by the method of permutations.

Let $M = \langle \omega, =, \in_M \rangle$ be a non-standard countable model of NF.

Consider $N = \{ \alpha \mid \alpha = \{z \mid z \in_M x\}$ for some $x \in \omega\}$. By Lemma 3 N satisfies all the conditions 1-4 and is, thus, countably saturated. By Lemma 2 the set of all recursive sets of natural numbers is also countably saturated. Denote the latter by RS. By Lemma 1 N and RS are isomorphic with respect to inclusion. In other words there exists a one-to-one correspondence φ between N and RS such that $\alpha \subseteq \beta \equiv \varphi(\alpha) \subseteq \varphi(\beta)$ for all α, β belonging to N. Now it follows from the definition of N that there exists a one-to-one correspondence ψ between ω and N such that $\alpha \subseteq_M \beta \equiv \psi(\alpha) \subseteq \psi(\beta)$ for all α and β which belong to ω, where $\alpha \subseteq_M \beta$ is an abbreviation for $\forall z \in \omega[z \in_M \alpha \supset z \in_M \beta]$. Similarly, one can conclude that there exists a one-to-one correspondence ξ between N and ω such that $\alpha \subseteq \beta \equiv \xi(\alpha) \subseteq_N \xi(\beta)$ for all α and β which belong to N, where $\alpha \subseteq_N \beta$ is an abbreviation for $\forall z \in \omega[z \in_N \alpha \supset z \in_N \beta]$, while m $\in_N$n means that m is a member of the recursive set of natural numbers whose number in the given enumeration of RS is n. This implies the existence of a mapping F of ω onto itself such that $\alpha \subseteq_M \beta \equiv F(\alpha) \subseteq_N F(\beta)$ for all α and β from ω. Denote by $\{\alpha\}_M$ the natural number playing the part of $\{\alpha\}$ in M and by $\{\alpha\}_R$ the number of the recursive set, whose unique member is a. Obviously, $\alpha \in_M \beta \equiv \{\alpha\}_M \subseteq_M \beta$ and $\alpha \in_N \beta \equiv \{\alpha\}_R \subseteq_N \beta$ for arbitrary α and β from ω. Now let us define a mapping h of ω onto ω by the equivalence $\beta = h(\alpha) \equiv \{\beta\}_R = F(\{\alpha\}_M)$, where α and β are arbitrary natural numbers. So, for any natural numbers α and β

$$\alpha \in_M \beta \equiv \{\alpha\}_M \subseteq_M \beta \equiv F(\{\alpha\}_M) \subseteq_N F(\beta) \equiv \{h(\alpha)\}_R \subseteq_N F(\beta) \equiv h(\alpha) \in_N F(\beta).$$

Therefore, for any α and β belonging to ω $\alpha \in_M \beta \equiv h(\alpha) \in_R h(\beta)$, where the relation $\in_R$ is defined as follows: m $\in_R$n means that m belongs to the recursive set of natural numbers, whose number equals to $F(h^{-1}(n))$. This means that h is the desired isomorphism and the corresponding enumeration of all recursive sets of natural numbers is obtained from the original one in such a way that the set, whose number was k, obtains the number $F(h^{-1}(k))$. So, Theorem 1 holds and the structure $R = \langle \omega, =, \in_R \rangle$ described above is a model of *NF* (provided *NF* is consistent).

Corollary 1. All countable non-standard models of *NF* can be obtained from each other by some permutation.

As regards to standard models of *NF* the situation is slightly more complex. Let D be a formula stating that each infinite set is a union of two disjoint infinite sets. Using the denotations taken from [3] one can say that D is the formula

$$\forall \alpha[Nc(\alpha) \notin Nn \supset \exists x \exists y[Nc(x) \notin Nn \& Nc(y) \notin Nn \& x \cap y = \wedge \& x \cup y = \alpha]]$$

In such a case all countable standard models of *NF* ought to be partitioned into two disjoint classes. The first of them consists of all standard models of *NF* in which the formula D holds. The second one consists of all standard models of *NF* in which it does not hold. One can easily prove that each standard model of *NF* belonging to the first class is isomorphic to some structure $\langle \omega, =, \in_R \rangle$ of the type described above. This is implied by the following.

Theorem 2. Let $M = \langle \omega, =, \in_M \rangle$ be a countable model of *NF* + D. Then there exists an enumeration of all recursive sets of natural numbers such that M is isomorphic to the structure $R = \langle \omega, =, \in_R \rangle$, where for all natural numbers m and n m $\in_R$ n means that m is a member of the recursive set, whose number in this enumeration is n. One can easily prove this theorem repeating the proof of Theorem 1 given above with one important correction. Instead of using Lemma 3 one ought to use the following.

Lemma 4. Let $M = \langle \omega, =, \in_M \rangle$ be a countable model of *NF* + D. Let $N = \{\alpha \mid \alpha = \{z \mid z \in_M x\}$ for some $x \in \omega\}$. Then N is a countably saturated family of sets of natural numbers.

Proof. To prove that N satisfies the conditions 1-3 it suffices to repeat the same considerations as ones used in the proof of Lemma 3. So, it remains only to prove that N satisfies the condition 4. Let $\alpha \in N$ be an infinite set. By definition of N there exists $x \in \omega$ such that $\alpha = \{z \mid z \in_M x\}$. There are two possible cases: either $M \models Nc(x) \in Nn$ or $M \models Nc(x) \notin Nn$, where all denotations have the same sense as in the proof of Lemma 3. Proceeding as in the proof of Lemma 3 one can come to the conclusion that if the first case takes place, then there exists an infinite $\alpha_1 \in N$ such that $\alpha_1 \subseteq \alpha$ and $|\alpha_1| = |\alpha - \alpha_1|$. Suppose now that the second case takes place. In this case one can infer from D that x is the union of two disjoint sets y and x - y such that both $M \models Nc(y) \notin Nn$ and $M \models Nc(x - y) \notin Nn$. Thus, Lemma 4 holds.

Let AxC(Den) be the axiom of choice for denumerable sets. Since the implication AxC(Den) $\supset$ D is easily provable in *NF* one can conclude that if one replaces the formula D in the wording of Theorem 2 by AxC(Den), then the obtained theorem is also valid.

Standard models of the second class are slightly more complex structures. This is the consequence of the following.

Theorem 3. Let $M = \langle \omega, =, \in_M \rangle$ be a standard countable model of *NF* in which the formula D does not hold. Then there exists an infinite subset RS* of the set of all recursive sets of natural numbers RS and an enumeration of RS* such that M is isomorphic to the structure $R^* = \langle \omega, =, \in_{R^*} \rangle$, where for all $m, n \in \omega$ $m \in_{R^*} n$ means that m belongs to the set from RS* whose number in the given enumeration of RS* is n.

Proof. Let $N = \{\alpha \mid \alpha = \{z \mid z \in_M x\}$ for some $x \in \omega\}$. Let us define an infinite sequence of sets N_i $(i < \omega)$ as follows. Let $N_0 = N$. Let N_{2n+1} contains all members of N_{2n} and besides that for every infinite $\alpha \in N_{2n}$ for which there exists no $\alpha_1 \subseteq \alpha$ belonging to N_{2n} such that the cardinality of α_1 equals to one of its complement to α, N_{2n+1} contains a certain $\alpha_1 \subseteq \alpha$ whose cardinality equals to one of $\alpha - \alpha_1$. Let N_{2n+2} contains all members of N_{2n+1} and besides that it contains all intersections of pairs of members of N_{2n+1} as well as all complements of members of N_{2n+1} to ω. N_ω is now defined as the union of all N_i $(i < \omega)$, i.e. $N_\omega = \bigcup_{i<\omega} N_i$. By the assumption the set N is countable. This implies the countability of each of N_i as well as N_ω.

Since *NF* contains the axiom of existence of one-element set and since M is a model of NF, N_0 as well as N_ω contains all singleton subsets of ω. Since *NF* contains the axiom of existence of the complement of a set to the universal one and the axiom of existence of intersection of any two sets and since M is a model of NF, N_0 contains the intersection of any two of its members as well as the complement of any of its members to ω. It follows from the way of construction of N_ω that N_ω also possesses this property. Now from the way of construction of N_ω one can conclude that for every infinite $\alpha \in N_\omega$ there exists $\alpha_1 \in N_\omega$ such that $\alpha_1 \subseteq \alpha$ and the cardinalities of α_1 and $\alpha - \alpha_1$ are equal. So, N_ω satisfies all the conditions of Lemma 1 from [2]. As it has been proved in [2] the set of all recursive sets of natural numbers RS also satisfies the same conditions. Thus, N_ω and RS are isomorphic with respect to inclusion ([4]), i.e. there exists a one-to-one mapping φ from N_ω onto RS such that $\alpha \subseteq \beta \equiv \varphi(\alpha) \subseteq \varphi(\beta)$ for all a and b belonging to N_ω. If one takes instead of φ the function $\varphi|N$, i.e. the restriction of φ to N, one will obtain the isomorphism between N and some RS* $\subseteq$ RS with respect to inclusion. To finish the proof one ought to proceed in the same way as in the proof of Theorem 1.

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