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A NOTE ON FRAÏSSÉ'S CHARACTERIZATION OF ELEMENTARY EQUIVALENCE

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Abstract.

We give a new, syntactic, proof of Fraïsse's characterization of elementary equivalence.
This proof, together with a short discussion of elementary equivalence of intuitionistic models, allows to exhibit some properties of classical structures and of classical predicate calculus, which are necessary for Fraïssé's characterization to work.

1. *Expressing elementary equivalence in a two-sorted language*

Let $\mathscr{L}$ be a one-sorted language whose non logical symbols are a finite number $R_1, \ldots, R_n$ of relation symbols; one of the R_i's might be equality, but this is not mandatory. $\mathscr{L}$ has no constant nor function symbols.
$\mathscr{L}$ can be canonically turned into a many-sorted language $\mathscr{L}^{ABC}$ having three types of variables. The set of $\mathscr{L}^{ABC}$-variables is the set of all symbols of the form x^A, x^B or x^C, where x is a variable of $\mathscr{L}$. $\mathscr{L}^{ABC}$ has exactly the same non logical symbols as $\mathscr{L}$ and atomic formulae of $\mathscr{L}^{ABC}$ are of the form

$$R_i(x_1^T, \ldots, x_k^T)$$

for any $1 \leq i \leq n$ and any $T \in \{A, B, C\}$.

$\mathscr{L}^{AB}$ is then defined as the subset of $\mathscr{L}^{ABC}$ obtained by removing all type C variables. In fact, type C variables allows to simplify some definitions and proofs, which could have been also expressed in $\mathscr{L}^{AB}$. So using $\mathscr{L}^{ABC}$ is not necessary but simply useful.

Let us consider two $\mathscr{L}$-structures $\mathsf{A}=(|A|, \mathrm{R}_1^A, \ldots, \mathrm{R}_n^A)$ and $B=(|B|, \mathrm{R}_1^B,$

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..., R_n^B). We suppose that $|A| \cap |B| = \varnothing$. Then this pair of structures can be turned into an $\mathcal{L}^{AB}$-structure:

$$A \oplus B = (|A|, |B| \; ; R_1^{A\oplus B}, \ldots, R_n^{A\oplus B})$$

where $R_i^{A\oplus B} = R_i^A \cup R_i^B$, for each $i \leq n$. In $A \oplus B$, $|A|$ (resp. $|B|$) is the domain of objects of type A (resp. type B).

We are going to define several sets Σ of $\mathcal{L}^{AB}$-sentences such that $A \equiv B$ iff $A \oplus B \models \Sigma$. Thus, in some sense, such sets Σ "axiomatize" the elementary equivalence of two $\mathcal{L}$-structures.

But first, we need to introduce some definitions.

- If $T \in \{A, B, C\}$ and φ is a formula of $\mathcal{L}^{ABC}$, then φ is said to be *T-homogeneous* if T is the type of all variables occurring in φ. And φ is said to be *homogeneous* if it is T-homogeneous for some $T \in \{A, B, C\}$. So all atomic $\mathcal{L}^{ABC}$-formulae are homogeneous.

- If $T \in \{A, B, C\}$ and φ is a formula of $\mathcal{L}$, then φ^T denotes the formula obtained by adding T as a superscript to all variables occurring in φ. φ^T is thus a formula of $\mathcal{L}^{ABC}$.

- If T, $U \in \{A, B, C\}$ and if φ is a formula of $\mathcal{L}^{ABC}$, then $\varphi(T/U)$ is obtained in the following way:

  - first rename variables of φ in such a way that, if x^U is *bound* in φ, then x^T does not occur in φ, for all variables x of $\mathcal{L}$;
  - then replace with T all occurrences in φ of the superscript U.

  For example, if R_1 is a binary relation symbol and φ is

  $$(\forall y^A)(\exists y^B)(R_1(x^A, y^A) \Leftrightarrow R_1(x^B, y^B)),$$

  then $\varphi(A/B)$ is

  $$(\forall y^A)(\exists t^A)(R_1(x^A, y^A) \Leftrightarrow R_1(x^A, t^A)).$$

  In this case, there are two variables occurring free in φ, while there is only one such variable in $\varphi(A/B)$. Remark also, in this example, that $\varphi(A/B)$ is a theorem of predicate calculus, while φ is not.

If σ is an $\mathscr{L}$-sentence, then, clearly

$$A \models \sigma \text{ iff } A \oplus B \models \sigma^A \text{ and } B \models \sigma \text{ iff } A \oplus B \models \sigma^B.$$

Thus,

$$A \equiv B \text{ iff for all } \mathscr{L}\text{-sentences } \sigma,\ A \oplus B \models \sigma^A \Leftrightarrow \sigma^B$$

This can also be stated in the following way:

$$A \equiv B \text{ iff for all } C\text{-homogeneous } \mathscr{L}^{ABC}\text{-sentences } \sigma, \quad A \oplus B \models \sigma(A/C) \Leftrightarrow \sigma(B/C). \tag{1}$$

Now consider the set

$$\Sigma_H = \{\sigma(A/C) \Leftrightarrow \sigma(B/C) : \sigma \text{ is a } C\text{-homogeneous } \mathscr{L}^{ABC}\text{-sentence}\}.$$

(1) exactly means that Σ_H "axiomatizes" the elementary equivalence of two $\mathscr{L}^{AB}$-structures.

In the sequel, we are going to define two other sets of axioms, Σ and Φ, that we shall prove, in a purely syntactic way, equivalent to Σ_H. Φ will be the "axiomatization" of Fraïssé's characterization of elementary equivalence.

The definition of Σ is nearly the same as the definition of Σ_H:

$$\Sigma = \{\sigma(A/C) \Leftrightarrow \sigma(B/C) : \sigma \text{ is an } \mathscr{L}^{ABC}\text{-sentence}\}.$$

$\Sigma_H \subset \Sigma$. So $\Sigma \vdash \Sigma_H$. The other direction, $\Sigma_H \vdash \Sigma$, is not so trivial. The proof is based on the following remark. Suppose that R_1 is a unary predicate symbol and consider for example $\sigma \equiv (\forall x^A)(\exists y^C)(R_1(x^A) \Leftrightarrow R_1(x^C))$. Then

$$\sigma \Leftrightarrow [(\exists x^A)(R_1(x^A)) \Rightarrow (\exists y^C)(R_1(y^C))] \wedge [(\exists x^A)(\neg R_1(x^A)) \Rightarrow (\exists y^C)(\neg R_1(y^C))].$$

So there are four $\mathscr{L}$-sentences τ_1, τ_2, τ_3 and τ_4 such that $\sigma \Leftrightarrow [\tau_1^A \Rightarrow \tau_2^C] \wedge [\tau_3^A \Leftrightarrow \tau_4^C]$.

In other words, σ is equivalent to a *boolean combination of homegeneous sentences*.

In the same way, $\sigma(A/C) \Leftrightarrow [\tau_1^A \Rightarrow \tau_2^A] \wedge [\tau_3^A \Leftrightarrow \tau_4^A]$ and $\sigma(B/C) \Leftrightarrow [\tau_1^A \Rightarrow \tau_2^B] \wedge [\tau_3^A \Leftrightarrow \tau_4^B]$. As $\Sigma_H \vdash \tau_2^A \Leftrightarrow \tau_2^B$ and $\Sigma_H \vdash \tau_4^A \Leftrightarrow \tau_4^B$, then $\Sigma_H \vdash \sigma(A/C) \Leftrightarrow \sigma(B/C)$.

This remark can be generalized for all σ's, as proved in the next Lemma.

Lemma 1

If φ is an $\mathcal{L}^{ABC}$-formula, then φ is equivalent to an $\mathcal{L}^{ABC}$-formula in which exactly the same variables occur freely and which is a conjunction of disjonctions of homogeneous formulae.

PROOF: By induction on the length of φ.

- If φ is atomic, it is trivial because atomic $\mathcal{L}^{ABC}$-formulae are homogeneous.

- If φ is $\varphi_1 \wedge \varphi_2$, it is trivial by the induction hypothesis.

- If φ is $\neg\psi$, then by the induction hypothesis, ψ is equivalent to some $\bigvee_{i<l} \bigwedge_{j<k_i} \psi_{ij}$, where each ψ_{ij} is homogeneous. Then $\neg\psi$ is equivalent to $\bigwedge_{i<l} \bigvee_{j<k_i} \neg\psi_{ij}$, and thus also to

$$\bigvee_{j_0<k_0} \cdots \bigvee_{j_{l-1}<k_{l-1}} \bigwedge_{i<l} \neg\psi_{ij_i}$$

  This is exactly what we wanted to prove.

- If φ is $(\exists x^T)\psi$, for some $T \in \{A, B, C\}$, then, by the induction hypothesis, ψ is equivalent to some $\bigvee_{i<l} \bigwedge_{j<k_i} \psi_{ij}$, where each ψ_{ij} is homogeneous. We may suppose that the ψ_{ij}'s are numbered in such a way that for each $i<l$, there exists l_i such that x^T occurs free in ψ_{ij} iff $j<l_i$. We may also suppose that $l_i \neq 0$ iff $i<l'$, for some l'. Then φ is equivalent to

$$\bigvee_{i<l'} \left[\left(\exists x^T \bigwedge_{j<l_i} \psi_{ij} \right) \wedge \bigwedge_{l_i \leq j \leq k_i} \psi_{ij} \right] \vee \bigvee_{l' \leq i < l} \bigwedge_{j \leq k_i} \psi_{ij}$$

which is exactly the requested boolean combination.

□

Theorem 2: Σ_H and Σ *prove exactly the same theorems.*

PROOF: We already remarked that $\Sigma \vdash \Sigma_H$ because $\Sigma_H \subset \Sigma$.

And Lemma 1 proves that the example given just before Lemma 1 can be generalized so that $\Sigma_H \vdash \Sigma$. □

2. *Fraïssé's characterization of elementary equivalence*

Let T, $U \in \{A, B, C\}$.
Suppose $x_1^T, \ldots, x_k^T$ be a sequence of k distinct variables of type T and $y_1^U, \ldots, y_k^U$ be a sequence of k distinct variables of type U. Then, for $r \in \mathbb{N}$,

$$x_1^T, \ldots, x_k^T \equiv_r y_1^U, \ldots, y_k^U$$

denotes a *Fraïssé formula* and is defined in the following way, by induction on r:

- $x_1^T, \ldots, x_k^T \equiv_0 y_1^U, \ldots, y_k^U$ is

$$\bigwedge_{\varphi \in \mathrm{At}(\vec{x}T)} \varphi(y_1^U/x_1^T, \ldots, y_k^U/x_k^T) \Leftrightarrow \varphi,$$

  where $\mathrm{At}(\overrightarrow{x^T})$ is the finite set of all $\mathcal{L}^{ABC}$-formulae whose free variables are among $x_1^T, \ldots, x_k^T$ and y_i^U/x_i^T denotes the substitution of y_i^U for x_i^T in φ.

- $x_1^T, \ldots, x_k^T \equiv_{r+1} y_1^U, \ldots, y_k^U$ is

$$(\forall x^T)(\exists y^U)(x^T, x_1^T, \ldots, x_k^T \equiv_r y^U, y_1^U, \ldots, y_k^U)$$
$$\wedge\ (\forall y^U)(\exists x^T)(x^T, x_1^T, \ldots, x_k^T \equiv_r y^U, y_1^U, \ldots, y_k^U),$$

  where, to avoid clashes, x^T and y^U are supposed not to occur in $x_1^T, \ldots, x_k^T \equiv_r y_1^U, \ldots, y_k^U$.

In the previous definition, k could be equal to zero. Precisely, if $\varnothing^T$ denotes the empty sequence of variables of type T (for any T), then

- we define $\varnothing^T \equiv_0 \varnothing^U$ to be any provable sentence, as for example,

$$(\forall x^T)(R_1(x^T, \ldots, x^T) \Rightarrow R_1(x^T, \ldots, x^T))$$

- and we define $\varnothing^T \equiv_{r+1} \varnothing^U$ to be

$$(\forall x^T)(\exists y^U)(x^T \equiv_r y^U)\ \wedge\ (\forall y^U)(\exists x^T)(x^T \equiv_r y^U).$$

Now let

$$\Phi = \{\varnothing^A \equiv_r \varnothing^B : r \in \mathbb{N}\}.$$

By using purely *semantic* arguments, Fraïssé proved that $A \equiv B$ iff $A \oplus B \models \Phi$ (see [4], [3, Chap. XI] or [2, exercices 1.3.15 to 1.3.20]). We shall prove, by using purely *syntactic* arguments, that Φ is equivalent to Σ (and thus Σ_H). But first, we shall state a monotonicity property of Fraïssé formulae.

Lemma 3

Let T, $U \in \{A, B, C\}$, k, $r \in \mathbb{N}$, $1 \leq k_1 < \ldots < k_l \leq k$ and $r' \leq r$. Then,

$$x^T_1, \ldots, x^T_k \equiv_r y^U_1, \ldots, y^U_k \text{ implies } x^T_{k_1}, \ldots, x^T_{k_l} \equiv_{r'} y^U_{k_1}, \ldots, y^U_{k_l}.$$

In particular, $\varnothing^T \equiv_r \varnothing^U$ implies $\varnothing^T \equiv_{r'} \varnothing^U$.

PROOF: We just sketch the proof, which is easy but lengthy.

1. First prove, by induction on r, that $x^T_1, \ldots, x^T_k \equiv_r y^U_1, \ldots, y^U_k$ implies

$$x^T_1, \ldots, x^T_{i-1}, x^T_{i+1}, \ldots, x^T_k \equiv_r y^U_1, \ldots, y^T_{i-1}, y^T_{i+1}, \ldots, y^U_k.$$

2. Then, by applying (1), prove, by induction on r, that

$$x^T_1, \ldots, x^T_k \equiv_{r+1} y^U_1, \ldots, y^U_k \text{ implies } x^T_1, \ldots, x^T_k \equiv_r y^U_1, \ldots, y^U_k.$$

3. The lemma is proved by applying (1) and (2) several times.

□

Lemma 4: $\Sigma \vdash \Phi$.

PROOF: First prove by induction on r that for any sequence $x^A_1, \ldots, x^A_k$, the formula $x^A_1, \ldots, x^A_k \equiv_r x^A_1, \ldots, x^A_k$ is a theorem of predicate calculus (including the cases where $k=0$). So, in particular, for all $r \in \mathbb{N}$, $\vdash \varnothing^A \equiv_r \varnothing^A$, and thus $\Sigma \vdash \varnothing^A \equiv_r \varnothing^A$.
Then, let τ be the sentence $\varnothing^A \equiv_r \varnothing^C$ and remark that $\varnothing^A \equiv_r \varnothing^B$ is $\tau(B/C)$, while $\varnothing^A \equiv_r \varnothing^A$ is $\tau(A/C)$. So,

$$\Sigma \vdash (\varnothing^A \equiv_r \varnothing^A) \Leftrightarrow (\varnothing^A \equiv_r \varnothing^B).$$

By Modus Ponens, this implies that $\Sigma \vdash \varnothing^A \equiv_r \varnothing^B$. QED. □

In the other direction, in Lemma 5, we follow the usual proof. But first, we need to introduce the notion of *quantifier rank* $\mathrm{qr}(\varphi)$ of an $\mathcal{L}^{ABC}$-formula φ, which is defined by induction on the length ofφ.

- If φ is atomic, then $\mathrm{qr}(\varphi) = 0$.
- $\mathrm{qr}(\psi_1 \wedge \psi_2) = \mathrm{qr}(\psi_1 \vee \psi_2) = \mathrm{qr}(\psi_1 \Rightarrow \psi_2) = \max\{\mathrm{qr}(\psi_1), \mathrm{qr}(\psi_2)\}$.
- $\mathrm{qr}(\neg\psi) = \mathrm{qr}(\psi)$.
- $\mathrm{qr}((\exists x^A)\psi) = \mathrm{qr}((\exists x^B)\psi) = \mathrm{qr}(\psi)$.
- $\mathrm{qr}((\exists x^C)\psi) = \mathrm{qr}(\psi) + 1$.

Lemma 5

If φ is an $\mathcal{L}^{ABC}$-formula whose free variables of type C are among $x_1^C, \ldots, x_k^C$, then

$$x_1^A, \ldots, x_k^A \equiv_{\mathrm{qr}(\varphi)} x_1^B, \ldots, x_k^B \textit{ implies } \varphi(A/C) \Leftrightarrow \varphi(B/C).$$

PROOF: By induction on the length of φ. We use such a notation as $\vec{x}^T$ to denote the sequence $x_1^T, \ldots, x_k^T$.

- f φ is atomic.
  There are two cases:

  - If φ is A-homogeneous or B-homogeneous, then $\varphi(A/C)$ and $\varphi(B/C)$ are both identical with φ. So $\vdash \varphi(A/C) \Leftrightarrow \varphi(B/C)$. And thus, $\vec{x}^A \equiv_{\mathrm{qr}(\varphi)} \vec{x}^B$ implies $\varphi(A/C) \Leftrightarrow \varphi(B/C)$.

  - If φ is C-homogeneous, then $\mathrm{qr}(\varphi)=0$ and $\vec{x}^A \equiv_0 \vec{x}^B$ is a conjunction one of whose terms is exactly $\varphi(A/C) \Leftrightarrow \varphi(B/C)$. Thus $\vec{x}^A \equiv_0 \vec{x}^B$ implies $\varphi(A/C) \Leftrightarrow \varphi(B/C)$.

- If φ is some $\varphi_1 \wedge \varphi_2$ or $\neg\varphi$, then it is trivial by the induction hypothesis and Lemma 3.

- If φ is $(\exists x^T)\psi$, where $T \in \{A, B\}$, then $\mathrm{qr}(\varphi) = \mathrm{qr}(\psi)$. We may suppose that x^T does not occur in $\vec{x}^A \equiv_{\mathrm{qr}(\varphi)} \vec{x}^B$. So, by the induction hypothesis, we know that $\vdash (\forall x^T)[\vec{x}^A \equiv_\varphi \vec{x}^B \Rightarrow (\psi(A/C) \Leftrightarrow (\psi(B/C)))]$. From this we infer that $\vec{x}^A \equiv_\varphi \vec{x}^B$ implies $(\exists x^T)(\psi(A/C) \Leftrightarrow (\exists x^T)(\psi(B/C)))$. QED.

- If φ is $(\exists x^C)\psi$, then $\text{qr}(\varphi)=\text{qr}(\psi)+1$.
  So, by the definition of Fraïssé formulae, $\vec{x}^A \equiv_{\text{qr}(\varphi)} \vec{x}^B$ implies $(\forall x^A)(\exists x^B)(x^A, \vec{x}^A \equiv_{\text{qr}(\psi)} x^B, \vec{x}^B)$. Then

  $$\begin{aligned}
  &\vec{x}^A \equiv_{\text{qr}(\varphi)} \vec{x}^B \wedge (\exists x^A)(\psi(A/C)) \\
  &\Rightarrow (\forall x^A)(\exists x^B)(x^A, \vec{x}^A \equiv_{\text{qr}(\psi)} x^B, \vec{x}^B) \wedge (\exists x^A)(\psi(A/C)) \\
  &\Rightarrow (\exists x^A)(\exists x^B)[x^A, \vec{x}^A \equiv_{\text{qr}(\psi)} x^B, \vec{x}^B \wedge \psi(A/C)] \\
  &\Rightarrow (\exists x^A)(\exists x^B)\ [\psi(A/C) \Leftrightarrow \psi(B/C) \wedge \psi(A/C)] \\
  &\quad \text{(by the induction hypothesis)} \\
  &\Rightarrow (\exists x^B)(\psi(B/C)).
  \end{aligned}$$

  Thus $\vec{x}^A \equiv_{\text{qr}(\varphi)} \vec{x}^B$ implies $(\exists x^A)\psi(A/C) \Rightarrow (\exists x^B)\psi(B/C)$.
  In the same way, $\vec{x}^A \equiv_{\text{qr}(\varphi)} \vec{x}^B$ implies $(\exists x^B)\psi(B/C) \Rightarrow (\exists x^A)\psi(A/C)$.
  QED. □

Theorem 6

Φ and Σ prove *exactly the same theorems*.

PROOF: This is nothing but a corollary of Lemmas 4 and 5.

3. *Generalization to any language $\mathcal{L}$*

Up to now, we supposed that the non logical symbols of $\mathcal{L}$ are a finite number of relation symbols. The definitions and proofs we gave can be generalized to other languages in the following way.
The problem is to keep $\vec{x}^A \equiv_0 \vec{y}^B$ a *finite* conjunction. This is why we introduce the notion of locally finite set of formulae.
We define a set $\mathfrak{C}$ of $\mathcal{L}^{ABC}$-formulae to be *locally finite* iff it satisfies the following conditions:

- If $\varphi \in \mathfrak{C}$, then φ is a C-homogeneous atomic formula.

- Let $\varphi \in \mathfrak{C}$ and $x_1^C, \ldots, x_k^C$ be all free variables of φ. If $y_1^C, \ldots, y_k^C$ is a sequence of variables, then $\varphi(y_1^C/x_1^C, \ldots, y_k^C/x_k^C) \in \mathfrak{C}$. In this condition, we suppose that x_i^C is distinct from x_j^C if $i \neq j$, but y_i^C might be identical with y_j^C if $i \neq j$.

- If $x^C_1, \ldots, x^C_k$ is a *finite* set of type C variables, then the number of $\varphi \in \mathfrak{C}$ such that all free variables of φ are among $x^C_1, \ldots, x^C_k$ is finite.

Now we define the *closure* $\overline{\mathfrak{C}}$ of $\mathfrak{C}$ as the smallest set satisfying the following conditions:

- $\mathfrak{C} \subset \overline{\mathfrak{C}}$.

- If φ is atomic and φ is A-homogeneous or B-homogeneous, then $\varphi \in \overline{\mathfrak{C}}$.

- If $\varphi, \psi \in \overline{\mathfrak{C}}$, then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$, $\varphi \Rightarrow \psi$, $(\exists x^T)\varphi$ and $(\forall x^T)\varphi$ belong to $\overline{\mathfrak{C}}$, for all $T \in \{A, B, C\}$.

Finally, let T, $U \in \{A, B, C\}$ and suppose that $x^T_1, \ldots, x^T_k$ is a sequence of k distinct variables of type T and that $y^U_1, \ldots, y^U_k$ is a sequence of k distinct variables of type U. Then, for $r \in \mathbb{N}$, we are going to define

$$\mathrm{x}^T_1, \ldots, x^T_k \equiv_r y^U_1, \ldots, y^U_k \ (\mathrm{mod}\ \mathfrak{C})$$

by induction on r:

- $\mathrm{x}^T_1, \ldots, x^T_k \equiv_0 y^U_1, \ldots, y^U_k$ is

  $$\bigwedge_{\varphi \in \mathrm{At}(\overrightarrow{z^C})} \varphi(x^T_1/z^C_1, \ldots, x^U_k/z^C_k) \Leftrightarrow \varphi(y^U_1/z^C_1, \ldots, y^U_k/z^C_k),$$

  where $\mathrm{At}(\overrightarrow{z^C})$ is the *finite* set of all $\varphi \in \mathfrak{C}$ whose free variables are among $z^C_1, \ldots, z^C_n$.

- $x^T_1, \ldots, x^T_k \equiv_{r+1} y^U_1, \ldots, y^U_k$ (mod $\mathfrak{C}$) is

  $$(\forall x^T)(\exists y^U)(x^T, x^T_1, \ldots, x^T_k \equiv_r y^U, y^U_1, \ldots, y^U_k \ (\mathrm{mod}\ \mathfrak{C}))$$
  $$\wedge\ (\forall y^U)(\exists x^T)(x^T, x^T_1, \ldots, x^T_k \equiv_r y^U, y^U_1, \ldots, y^U_k \ (\mathrm{mod}\ \mathfrak{C})),$$

  where, to avoid clashes, x^T and y^U are supposed not to occur in $x^T_1, \ldots, x^T_k \equiv_r y^U_1, \ldots, y^U_k$.

We now define

$$\Phi_{\mathfrak{C}} = \{\emptyset^A \equiv_r \emptyset^B \ (\mathrm{mod}\ \mathfrak{C}) : \mathrm{r} \in \mathbb{N}\}.$$

Lemma 7:

If $\mathfrak{C}$ is a locally finite set of formulae, then $\Sigma \vdash \Phi_{\mathfrak{C}}$.

PROOF: identical with the proof of Lemma 4.

Lemma 8

Let $\mathfrak{C}$ be a locally finite set of formulae. If $\varphi \in \overline{\mathfrak{C}}$ is a formula whose free variables of type C are among $x_1^C, \ldots, x_k^C$, then

$$x_1^A, \ldots, x_k^A \equiv_{\mathrm{qr}(\varphi)} x_1^B, \ldots, x_k^B \ (\mathrm{mod}\ \mathfrak{C}) \textit{ implies } \varphi(A/C) \Leftrightarrow \varphi(B/C).$$

PROOF: identical with the proof of Lemma 5.

Theorem 9

Let A and B be two $\mathscr{L}$-structures.
Then $A \equiv B$ iff for all locally finite sets of formulae $\mathfrak{C}$, $A \oplus B \models \Phi_{\mathfrak{C}}$.

PROOF: Remark that the set of all $\mathscr{L}^{ABC}$-formulae is equal to

$$\bigcup_{\mathfrak{C} \text{ locally finite}} \overline{\mathfrak{C}}.$$

Then the theorem is a corollary of Lemmas 7 and 8.

4. *Elementary equivalence of intuitionistic models*

It is an interesting problem to see whether the previous results can be adapted from usual classical models to Kripke models for intuitionistic logic. On the one hand, it gives some results about elementary equivalence of Kripke models. On the other hand, it exhibits which properties of classical predicate calculus and of usual classical models are needed for Fraïssé's characterization to work.
We suppose that a Kripke model A is a family of classical structures with an underlying partial order:

$$\mathcal{A} = \langle (\mathcal{A}_k)_{k \in K}, (K, \leq_K, 0_K) \rangle.$$

We suppose that $(\forall k \in K)(0_K \leq k)$ and forall $k, l \in K$, $\mathcal{A}_k$ is included in $\mathcal{A}_l$ if $k \leq_K l$ (see [1] or [5] for more definitions and results on Kripke models).
When we try to adapt the results of previous sections to Kripke models, several problems arise.
Given two Kripke models $\mathcal{A}$ and $\mathcal{B}$, the *first problem* is to define $\mathcal{A} \oplus \mathcal{B}$. Here, we shall restrict ourselves to the easy case by supposing that $\mathcal{A}$ and $\mathcal{B}$ have the same underlying partial order, i. e.

$$\mathcal{A} = \langle (\mathcal{A}_k)_{k \in K}, (K, \leq_K, 0_K) \rangle \text{ and } \mathcal{B} = \langle (\mathcal{B}_k)_{k \in K}, (K, \leq_K, 0_K) \rangle.$$

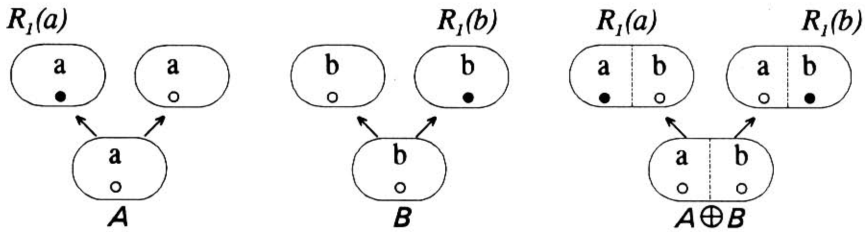

Figure 1: $\mathcal{A} \equiv \mathcal{B}$ is not equivalent to $\mathcal{A} \oplus \mathcal{B} \Vdash \Sigma_H$

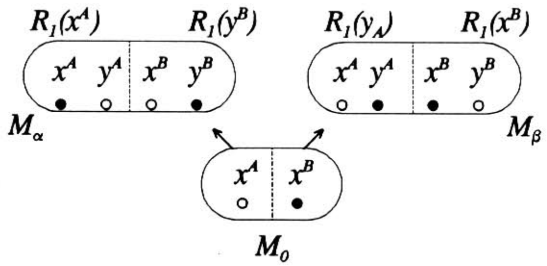

Figure 2: Σ_H is not equivalent to Σ

Then we can define

$$\mathcal{A} \oplus \mathcal{B} = \langle (\mathcal{A}_k \oplus \mathcal{B}_k)_{k \in K}, (K, \leq_K, 0_K) \rangle.$$

In the sequel, we shall omit the K subscripts and write $\leq$ instead of $\leq_K$ and

0 instead of 0_K.

The *second problem* is that, in general, if A is a Kripke model and σ, τ are two sentences, then $A \Vdash (\sigma \Leftrightarrow \tau)$ is not equivalent to $(A \Vdash \sigma) \Leftrightarrow (A \Vdash \tau)$. This is why $A \equiv B$ is implied by, but not equivalent to $A \oplus B \Vdash \Sigma_H$. Suppose that $\mathcal{L}$ contains a unary predicate symbol R_1. Then Figure 1 gives two Kripke models A and B such that $A \equiv B$ but $A \oplus B \nVdash \Sigma_H$. Indeed, $A \equiv B$ because A is isomorphic to B. And $A \oplus B \nVdash \Sigma_H$ because $A \oplus B \nVdash (\exists x^A)(R_1(x^A)) \Leftrightarrow (\exists x^B)(R_1(x^B))$. Remark also that $\oplus$ is not stable under isomorphisms: $A \cong B$ but $A \oplus A \ncong A \oplus B$.

The *third problem* is that, in intuitionistic predicate calculus, $\Sigma_H \nvdash \Sigma$. One could expect such a negative result by considering the proof of Lemma 1, which uses "very classical" arguments.

The proof of $\Sigma_H \nvdash \Sigma$ is tricky. Consider the $\mathcal{L}^{AB}$-Kripke model given in Figure 2. We are going to prove that $M \Vdash \Sigma_H$ and $M \nVdash \Sigma$.

$M \nVdash \Sigma$ because

$$M \nVdash_0 [(\exists x^A)(\exists y^A)(R_1(x^A) \Leftrightarrow R_1(y^A))] \Leftrightarrow [(\exists x^A)(\exists y^B)(R_1(x^A) \Leftrightarrow R_1(y^B))].$$

The tricky part consists in checking that $M \Vdash \Sigma_H$. Here is a sketch of proof:

1. For any $\mathcal{L}$-formula φ, prove that $M \Vdash_\alpha \varphi^A[x^A, y^A]$ iff $M \Vdash_\alpha \varphi^B[y^B, x^B]$. Thus, for any $\mathcal{L}$-*sentence* σ, $M \Vdash_\alpha \sigma^A$ iff $M \Vdash_\alpha \sigma^B$. In the same way, $M \Vdash_\beta \sigma^A$ iff $M \Vdash_\beta \sigma^B$.

2. For any $\mathcal{L}$-formula φ, prove that $M \Vdash_\alpha \varphi^A[x^A, y^A]$ iff $M \Vdash_\beta \varphi^B[x^B, y^B]$. In the same way, $M \Vdash_\alpha \varphi^B[x^B, y^B]$ iff $M \Vdash_\beta \varphi^A[x^A, y^A]$. Thus,

$$(\forall k > 0\ M \Vdash_k \varphi^A[x^A, y^A]) \text{ iff } (\forall k > 0\ M \Vdash_k \varphi^B[x^B, y^B]).$$

3. Using (2), prove by induction on the length of φ that $M \Vdash_0 \varphi^A[x^A]$ iff $M \Vdash_0 \varphi^B[x^B]$, for any $\mathcal{L}$-formula φ with only one free variable. (For example, remark that $M \Vdash_0 \neg\psi_A$ iff $(M \nVdash_0 \psi) \wedge (\forall k > 0)(M \Vdash_k \neg\psi)$; thus $M \Vdash_0 \neg\psi^A$ implies $M \Vdash_0 \neg\psi^B$ by the induction hypothesis and (2).)

4. Using (1) and (3), prove that, for any $\mathcal{L}$-sentence σ, $M \Vdash_k \sigma^A$ iff M

$\Vdash_k \sigma^B$, for all $k \in K$. Thus, $M \Vdash_0 \sigma^A \Leftrightarrow \sigma^B$.

Nevertheless, a positive result, that the reader can check, is that Σ and Φ still prove the same theorems when the underlying predicate calculus is intuitionistic. More precisely, the proofs of all the lemmas, except Lemma 1, are intuitionistic.

We could define a new elementary equivalence relation $\equiv_F$ on Kripke models by

$$\mathcal{A} \equiv_F \mathcal{B} \text{ iff } \mathcal{A} \oplus \mathcal{B} \Vdash \Sigma.$$

On the one hand, this definition does not seem convenient:

- $\mathcal{A} \oplus \mathcal{B}$ is naturally defined only if $\mathcal{A}$ and $\mathcal{B}$ have the same underlying partial order;

- we gave an example showing that if $\mathcal{A} \equiv_F \mathcal{B}$ and $\mathcal{A} \cong \mathcal{A}'$, then it could be the case that $\mathcal{A}' \not\equiv_F \mathcal{B}$.

But on the other hand, we conjecture that several interesting properties of *classical* elementary equivalence are satisfied by $\equiv_F$. For example, with suitable conditions on $\mathcal{A} \oplus \mathcal{B}$ (cardinality, saturation, etc.) $\mathcal{A} \equiv_F \mathcal{B}$ implies $\mathcal{A} \cong \mathcal{B}$, by back-and-forth arguments.

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