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TARSKI AND THE CONCEPT OF LOGICAL CONSTANT

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This paper examines some philosophical questions about Tarski's explication of the concept of logical constant. Tarski's explication of logical constancy is presented as a precision of the condition of topic-neutrality which has been traditionally associated with logical constants. The extensional adequacy of Tarski's explication is discussed, especially with respect to first-order logic and to the distinction between set-theoretical (and hence mathematical) notions and logical notions. Lastly, I try to solve an apparent inconsequence of Tarski's statements about the problem of drawing a boundary between logical and non-logical constants.

1. *Introduction*

Tarski formulated his explication of logical consequence and logical truth in [1936]; however, he concedes that his definitions of these notions leave several open questions and he attends explicitly to one of those questions, of which he says it is "perhaps the most important" one.(1) This question consists in justifying the *boundary* between logical and non-logical (extra-logical) terms, on which the definition is based. Tarski considers that this boundary is not quite arbitrary, but he adds that he does not know any objective grounds which allow us to draw a sharp boundary between logical and non-logical constants. He admits the possibility that the inclusion of some terms which are usually regarded as non-logical among the logical ones would not lead to counterintuitive results, but unfortunately Tarski does not argue this point.(2) Tarski's concluding remarks in [1936] on this question are the following:

> Further research will doubtless greatly clarify the problem which in-

(1) Tarski [1936], p. 10; Tarski [1956a], p. 418.

(2) Tarski [1936], p. 10; Tarski [1956a], pp. 418 f.

> terests us. Perhaps it will be possible to find important objective arguments which will enable us to justify the traditional boundary between logical and extra-logical expressions. But I also consider it to be quite possible that investigations will bring no positive results in this direction, so that we shall be compelled to regard such concepts as "logical consequence" [...] [and "logical truth"] as relative concepts which must, on each occasion, be related to a definite, although in greater or less degree arbitrary, division of terms into logical and extra-logical.(3)

The establishment of a list of terms divided into logical and non-logical ones will only help to solve the problem that Tarski states if this list is not arbitrary and hence if the list at issue is based on an account of the concept of logical term or of non-logical term. But it seems that non-logical terms are very heterogeneous; the only thing they seem to share is, according to their name, that they are just *not* logical terms. For this reason a division of terms into logical and non-logical ones which is not arbitrary seems to require an explication of the concept of logical constant. Therefore we can conclude that one adequate account of logical consequence and logical truth presupposes an explication of the concept of logical constant, i.e. of the concept of logical constancy or logicality.

2. *Tarski's explication of logical constancy*

Tarski presented explicitly an explication of the concept of logical constant in a posthumously published paper, Tarski [1986].(4) His proposal for the explication of logical constancy is the following: "we call a notion 'logical' if it is invariant under all possible one-one transformations of the world onto itself",(5) where by "world" one understands here the basic universe of

(3) Tarski [1936], p. 11; Tarski [1956a], p. 420.

(4) The content of Tarski [1986] is based on two lectures given respectively in 1966 und 1973.

(5) Tarski [1986], p. 149. In Tarski [1936] und [1987] he uses the words "logical term", "logical expression" and "logical constant". In Tarski [1986] he uses only the expresion "logical notion" and says on his use of the term "notion": "I use the term 'notion' in a rather loose and general sense, to mean, roughly speaking, objects of all possible types in some hierarchy of types like that in *Principia mathematica*. Thus notions include individuals [...],

discourse, to wit, the universe of individuals.

With respect to this formulation of the explication of logical constant it is appropiate to make two remarks. Firstly, a one-one mapping (transformation) of a universe of discourse onto itself is a bijective function whose domain and range coincide with the universe of discourse; such functions are called permutations. Secondly, from the universe of individuals derivative universes of higher order are constructed, and every permutation of the universe of individuals induces a permutation of any derivate universe; this is the reason why in the explication of the concept of logical constant it is sufficient to take into account the permutations of the basic universe, the universe of individuals. Therefore Tarski's explication of logical constant amounts to the following one: logical constants are the expressions which are invariant under all permutations of the universe of discourse — where by "universe of discourse" one understands the universe of individuals.(6)

An antecedent of that account of logical constancy is found in Lindenbaum/Tarski [1936]. In this article it is said that "every relation between objects (individuals, classes, relations, etc.) which can be expressed by purely logical means is invariant with respect to every one-one mapping of the 'world' (i.e. the class of all individuals) onto itself [...]".(7)

But neither in the article of 1986 nor in the one of 1936 is something said about the relation between the explication of the concept of logical constant and features traditionally associated with the logical constants, especially their topic-neutrality. Very little is said about the adequacy of that explication of logicality with respect to the logical constants of classical first-order logic, which seems to be the minimum condition required of the extensional

classes of individuals, relations of individuals, and so on"(Tarski [1986], p. 147). In Tarski/Givant [1987] it is spoken about logical objects and about logical constants, where a logical constant is a symbol which denotes a logical object. So logical notions in Tarski [1986] are logical objects and not logical constants, i.e. they are not expressions, but objects denoted by expressions. In this paper I will use indifferently the terms "logical constant" and "logical notion" to refer both to logical objects and to logical constants - in the usual sense of logical terms or logical expressions. This creates no ambiguity because the context makes clear in which sense the terms "logical constant" and "logical notion" are being used. I have taken this decision because on Tarski's explication of logicality the term "logical" applies primarily to objects and only derivative to expressions — namely an expression is logical if it denotes a logical object —, but in the actual discussion about logicality is more natural to apply the term "logical" to expressions.

(6) Cf. Tarski/Givant [1987], p. 57.

(7) Lindenbaum/Tarski [1936], p. 206; Lindenbaum/Tarski [1956], p. 385.

adequacy of an explication of logical constancy. Before I approach these questions it is, however, proper to make one remark.

To the question what is a logical constant there is in principle an obvious answer: a logical constant is an expression which receives a *constant* interpretation in the interpretation of formal languages. This answer is undoubtedly true, but it is also to a great extent *trivial*. When a first-order language is interpreted in a structure neither the negation nor the conjunction nor the universal quantifier, etc. are interpreted in the structure, but the interpretation of these expressions is considered fixed in advance and is independent of every structure. However, it is so simply because it is in advance stipulated that those expressions are just the logical constants of the language. In short, I do not think that this answer can be regarded as satisfactory by someone who questions the distinction between logical and non-logical constants. For this reason an ulterior justification why some expressions are to be regarded as logical constants, seems to be required.

3. *Topic-neutrality*

In the search of a possible justification one can take as starting point one of the features which accostums to be regarded as central to the notion of logical constant. This feature can be presented by mentioning a traditional distinction like that between the form and the subject matter of an argument. The logical evaluation of an argument considers exclusively its form and is independent of its subject matter, i.e. of the specific content of the sentences which constitute the argument. The form of the argument which interests the logic, to wit, its logical form is expressed precisely by logical constants. So logical constants will not involve particular individuals, particular properties of individuals, etc. which will concern the subject matter of the sentences. This feature of the logical constants is usually called *topic-neutrality*. From an explication of logical constancy one should expect a precision of this feature of the logical constants. In fact, Tarski's explication can be just interpreted as a precision of the topic-neutrality which seems to be characteristic of logical constants.

To this end it is sufficient to indicate that a way of determinating if a term is topic-neutral and hence a logical constant is to introduce modifications in the individuals and in the features of the individuals of the universe of discourse and to attend to whether what is expressed by the term remains in spite of these modifications invariant. A procedure which permits to introduce such modifications is to carry out permutations of the universe of discourse. Hence Tarski's proposal: logical constants are the expressions which are invariant under every permutation of the universe of discourse. This feature guarantees that the notion at issue is independent of the specific content of the structures, i.e. of the specific individuals of the universe of discourse, of the specific properties of those individuals, of the specific relations between the individuals, etc. Through a permutation of the universe of discourse some of its individuals can be transformed into different individuals, from which it follows that at the level of the individual constants there are not any logical constants.

4. *Tarski's explication and first-order logic*

An adequate, however partial, criterion to determinate the set of the logical constants is the following one: a constant which is definable *exclusively* on the basis of constants which are already accepted as logical ones is a logical constant.(8) But we know that one can regard as primitive logical constants of first-order logic(9) the negation, the conjunction and the universal quantifier, because the other logical constants of a first-order language are definable by the use of these constants only. In this way the examination of the explications of logical constancy with respect to first-order logic is simplified: to decide if an explication of the concept of logical constant is adequate with respect to first-order logic, we can confine our examination to the application of that account to the negation, the conjunction and the universal quantifier. These three constants are the basis of the adequacy condition of the explication sought. Let us now examine if Tarski's explication is adequate with respect to first-order logic. I examine if the negation, the conjunction and the universal quantifier satisfy Tarski's account.

(8) Cf. Carnap [1942], p. 58.

(9) I understand here by "first-order logic" classical first-order logic without identity. On the other hand, it is rather obvious that the identity is according to Tarski's explication a logical constant.

The reason is that a permutation is a bijection and a bijective function is injective (one-to-one), i.e. a function which assigns distinct elements of the range to distinct elements of the domain; therefore identical elements of the range will always correspond to identical elements of the domain. So the identity relation (and the same holds for the diversity relation) is invariant under all permutations of the universe of discourse and is hence a logical constant.

The negation and the conjunction can be regarded as functions whose domain and range consists of the truth-values "true" and "false". The problem which arises here is how to unify the domain of the truth-values with the universe of discourse characteristic for first-order logic, i.e. with a domain of individuals. A possibility would be to regard the truth-values as individuals, although as peculiar individuals. But in this case one comes to a certainly counterintuitive result, namely that every individual of the universe of discourse could be regarded as a truth-value, because it could occur that in different permutations of the universe of discourse the two truth-values are transformed into different individuals.

An alternative which allows to justify Tarski's explication consists in regarding the truth-values not as individuals, i.e. as first-order objects, but as classes or sets of individuals, to wit, as second-order objects, and, in particular, the truth-value "true" as the universe of discourse and the truth-value "false" as the empty set.(10)

This interpretation of the truth-values implies that the truth-values must be regarded as *logical constants*, since the universe of discourse and the empty set are invariant under all permutations of the universe of discourse. A permutation of an universe of discourse results in the same universe of discourse; on the other hand, a permutation of the empty set results in the empty set, and this can not be in another way, as the empty set has no elements. It is the invariance of the universe of discourse and of the empty set under all permutations of the universe of discourse that guarantees us that the negation and the conjunction as functions defined on the truth-values so interpreted will have that invariance too, since the arguments and values of these functions will be invariant.

What can be said about the universal quantifier? A quantifier is, roughly speaking, an operator which binds variables. But from another point of view a quantifier can be regarded as a predicate of predicates. The idea which justifies this conception can be illustred with the universal quantifier. Let "x" be an individual variable and "P" a predicate constant, then the quantification "∀x Px" says that the predicate "P" applies to every object of the universe of discourse, i.e. that every individual of the universe of discourse belongs to the set of the objects which are P, and the quantifications "∀x Qx", "∀x Rx", etc. are correspondently interpreted. The universal quantifier

(10) Tarski seems to have hinted at this possibility; see Tarski [1986], p. 150, footnote 6. However, one must concede that this identification is rather a formal trick and lacks any independent justification (see Simons [1987], p. 389).

can be regarded hence as a set of sets, i.e. as the set of the sets whose extension coincides with the universe of discourse. However the extensionality axiom of set theory says that two sets which contain the same elements are identical, i.e. are the same set. Therefore we reach the conclusion that the universal quantifier can be regarded as the set whose only element is the universe of discourse. And from our remarks on the invariance of the universe of discourse under all permutations it follows that the universal quantifier also has this invariance. So one can conclude that the explication of the notion of logical constant as invariance under all permutations of the universe of discourse is extensionally adequate with respect to first-order logic.

5. *Logical and set-theoretical constants*

However, even if the result obtained is the desired one with respect to first-order logic, it can be objected that in order to obtain it we have admitted as logical constants set-theoretical notions; in fact, we have seen that from Tarski's explication of logical constancy it follows that at least some set-theoretical notions are logical constants. This consequence will be in the opinion of many of us unacceptable, since from the extensional adequacy of an explication of logical constancy it must be not only required that it certifies the logical character of the constants which are usually regarded as logical ones, but also that it excludes notions which are not considered in a strict sense as logical constants, but rather as mathematical notions, for instance, set-theoretical constants.

Tarski regarded the problem of drawing a justified boundary between logical and non-logical constants as important (see text of footnote 1), because his definitions of logical consequence and logical truth are based on the division of the constants of the object-language into logical and non-logical. But he apparently did not regard the problem of drawing a boundary between logical and set-theoretical (and hence mathematical) notions as important; so in his letter to Morthon White of 23 September 1944 Tarski says:

> [...] we can consider even the possibility of several non-equivalent definitions of "logical terms" [...]; e.g., sometimes it seems to me convenient to include mathematical terms, like the $\in$-relation, in the class of logical ones, and sometimes I prefer to restrict myself to

terms of "elementary logic". Is any problem involved here?([11])

It is indeed rather disappointing that Tarski leaves this question open, as though it would be only a matter of convenience whether set-theoretical terms and hence mathematical terms are regarded as logical ones. I think he does not see here any problem because he would have considered both a negative and an affirmative answer to the question whether set-theoretical terms are logical terms as legitimate according to whether one uses the expressions "logic" and "logical" in a broad or in a narrow sense; Tarski apparently finds sometimes the first use convenient, sometimes the second one. Concerning this point it is proper to pay attention to a text of Tarski [1944] (published in March 1944), where he says:

> The terms "logic" and "logical" are used in this paper in a broad sense, which has become almost traditional in the last decades; logic is assumed here to comprehend the whole theory of classes and relations (i.e., the mathematical theory of sets). For many different reasons I am personally inclined to use the term "logic" in a much narrower sense, so as to apply it only to what is sometimes called "elementary logic", i.e. to the sentential calculus and the (restricted) predicate calculus.([12])

On this text one should firstly remark that Tarski's inclination to distinguish between set theory and logic (or logic in a narrow sense) has nowadays become usual. But this text can support the interpretation that it was Tarski's frequent use (especially conspicuous in his papers on semantics) of the expression "logic" in a wide sense including set theory, which has probably led him not to concede importance to the problem of drawing a boundary between logic (in a narrow sense) and set theory or rather to admit as legitimate both a negative and an affirmative answer to the question whether set-theoretical and hence mathematical constants are logical. The affirmative answer is legitimate if one uses the term "logic" in a broad sense, so that logic includes set theory, but in accordance with Tarski's use of "logic" in a narrow sense he should have given, of course, a negative answer to this question.

([11]) Tarski [1987], p. 29.

([12]) Tarski [1944], p. 371, footnote 12.

In Tarski [1986] he also leaves the question open whether set-theoretical and hence mathematical constants are logical constants. In this paper Tarski, at first, formulates his remarks, as I have done till now, in the set theory developed in the framework of the simple theory of types;(13) from that it follows that set-theoretical notions are logical notions.

However, in the last section of Tarski [1986] he formulates explicitly the question whether mathematical notions are logical; Tarski reduces this problem to the question of whether set-theoretic notions are logical notions, and Tarski's final formulation of the question is whether the membership relation is a logical notion. He says that on the basis of his explication of logicality one can answer this question affirmatively or negatively according to the method used in developing set theory. On the one hand, if one develops set theory according to the theory of types, the membership relation is a logical notion and the same holds for the other set-theoretical notions. But, on the other hand, Tarski maintains that one comes to a different result if set theory is developed in a framework "where we have no hierarchy of types, but only one universe of discourse [...] and individuals and sets are considered as belonging to the same universe of discourse".(14) In this framework the membership relation and, more generally, set-theoretic (and hence mathematical) notions are not logical notions, i.e. invariant under all permutations of the universe of discourse. Tarski concludes his paper pointing out explicitly that his explication of logical constancy "does not, by itself, imply any answer to the question of whether mathematical notions are logical".(15) On this result I will make two remarks.

Firstly, it is interesting to notice that it agrees with Tarski's distinction between a broad and a narrow sense of the term "logic", since if one admits these two senses of the word, one should look for an explication of logicality which permits *both* possibilities and so both an affirmative and a negative answer to the question of whether set-theoretical (and hence mathematical) notions are logical notions. Secondly, the answer to this question will depend according to Tarski on the method assumed to construct set theory. However, I think this is not right. Indeed, according to the last mentioned framework for set theory there are many set-theoretical notions,

(13) Cf. text cited in footnote 5 about Tarski's use of the term "notion" and Tarski/Givant [1987], p. 57.

(14) Tarski [1986], p. 153.

(15) *Ibid.*

especially the membership relation, that are no longer logical ones. But even according to that conception of set theory there are still some set-theoretical concepts such as the universe of discourse and the cardinality of the universe of discourse that are invariant under every permutation of the universe of discourse and hence according to Tarski's explication are logical constants. So one must conclude that according to Tarski's account at least *some* set-theoretical notions are logical ones.

But if one agrees that set-theoretical notions are not logical notions — or not logical notions in a narrow sense of "logic" and "logical", which Tarski was "personally inclined to use" — , then one must conclude that, although Tarski considered invariance under all permutations of the universe of discourse as a sufficient condition or even as a necessary and sufficient condition of logicality,([16]) it seems to be more reasonable to consider it only as a *necessary* condition.

Tarski's explication of logical constancy is hence not quite satisfactory. The problem it presents us with, is how to strengthen the concept of logicality as invariance under all permutations of the universe of discourse or alternativaly, how to complement that explication with another account of logicality so that the proper logical notions are specified and the set-theoretical notions are excluded.

6. *An inconsequence?*

We have observed that Tarski regarded as admissible both an affirmative and a negative answer to the question of whether set-theoretical and hence mathematical notions are logical notions. But I do not think that he could have maintained this opinion consequently in Tarski [1936].

This paper is the summary of a lecture delivered by Tarski in September 1935. As already noted, Tarski affirmed there that he does not know any

([16]) In the formulation of the explication of logical constancy in Tarski [1986] the conditional is formulated only in one direction: *if* a notion is invariant under all permutations of the universe of discourse, then it is a logical constant (see text corresponding to footnote 5 *infra*). The same occurs in Tarski/Givant [1987], p. 57. For that reason one could maintain that Tarski considered that his explication provides only sufficient conditions of logicality. However, several remarks in Tarski [1986], for instance, the introductory remarks in which Tarski says he proposes to present a definition of the term "logical notion" are to be interpreted rather in the sense that Tarski regarded his explication as providing necessary and sufficient conditions of logicality and not only sufficient conditions.

objective grounds which permit us to draw a boundary between logical and non-logical constants and he expressed doubts about the possibility of finding such arguments. On the other hand, Lindenbaum/Tarski [1936] is the text of a lecture delivered by Tarski in June 1935 and, as we have seen, in this paper it is maintained that every relation between individuals, classes, relations, etc. which can be formulated with logical expressions, is invariant under all permutations of the universe of discourse. But this statement suggests immediately the definition of logical constants as the expressions invariant under all permutations of the universe of discourse and it provides a criterion to distinguish between logical and non-logical constants — if one assumes that set-theoretical constants are logical ones.

Therefore one could argue that if Tarski regarded in Tarski [1936] as admissible to maintain that set-theoretical constants are logical constants, then he was inconsequent in expressing doubts about the possibility of finding objective arguments to draw a boundary between logical and non-logical constants because some *months* before he had already formulated a criterion which enables this boundary to be drawn.

The only way I see to resolve this apparent inconsequence is to assume that Tarski in [1936] pressuposes that set-theoretical constants are not logical ones, so that the criterion of logicality sketched in Lindenbaum/Tarski [1936] cannot constitute by itself the criterion to draw the distinction between logical and non-logical constants that Tarski was looking for and that he never found.

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