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A NOTE ON UNPROVABILITY-PRESERVING SOUND TRANSLATIONS(1)

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1. *Introduction*

In this note, which may be regarded as a friend of Inoué [15, 16, 17, 18, 19, 20](1), we shall show curious nontrivial *syntactical* subtleties of unprovability-preserving sound translation. The technicalities required to understand this note are moderate. By F_X we denote the set of all well-formed formulas of a formal system X. By an *unprovability-preserving sound translation* $\tau(\cdot)$ from a formal system X to a formal system Y, we mean a mapping τ from F_X to F_Y such that it satisfies the following two conditions:

(i) τ is sound: i.e. for any formula A of X,

$\vdash_X A$ implies $\vdash_Y \tau(A)$,

where $\vdash_X A$ means that A is a theorem of X.

(ii) τ preserves unprovability: i.e. for any formula A of X,

$\dashv_X A$ implies $\dashv_Y \tau(A)$,

where $\dashv_X A$ means that A is not a theorem of X.

The notion of unprovability-preserving sound translation is not a trivial one. If a translation is unprovability-preserving, then it is faithful, and vice versa. So an unprovability-preserving sound translation is logically nothing

(1) This paper is dedicated to Edina Jung.

(1) In the sense that this note is concerned with the syntax of unprovability ([15, 16, 17, 19, 20]) and that it also is concerned with a way to construct new embeddings in terms of a given one ([18]).

but an *embedding* in the ordinary terminology. But, we take a different terminology in order to emphasize unprovability-preservation of a translation, because one has been little concerned with unprovability seriously. To make sure, we shall illustrate it with an example. Let X be a classical consistent formal system. Define three (auto)translations $\mathrm{id}(\cdot)$, $\mathrm{const}_1(\cdot)$, $\mathrm{const}_0(\cdot)$ from X to X as follows: for any formula A of X,

$$\begin{aligned} \mathrm{id}(A) &= A, \\ \mathrm{const}_1(A) &= A \vee \sim A, \\ \mathrm{const}_0(A) &= A \wedge \sim A. \end{aligned}$$

Then, id is an unprovability-preserving sound translation. The translation const_1 is sound but not unprovability-preserving. On the contrary, const_0 is unprovability-preserving but not sound. So we have the following trivial but nice theorem.

THEOREM 1. 1. *The unprovability-preserving soundness of the identity autotranslation* id *is an invariant of formal systems.*

Let X, Y be given consistent formal systems. Suppose that an unprovability-preserving sound translation τ from X to Y is given. Can we then construct countably many *new* unprovability-preserving sound translations from τ? The difficulty of this question depends on the systems X, Y and the given translation, of course. But we first have to specify a precise meaning of the ambiguous word 'new' in the question, before we give some nontrivial examples of such a construction of countably many new unprovability-preserving sound translations from a given one.

DEFINITION 1. 2. Let X, Y be arbitrary consistent formal systems. Suppose that Y has truth-functional equivalence $\equiv$ in its language. For any unprovability-preserving translations $\tau_1(\cdot)$, $\tau_2(\cdot)$ from X to Y, τ_1, τ_2 are said to be *non-C-equivalent* if there is a formula A of X (called a *ditcher*) such that (i) A is not a theorem of X and (ii) $\tau_1(A) \equiv \tau_2(A)$ is not a theorem of Y.(2) Otherwise we say that they are *C-equivalent*).

As suggested in the above definition, we are interested in non-C-equivalent unprovability-preserving translations. In other words, because we might

(2) In this definition, we implicitely assume that Y is an extension of a classical theory. But this assumption is not essential for more general considerations. It depends on a given system and its language. For example, if Y is an intuitionistic system, then we shall regard $\equiv$ as intuitionistic equivalence in place of classical one.

identify C-equivalent unprovability-preserving translations with each other, we are not curious about the same (in the above sense) translation as a given one. In the sequel, we will see a lot of examples of non-C-equivalent unprovability-preserving translations. It is also easy to give an example of C-equivalent unprovability-preserving translations.(3) We shall make the above informal descriptions of the question precise as follows:

Question 1: Let X, Y be given consistent formal systems. For any unprovability-preserving sound translation τ from X to Y, is there a construction $\mathbb{C}$ such that by the construction $\mathbb{C}$, we can obtain *countably many mutually non-C-equivalent unprovability-preserving sound translations* (= *new ones*) from τ?(4)

This Question 1 creates many open problems if we vary X and Y. Because it seems hard to answer Question 1 in general, we shall pose a rather moderate version of it, as a first step to answer the hard one.

Question 2: Let X, Y be given consistent formal systems. Suppose that an unprovability-preserving sound translation τ from X to Y is given. Is there a construction $\mathbb{C}$ such that by the construction $\mathbb{C}$, we can obtain countably many mutually non-C-equivalent unprovability-preserving sound translations from τ?

Even this rather moderate Question 2 provides a lot of open problems.

My plan for the rest of this note is the following. We shall give an affirmative answer to Question 2, when given X, Y, τ are the following:

(P1) $X = Y =$ intuitionistic first-order predicate (propositional) logic IQC (IPC),
$\tau =$ the identity translation (*idem*),

(P2) $X =$ classical first-order predicate (propositional) logic CQC (CPC),
$Y =$ IQC (IPC),

(3) For example, from LEMMA 43a in Kleene [24, p. 495], it follows that unprovability-preserving translations $(\cdot)^\circ$, $\neg\neg((\cdot)^\circ)$ from the classical to intuitionistic number-theoretic formal systems are C-equivalent. Another example: from a remark in Troelstra [38, p. 296], unprovability-preserving translations $(\cdot)^{'}$, $\Box((\cdot)^{'})$ from intuitionistic propositional logic IPC to modal logic S4 are C-equivalent (cf. Gödel [12] and McKinsey-Tarski [27]).

(4) A trivial remark: If in the question, we replace "mutually non-C-equivalent" by "C-equivalent", then the question becomes a trivial one so that the word "countably many" also becomes meaningless. Given a translation τ, we can, for example, define a C-equivalent translation $\neg\neg\tau$ with τ in classical consistent formal system.

τ = Gödel translation (*idem*),
(see Gödel [11] and also refer e.g. to Kleene [24] or Troelstra-van Dalen [39]),

(P3) $X = Y$ = Lemmon-Scott's normal propositional modal logic K,
τ = the identity translation,

(for K, see e.g. Hughes-Cresswell [14], Segerberg [31] and Van Benthem [40]). We shall first work out the cases (P1) and (P2) in the following second section, because the used technique is common to them. After that we shall deal with the case (P3) in the third section. In Appendix, we recall Gentzen-style sequent calculus G3 for the convenience of the reader.

2. *A study of τ_p^i, an analysis in Gentzen system*

In this section, we give affirmative answers to Question 2 in the cases of (P1) and (P2), respectively. We will deal only with the predicate case of them, i.e. that of (P1) in great detail. The other cases of them are, *mutatis mutandis*, similarly taken care of. We take a formulation without $\perp$ (falsum) for IQC. Let τ be the identity translation id. A construction for an answer for (P1) is the following:

(*) $\tau_p^i(A) = \mathrm{id}(A) \supset p_i \,.\!\supset p_i \ (i \geq 0)$,

where p_i is an atomic formula of IQC and $p_i \neq p_j$ $(i \neq j)$.(5)

It is obvious that for any $i \geq 0$, τ_p^i is sound (prove it in your favorite syntactical way). We shall prove that for any $i \geq 0$, τ_p^i is unprovablility-preserving. Let $i \geq 0$ be an arbitrary index (integer). Suppose that $\tau_p^i(A)$ is a theorem of IQC. We shall consider the provability of $\tau_p^i(A)$ in a Gentzen-style sequent calculus. For a Gentzen-style formulation for IQC, we here adopt Kleene's G3 (see Kleene [24, p. 481] and Appendix of the present paper) for which Gentzen's normal form theorem (Hauptsatz or cut elimination theorem)(6) holds. So we are given a proof (figure) $\mathbf{R}$ of $\tau_p^i(A)$ in G3. The proof $\mathbf{R}$ should look the following:

(5) This type of construction was to some extent considered in Flagg [3, 4] in the context of type theory and set theory. But in the present paper we can obtain more about it.

(6) For proof theory in general, one may consult e.g. Gentzen [6], Girard [7], Girard et al. [8], Kleene [24], Mints [28], Schütte [29, 30] and Takeuti [37].

$$\dfrac{\dfrac{\begin{array}{c}\mathrm{D}_1\\ A \supset p_i \to A\end{array} \quad A \supset p_i, p_i \to p_i}{A \supset p_i \to p_i}}{\to A \supset p_i \,.\!\supset p_i,}$$

where except for a proof D_1 of $A \supset p_i \to A$, the above cut-free proof of τ_p^i (A) in G3 is unique (note that there are no structural rules in G3).

We shall further analyse the structure of D_1. First, as a remark, we keep in mind that D_1 is constructed by a *finite* number of applications of the rules in G3. Suppose that D_1contains no application of the rule introducing $\supset$ to A and p_i, in order to introduce $A \supset p_i$ in the antecedent of a sequent. Then, it is obvious that $A \supset p_i$ does not contribute the provability of $A \supset p_i \to A$. In this case, by deleting all $A \supset p_i$'s from D_1, we can obtain a proof D_2 of $\to A$ in G3. So A is a theorem of IQC.

Next suppose that D_1 contains an application of the rule introducing $\supset$ to A and p_i in order to introduce $A \supset p_i$ in the antecedent of a sequent. The number of such applications is finite because of the above remark. Then we can find a sequent

(#) $\quad A \supset p_i, B_1, \ldots, B_n \to A,$

and its proof D_3 such that the following conditions (C1)-(C4) are satisfied:

(C1) D_3 contain no application of the rule introducing $\supset$ for introducing $A \supset p_i$ in the antecedent of a sequent,

(C2) $B_1, \ldots, B_n$ $(n \geq 0)$ are proper subformulas of A,

(C3) Below D_3, for any i $(1 \leq i \leq n)$, B_i is not used as p_i for an application of the rule introducing $\supset$ for introducing $A \supset p_i$ in the antecedent of a sequent (note that B_i may be p_i for any i $(1 \leq i \leq n)$),

(C4) D_1 is analysed as follows:

$$\dfrac{\begin{array}{c}\mathrm{D}_3\\ A \supset p_i, B_1, \ldots, B_n \to A\end{array} \qquad \begin{array}{c}\mathrm{D}_4\\ A \supset p_i, B_1, \ldots, B_n, p_i \to C\end{array}}{A \supset p_i, B_1, \ldots, B_n \to C}$$

D_5

$$A \supset p_i \to A.$$

In other words, such a sequent (#) can be found at a place in D_1 where the most left application of the rule in question occurs. Considering the part of D_1 below (#), from (C1)-(C3), it follows that $\to A$ should have a proof of it in G3. Hence, A is provable in IQC.

Take a contraposition of the argument considered above. Then we can conclude that τ_p^i is unprovability-preserving. For any $i \neq j$, $\tau_p^i(p_k) \equiv \tau_p^j(p_k)$ is not a theorem of IQC, if $k \neq i$ or $k \neq j$ (this atomic formula p_k is a dichter for the non-C-equivalence of τ_p^i and τ_p^j).(7) Thus the construction (*) yields countably many mutually non-C-equivalent unprovability-preserving sound translations from IQC to IQC. The case (P2) can similarly be taken care of. So it is left to the reader. Note that the construction (*) even provides a way to construct new embeddings from an *arbitrary* embedding (need not to be restricted to the identity translation) if the above argument can be applied. I think that this is a very remarkable fact. Also note that we do not need to stick to predicate logics as treated above and that the same holds for extensions of intuitionistic propositional logic IPC as Y so long as the above argument is applied.

Although our treatment is here proof-theoretic, it would be possible to give a semantic argument by means of Kripke-style semantics (Kripke [26]), for example. However, if we deal with them semantically, in the case of predicate logic, we have to take a restriction of the translation in question to the set of all sentences of the logic in question (for the preservance of unprovability) and the p_i of τ_p^i must be a propositional letter, in order to make use of the completeness theorem of the logics. We have no such a problem in the case of propositional logic for a semantic treatment.

Summing up the above, we have the following theorems. (By CPC we denote classical propositional logic.)

THEOREM 2. 1. *For any atomic formula (propositional letter) p but not $\bot$ of* IQC (IPC), *a translation* $(\cdot) \supset p. \supset p$ *is an embedding of* IQC (IPC) *in* IQC (IPC). *For any pair of distict atomic formulas (propositional letters) p, q with $p \neq \bot, q \neq \bot$, translations* $(\cdot) \supset p. \supset p$ *and* $(\cdot) \supset q. \supset q$ *from* IQC (IPC) *to* IQC (IPC) *are non-C-equivalent.*

(7) The equivalence $\equiv$ in $\tau_p^i(p_k) \equiv \tau_p^j(p_k)$ is intuitionistic.

COROLLARY 2. 2. *Let X be a consistent formal system. Suppose that we have an embedding* τ *of X in* IQC (IPC). *Then, for any atomic formula (propositional letter) p but not* $\bot$ of IQC (IPC), $\tau(\cdot) \supset p. \supset p$ *is an embedding of X in* IQC (IPC).

PROOF. Immediate from THEOREM 2. 1. $\square$

THEOREM 2. 3. *There are an unprovability-preserving sound translation* τ *from* IQC (IPC) *to* IQC (IPC) *and a construction* $\mathbb{C}$ such that by the *construction* $\mathbb{C}$, *we can obtain, from* τ, *countably many mutually non-C-equivalent unprovability-preserving sound translations from* IQC (IPC) *to* IQC (IPC).

COROLLARY 2. 4. *There are countably many mutually non-C-equivalent embeddings of* IQC (IPC) *in* IQC (IPC).

THEOREM 2. 5. *There are an unprovability-preserving sound translation* τ *from* CQC (CPC) *to* IQC (IPC) *and a construction* $\mathbb{C}$ *such that by the construction* $\mathbb{C}$, *we can obtain, from* τ, *countably many mutually non-C-equivalent unprovability-preserving sound translations from* CQC (CPC) *to* IQC (IPC).

COROLLARY 2. 6. *There are countably many mutually non-C-equivalent embeddings of* CQC (CPC) *in* IQC (IPC).

3. *A study of* $\tau_N{}^k{}_{id}$, *an analysis in refutation system*

Now we shall take care of the case (P3) of Introduction. Let X be Lemmon-Scott's normal propositional modal logic K. Take $Y = X$. We take $\square$ (the necessity operator) as primitive for our arguments, and $\square^k$ stands for $\square ... \square$ with the k copies of $\square$ ($k \geq 0$). The possibility operator $\diamond$ is defined in terms of $\square$ as usual. Formulas containing no modal operator, are called $\square$*-free*. As a given unprovability-preserving sound translation, we take the identitiy translation id from K to K. For a construction $\mathbb{C}$ for an answer to the question, define $\tau_N{}^k{}_{id} = \square^k$ id ($k \geq 0$). It is immediate by the rule of necessitation that for each $k \geq 0$, $\tau_N{}^k{}_{id}$ is a sound translation. We shall show *in a syntactical way* that for each $k \geq 0$, $\tau_N{}^k{}_{id}$ preserves unprovability, because I am interested in the syntax of the unprovability of K. For that purpose, we need some preparation. In Goranko [9, 10], a method of axiomatic rejection (or refutation calculus (system)) for K was proposed. An *axiomatic rejection* (or *refutation calculus* (*system*)) for a (consistent) formal system X is a (consistent) formal system to yield all unprovable formulas (i.e. *rejected formulas*) of X but not provable ones (for the recent developments and the literature on axiomatic rejection, see e.g. [9, 10], Inoué [16], Skura [32, 33] and Słupecki-Bryll [35]). The Goranko's system, which is here denoted by GRK, consists of K plus the following axioms and rules:

Axioms:

(i) $\dashv_{\mathrm{GRK}} \bot$,
(ii) $\dashv_{\mathrm{GRK}} \Diamond \top$,

Rules:

(iii) Reverse substitution: $\dashv_{\mathrm{GRK}} \sigma(A) \Rightarrow \dashv_{\mathrm{GRK}} A$, for any uniform substitution σ,

(iv) Łukasiewicz's rule: $\vdash_{\mathrm{K}} A \supset B, \dashv_{\mathrm{GRK}} B \Rightarrow \dashv_{\mathrm{GRK}} A$,
(v) For any $\Box$-free formula A,

$$R_{\mathrm{K}} \quad \frac{\dashv_{\mathrm{GRK}} A, \ \dashv_{\mathrm{GRK}} B \vee C_1, \ldots, \ \dashv_{\mathrm{GRK}} B \vee C_k}{\dashv_{\mathrm{GRK}} A \vee \Box C_1 \vee \ldots \vee \Box C_k \vee \Diamond B,} \quad \text{(for all } k \geq 1)$$

where $\bot$ is *falsum* and τ is *verum*. Note that R_{K} is a rule-schema with respect to k (cf. Remark in [9, p. 28]).

DEFINITION 3. 1. Let X be a formal system. Suppose that an axiomatic rejection ARX for X is given. Then X is said to be *Ł-complete with respect to ARX* (*Ł-decidable* in Słupecki [34] (cf. Dutkiewicz [1])) if for any formula A, exactly one of $\vdash_X A$ and $\dashv_{ARX} A$ holds, where $\dashv_{ARX} A$ means that A is a theorem of ARX, i.e. that A is eventually not a theorem of X.

We know the following theorem.

THEOREM 3. 2. (Goranko [9, 10]) K *is Ł-complete with respect to* GRK.

By THEOREM 3. 2, we may say that a relation $\dashv_{\mathrm{GRK}}$ completely well expresses the unprovability of K. So we can regard $\dashv_{\mathrm{GRK}} A$ as a synonym of $\dashv_{\mathrm{K}} A$, which means that A is not a theorem of K.

Now we are in a position to show in a syntactical way that for each $k \geq 0$, ${\tau_N}^k_{id}$ preserves unprovability. We shall show it by induction on k. The basis of induction ($k = 0$) is trivial. For the induction-step, suppose that for $i \geq 0$, ${\tau_N}^i_{id}$ preserves unprovability. Assume that $\dashv_{\mathrm{GRK}} A$ holds. Then we see that

(1) $\dashv_{\mathrm{GRK}} A$	Assumption
(2) $\dashv_{\mathrm{GRK}} \bot$	Axiom (i)
(3) $\dashv_{\mathrm{GRK}} \Box^i A$	Induction hypothesis,(1)
(4) $\vdash_{\mathrm{K}} \bot \vee \Box^i A . \supset \Box^i A$	A theorem of K
(5) $\dashv_{\mathrm{GRK}} \bot \vee \Box^i A$	Łukasiewicz's rule,(3),(4)
(6) $\dashv_{\mathrm{GRK}} \bot \vee \Box^{i+1} A \vee \Diamond \bot$	R_{K}, (2), (5)
(7) $\vdash_{\mathrm{K}} \Box^{i+1} A \supset . \bot \vee \Box^{i+1} A \vee \Diamond \bot$	A theorem of K
(8) $\dashv_{\mathrm{GRK}} \Box^{i+1} A$	Łukasiewicz's rule,(6),(7).

It is also easy to see that any $i \neq j$, $\tau_N{}^i{}_{id}$, $\tau_N{}^j{}_{id}$ are non-C-equivalent.([8]) For a proof of it, take just falsum $\bot$ as a ditcher. It is sufficient to prove that $\dashv_{\mathrm{GRK}} \Box^{k+1} \bot \supset \bot$ holds for all $k \geq 0$, since the following derived rule holds (please check it *in a syntactical way*):

$$\dashv_{\mathrm{GRK}} A \supset B \text{ implies } \dashv_{\mathrm{GRK}} \Box A \supset \Box B .$$

We shall show it by induction on k. The basis of induction is just Axiom (ii), i.e. $\Diamond \top$ ($= \Box \bot \supset \bot$). For the induction-step, suppose that $\dashv_{\mathrm{GRK}} \Box^{i+1} \bot \supset \bot$ holds for $i \geq 0$. We see that

(1) $\dashv_{\mathrm{GRK}} \Box^{i+1} \bot \supset \bot$	Assumption
(2) $\dashv_{\mathrm{GRK}} \Diamond \sim \Box^i \bot \vee \bot$	(1), definition
(3) $\dashv_{\mathrm{GRK}} \bot$	Axiom (i)
(4) $\dashv_{\mathrm{GRK}} \bot \vee \Box \bot \vee \Diamond \Diamond \sim \Box^i \bot$	R_{K}, (2), (3)
(5) $\vdash_{\mathrm{K}} \Diamond \Diamond \sim \Box^i \bot \vee \bot . \supset . \bot \vee \Box \bot \vee \Diamond \Diamond \sim \Box^i \bot$	A theorem of K
(6) $\dashv_{\mathrm{GRK}} \Diamond \Diamond \sim \Box^i \bot \vee \bot$	Łukasiewicz's rule,(4),(5)
(7) $\dashv_{\mathrm{GRK}} \Box^{i+2} \bot \supset \bot$	(6), definition.

This construction of $\tau_N{}^k{}_{id}$ is therefore an affirmative answer to Question 2, when $X = Y = \mathrm{K}$ and $\tau = \mathrm{id}$.

Summing up the above, we have the following theorems.

THEOREM 3. 3. *For any integer $k \geq 0$, $\tau_N{}^k{}_{id}$ is an embedding of* K *in* K. *For any integers $i \neq j \geq 0$, $\tau_N{}^i{}_{id}$ and $\tau_N{}^j{}_{id}$ are non-C-equivalent.*

THEOREM 3. 4. *There are an unprovability-preserving sound translation*

([8]) The equivalence here used for non-C-equivalence is, of course, classical (truth-functional).

τ from K *to* K *and a construction* $\mathbb{C}$ *such that by the construction* $\mathbb{C}$, *we can obtain, from* τ, *countably many mutually non-C-equivalent unprovability-preserving sound translations from* K *to* K.

COROLLARY 3. 5. *There are countably many mutually non-C-equivalent embeddings of* K *in* K.

As a remark, we have the following theorem.

THEOREM 3. 6. *There are countably many mutually non-C-equivalent unprovability-preserving sound translations from* K *to* K, *each of which can be used to give an affirmative answer to Question 2 with a common associated construction.*

PROOF. Take $\tau_N{}^k{}_{id}$ ($k \geq 0$) as a given unprovability-preserving sound translation with the same construction $\mathbb{C}$ as above. □

There would be many modal logics to which we can apply the above argument to obtain the same result.

In a further search for answers to Questions 1 and 2 in general, as well as model-theoretic techniques, syntactical ones, e.g. Gentzen's normal form theorem (Hauptsatz) and Kleene's permutability theorem would be useful (for the permutability theorem, in particular, see Kleene [23, 25] and Ishimoto [21; 22, p. 59-60]).

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APPENDIX

Here for the convenience of the reader, we shall recall Kleene's Gentzen-style calculus G3 for intuitionistic predicate logic IQC, which will be used as the most suitable formulation for our proof in the second section (for G3, see Kleene [24, p. 481] and for variants of G3, refer to Fitting [3] and Troelstra-Van Dalen [39]). Let Γ, Θ be finite sets of formulas of IQC with Θ empty or consisting of one formula. The system G3 consists of the following axiom-schema and rules:

Axiom schema: $C, \Gamma \rightarrow C.$

Rules of inference:

$(\vee \to)$
$$\frac{A, A \vee B, \Gamma \to \Theta \quad B, A \vee B, \Gamma \to \Theta}{A \vee B, \Gamma \to \Theta.}$$

$(\to \vee)$
$$\frac{\Gamma \to A \text{ or } \Gamma \to B}{\Gamma \to A \vee B.}$$

$(\wedge \to)$
$$\frac{A, A \wedge B, \Gamma \to \Theta \text{ or } B, A \wedge B, \Gamma \to \Theta}{A \wedge B, \Gamma \to \Theta.}$$

$(\to \wedge)$
$$\frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge B.}$$

$(\supset \to)$
$$\frac{A \supset B, \Gamma \to A \quad B, A \supset B, \Gamma \to \Theta}{\Gamma, A \supset B \to \Theta.}$$

$(\to \supset)$
$$\frac{A, \Gamma \to B}{\Gamma \to A \supset B.}$$

$(\neg \to)$
$$\frac{\neg A, \Gamma \to A}{\neg A, \Gamma \to \Theta,}$$

$(\to \neg)$
$$\frac{A, \Gamma \to}{\Gamma \to \neg A.}$$

$(\forall \to)$
$$\frac{A(t), \forall x A(x), \Gamma \to \Theta}{\forall x A(x), \Gamma \to \Theta.}$$

$(\rightarrow \forall)$
$$\frac{\Gamma \rightarrow A(b)}{\Gamma \rightarrow \forall x\, A(x).} \quad (+)$$

$(\exists \rightarrow)$
$$\frac{A(b),\ \exists x A(x),\ \Gamma \rightarrow \Theta}{\exists x A(x),\ \Gamma \rightarrow \Theta.} \quad (+)$$

$(\rightarrow \exists)$
$$\frac{\Gamma \rightarrow A(t)}{\Gamma \rightarrow \exists x\, A(x).}$$

((+): subject to the well-known restriction on variables. (see [24, p.442]).

The cut-free calculus G3 is proof-theoretically equivalent to IQC. A provable sequent $A_1, \ldots, A_n \rightarrow B$ in G3 may be regarded as a provable formula $A_1 \wedge \ldots \wedge A_n . \supset B$ in IQC. Note that G3 has no structural rules. Note also that G3 is formulated without $\perp$, both of which are the reasons why we choose G3 for our arguments in the second section.

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