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ON BCC-ALGEBRAS

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1. *Introduction*

As is shown by the title of the first paper on this subject [3], much of the work on BCK and related algebras was motivated by known work on implicational logic. This is illustrated by the similarities between the names of some of the systems. We have BCI-algebra and BCI positive logic, BCK-algebra and BCK positive logic, Positive implicative BCK-algebra and Positive implicative logic and Implicative BCK-algebra and Implicative (classical) logic.

BCK-algebras form a quasivariety of algebras amongst whose subclasses can be found the earlier implicational models of Henkin [2], algebras of sets closed under set-substraction, and dual relatively pseudocomplemented upper semilattices. K. Iséki posed an interesting problem (solved by A. Wronski in [8]) whether the class of BCK-algebras is a variety. In connection with this problem Y. Komori introduced in [6] a notion of BCC-algebras and proved (using some Gentzen-type system LC) that the class of all BCC-algebras is not a variety, but the variety generated by BCC-algebras, that is, the smallest variety containing the class of all BCC-algebras, is finitely based.

In this paper we consider the connections between BCC-algebras and BCK-algebras. We consider also those binary relations on a set which trivially yield the structure of a BCC-algebra.

2. *BCK-algebras*

The algebras we consider are those based on a set G containing a constant 0, an operation $*$ or $\cdot$, such an algebra is an algebra of type (2,0) and is denoted by $(G, *, 0)$ or respectively by $(G, \cdot, 0)$. Each such algebra will have certain equality axioms (including $x = x$) and the rule of substitution of equality as well as perhaps some other rules. The logic are those that have implication $\supset$ as their only primitive connective and modus ponens as a rule. There may be axioms and (or) other rules.

A BCK-logic is an implicational logic based on modus ponens and the following axiom system.

B $A \supset B \,.\supset .\, (C \supset A) \supset (C \supset B)$
C $A \supset (B \supset C) \,.\supset .\, B \supset (A \supset C)$
K $A \supset (B \supset A)$.

By a BCK-algebra we mean a general algebra $(G, *, 0)$ of type (2,0) satisfying the following axioms:

(1) $((z*x)*(z*y))*(y*x) = 0$,
(2) $(x*(x*y))*y = 0$,
(3) $x*x = 0$,
(4) $0*x = 0$,
(5) $x*y = y*x = 0$ implies $x = y$.

M.W. Bunder proved in [1] that BCK-algebra and BCK-logic are isomorphic, i.e. theorems of one can be mapped into theorems of the other.
As it is well known (see for example [5]) every BCK-algebra satisfies

(6) $x*0 = x$.

Indeed, putting $x \leq y$ iff $x*y = 0$, we obtain $x \leq x*0$, since $x*(x*0) \leq 0$ (by (2)) and $0 \leq x*(x*0)$ by (4). Hence $x*(x*0) = 0$. Similarly (2) and (3) gives $x*0 \leq x$. Therefore, $x \leq x*0 \leq x$, which implies (6).
On the other hand, putting $x = 0$ in (1) and using (6) we obtain (2). Putting in (2) $y = 0$ we get (3). Thus the following theorem is true.

THEOREM 1. The class of all BCK-algebras is defined by (1), (4), (5) and (6). These axioms are independent.
Proof. To prove that these axioms are independent, we consider three algebras defined as follows.

*	0	1	2
0	0	0	0
1	2	0	0
2	2	2	0

Table 1

*	0	1	2
0	0	0	0
1	1	0	0
2	2	0	0

Table 2

*	0	1	2
0	0	0	0
1	1	0	2
2	2	2	0

Table 3

The algebra defined by Table 1 satisfies (1), (4) and (5). Indeed, if $z = x$, then

$((x*x)*(x*y))*(y*x) = (0*(x*y))*(y*x) = 0*(y*x) = 0.$

If $z = y$, then

$((y*x)*(y*y))*(y*x) = ((y*x)*0)*(y*x) = 0,$

because $y*x = 0$ or $y*x = 2$.

The case $x = y$ is obvious. If x, y, z are different, then $x = 0$ or $y = 0$ or $z = 0$, and direct computations show that in these cases (1) holds, too. Conditions (4) and (5) are obvious. But (6) is not true, since $1*0 \neq 1$.

In the same manner we can prove that the algebra defined by Table 2 satisfies (1), (4) and (6), but (5) is not satisfied.

Since in every Boolean group the conditions (1), (5) and (6) hold, but (4) holds only in one-element group, then (4) is independent.

Finally we remark that the algebra defined by Table 3 satisfies (4), (5) and (6), but $((1*2)*(1*0))*(0*2) = 2$, i.e. the axiom (1) is independent. This completes our proof.

As it is well known in BCK-algebras the equality

(7) $(x*y)*z = (x*z)*y$

holds for all elements x, y, z.

We give a simple proof of this identity. The original proof is given in [4]. To prove (7) observe first that from (1), (4), (5) and (6) follows that the relation $\leq$ is a partial order with 0 as a smallest element. Moreover,

(8) if $x \leq y$, then $x*z \leq y*z$ and $z*y \leq z*x$.

Since $x*(x*z) \leq z$ by (2), then (8) implies

$(x*y)*z \leq (x*y)*(x*(x*z)).$

This yields

$((x*y)*z)*((x*z)*y) \leq ((x*y)*(x*(x*z)))*((x*z)*y) = 0$ by (8) and (1). Hence $((x*y)*z)*((x*z)*y) \leq 0$, which by (4) and (5) completes the proof.

3. *BCC-algebras*

By a BCC-algebra we mean a non-empty set G together with a binary multiplication denoted by juxtaposition and a some distinguished element 0 such that the following axioms are satisfied for all $x, y, z \in G$:

(9) $(yz)((xy)(xz)) = 0$,
(10) $xx = 0$,
(11) $x0 = 0$,
(12) $0x = x$,
(13) $xy = yx = 0$ implies $x = y$.

THEOREM 2. The class of all BCC-algebras is defined by the independent axioms system: (9), (11), (12), (13).
Proof. Putting in (9) $x = y = 0$ we obtain $zz = 0$ which shows that (10) follows from (9) and (12).
The algebra defined by Table 4

	0	1	2
0	0	1	2
1	0	0	2
2	0	2	0

Table 4

	0	1	2
0	0	2	2
1	0	0	2
2	0	0	0

Table 5

	0	1	2
0	0	1	2
1	0	0	0
2	0	0	0

Table 6

satisfies (11), (12) and (13), but (9) is not satisfied since $(01)((20)(21)) = 2$. Every Boolean group satisfies (9), (12) and (13). Obviously, in these groups (11) is not satisfied. In the same manner as in the proof of Theorem 1 we can prove that (9) holds in the algebra defined by Table 5. In this algebra (11) and (13) hold, too. But (12) is not true. Similarly in the algebra defined by Table 6 the conditions (9), (11) and (12) are satisfied. Obviously (13) is not satisfied, which completes our proof.

Y. Komori noticed in [5] and [6] that if in the axiom system of BCC-algebras we replace (9) by $(xy)((yz)(xz)) = 0$, then we obtain the axiom system of BCK-algebras (but in the dual form).

Moreover, if $(G, *, 0)$ is a BCK-algebra and an algebra $(G, \cdot, 0)$ of type (2,0) is dual to $(G, *, 0)$, i.e. if $xy = y*x$, then by (7) and (1) we get

$$(yz)((xy)(xz)) = ((z*x)*(y*x))*(z*y) = ((z*x)*(z*y))*(y*x) = 0,$$

which proves the following theorem.

THEOREM 3. If $(G, *, 0)$ is a BCK-algebra, then its dual algebra $(G, \cdot, 0)$, where $xy = y*x$, is a BCC-algebra.

Note that the converse is not true. As an example we consider algebras defined by Table 7 and Table 8.

	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	1
3	0	0	1	0

Table 7

	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	1	0

Table 8

First we prove that the algebra given by Table 7 is a BCC-algebra. It is clear that in this algebra the conditions (10), (11), (12) and (13) are satisfied. Obviously (9) holds for $x = 0$ or $z = 0$. Similarly (9) is satisfied if $x = y$ or $x = z$ or $y = z$. For $y = 0$ (9) has the form $z(xz) = 0$. Direct computations show that this identity holds in the algebra defined by Table 7. Also it is easily verified that (9) holds if x, y, z are different. Hence this algebra is a BCC-algebra. But in the dual algebra we have $(3*1)*2 = 2(13) = 1$ and $(3*2)*1 = 1(23) = 0$, i.e. the identity (7) is not true. Therefore the dual algebra is not a BCK-algebra.

In the same manner we can prove that the algebra defined by Table 8 is a BCC-algebra, but in this algebra we have $1(32) \neq 3(12)$, which shows that in its dual algebra the condition (7) is not true.

In a similar way as Theorem 3 we can prove the following theorem.

THEOREM 4. A BCC-algebra $(G, \cdot, 0)$ is dual to some BCK-algebra iff it satisfies the identity $z(yx) = y(zx)$.

As an immediate consequence we obtain

COROLLARY. If a BCC-algebra has at most three elements, then it is dual to some BCK-algebra.

Proof. If a BCC-algebra has at most three elements, then in the word $z(yx)$ at least one element is 0 or at least two elements are equal. If $y = z$ or at least one from elements x, y, z is 0, then the identity $z(yx) = y(zx)$ is satisfied. If $x = y$ or $x = z$, then this identity is equivalent to the identity $a(ba) = 0$, which immediately follows from (9). Hence such BCC-algebra satisfies the condition $z(yx) = y(zx)$.

4. *Partial orders and BCC-structures*

Let $(G, \cdot, 0)$ be a BCC-algebra. We define a binary relation of G as follows

(14) $x \leq y$ iff $xy = 0$.

Direct computations show that this relation is a partial order on G with 0 as a largest element.

Now we give some fundamental properties of this relation.

THEOREM 5. The relation $\leq$ defined on a BCC-algebra by the formula (14) satisfies the following conditions:

(*i*) $x \leq y$ implies $zx \leq zy$ and $yz \leq xz$,
(*ii*) $x(yz) \leq x((uy)(uz))$,
(*iii*) $((xy)(xz))u \leq (yz)u$,
(*iv*) $(xy)z \leq yz$,
(*v*) $(xy)(xz) \leq y(xz)$,
(*vi*) $x \leq y$ implies $x \leq zy$,
(*vii*) $x \leq zx$,
(*viii*) $yz \leq x(yz)$,
(*ix*) $(xy)z \leq x(yz)$,
(*x*) $(xy)z \leq y(uz)$.

Proof. The first condition follows from (9) and (12), in particular, if $x \leq y$, then $yz \leq (xy)(xz) = 0(xz) = xz$. Conditions (*ii*) and (*iii*) follow from (9) and (*i*). From (*iii*) and (12) we obtain $(xy)z = (0(xy))z = ((x0)(xy))z \leq$

$(0y)z = yz$, which proves (*iv*). Putting $z = xz$ in (*iv*) we get (*v*). If $x \leq y$, then $zx \leq zy$ and $0 = (zx)(zy) \leq x(zy)$ by (*v*). But 0 is a largest element, then $0 \leq x(zy)$ implies $x(zy) = 0$, i.e. $x \leq zy$, which proves (*vi*). As a simple consequence of (*vi*) we obtain (*vii*) and (*viii*). From (*iv*) and (*viii*) follows (*ix*). Since $y \leq xy \leq u(xy)$, then $(u(xy))(uz) \leq y(uz)$ by (*i*). But $(xy)z \leq (u(xy))(uz)$ by (9). Hence $(xy)z \leq y(uz)$, which completes our proof.

As it is well known if (G, S) is a partially ordered set with a smallest element 0, then G can be made into a BCK-algebra by

$$x^*y = \begin{cases} 0 & \text{if } xSy \\ x & \text{otherwise.} \end{cases}$$

This is called the trivial structure on $(G, ^*)$. The partial order S on G considered as a BCK-structure coincides with the original partial order defined by the BCK-operation.

The similar result holds for BCC-algebras. Indeed, let G be a set with a distinguished element 0 and let R be a binary relation on G. In G we define a multiplication as follows

$$xy = \begin{cases} 0 & \text{if } xRy \\ y & \text{otherwise.} \end{cases}$$

THEOREM 6. Let R be a partial order on G. If there exists $0 \in G$ such that $xR0$ for all $x \in G$, then $(G, \cdot, 0)$ is a BCC-algebra. Moreover R coincides with the original partial order defined by the BCC-operation.

Proof. We prove only (9), since the other axioms are obvious. First we consider the case xRy. If also $0Rxz$, then $(yz)((xy)(xz)) = (yz)(0(xz)) = (yz)0 = 0$. If not$(0Rxz)$, then $(yz)((xy)(xz)) = (yz)(xz)$. For yRz the transitivity of R and xRy yields xRz. Thus $(yz)(xz) = 0$. If not(yRz), then $(yz)(xz) = z(xz)$, which gives 0, without regard to whether xRz or not(xRz).

In the case not(xRy) for xRz we obtain $(yz)((xy)(xz)) = (yz)(y(xz)) = (yz)(y0) = (yz)0 = 0$. If not$(xRz)$, then $(yz)((xy)(xz)) = (yz)(y(xz)) = (yz)(yz) = 0$, which completes the proof.

In the same manner we can prove that a BCC-algebra with the trivial structure satisfies the condition $y(zx) = z(yx)$. Hence as an immediate consequence of Theorem 4 we obtain

THEOREM 7. A BCC-algebra has a trivial structure iff it is dual to a BCK-algebra with a trivial structure.

In some cases on G there exists a partial order without the maximum condition (i.e. a partial order R such that not($xR0$) for some $x \in G$) which defines on G the BCC-structure. Obviously this order does not coincide with the natural partial order defined by the BCC-structure. For example, let $G = \{0, a\}$ and let $0R0$, aRa, not($aR0$) and not($0Ra$). Then $(G, \cdot, 0)$ is the unique two-elements BCC-algebra, but R does not coincide with the BCC-order $\leq$ since $a \leq 0$.
Now we give two methods of construction of BCC-algebras.

LEMMA 1. Let $(G, \cdot, 0)$ be a BCC-algebra and let $a \notin G$. Then the set $G \cup \{a\}$ with the operation $\Box$ defined as follows

$$x \Box y = \begin{cases} xy \text{ if } x, y \in G, \\ 0 \ \text{ if } x\ G \setminus \{0\}, y = a, \\ a \ \text{ if } x = 0, y = a, \\ 0 \ \text{ if } x = a, y = a, \\ y \ \text{ if } x = a, y \in G, \end{cases}$$

is a BCC-algebra.
Proof. We verify only the axiom (9). For $x, y, z \in G$ this axiom is satisfied. If $x = a$, then

$$(y \Box z) \Box ((a \Box y) \Box (a \Box z)) = \begin{cases} 0 & \text{if } z = a \\ (y \Box z) \Box ((a \Box y) \Box z) & \text{if } z \neq a \end{cases}$$

In the second case for $y \neq a$ we have $(y \Box z) \Box ((a \Box y) \Box z) = (yz)(yz) = 0$. For $y = a$ we get $(y \Box z) \Box ((a \Box y) \Box z) = z(0z) = 0$ since $z \neq a$. Hence (9) holds for $x = a$.

Now let $y = a$. If $x = 0$, then $(a \Box z) \Box ((x \Box a) \Box (x \Box z)) = (a \Box z) \Box (a \Box z) = 0$. For $x \neq 0$ we have $(a \Box z) \Box ((x \Box a) \Box (x \Box z)) = (a \Box z) \Box (x \Box z) = 0$.
The case $z = a$ is obvious.

LEMMA 2. If $\{(A_\xi, 0_\xi, \cdot_\xi)\}_{\xi \in \Lambda}$ is a non-empty family of BCC-algebras, where Λ is a totally ordered set with initial element α, then the set $\bigcup_{\xi \in \Lambda} A_\xi$, with the operation defined by the formula

$$xy = \begin{cases} x\cdot_\alpha y \text{ if } x, y \in A_\alpha, \\ x\cdot_\xi y \text{ if } x, y \in A_\xi, x\cdot_\xi y \neq 0_\xi, \alpha < \xi, \\ 0_\alpha \text{ if } x, y \in A_\xi, x\cdot_\xi y = 0_\xi, \alpha < \xi, \\ 0_\alpha \text{ if } x \in A_\eta, y \in A_\xi, \xi < \eta, \\ y \text{ if } x \in A_\xi, y \in A_\eta, \xi < \eta, \end{cases}$$

is a BCC-algebra.

The natural order induced by this operation coincides with the original order of each A_ξ. Obviously $x \leq y$ for all $x \in A_\eta$ and $y \in A_\xi$, where $\xi < \eta$. If $\Lambda = \{\alpha, \beta\}$, $\alpha < \beta$, $G_\alpha = G$, $G_\beta = \{0\}$ then this construction is not a generalization of a construction from Lemma 1.

Using the above construction we can prove the following theorem.

THEOREM 8. For any cardinal $n \geq 4$ there exists a BCC-algebra which has n elements and which is not dual to any BCK-algebra.

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