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MEREOLOGY AS A THEORY OF PART-WHOLE

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1. *Introduction*

Mereology is now a widely known general name for various theories concerning part-whole relationships (see Simons [13]). The notions of part and whole are highly placed among philosophical concepts and they have been regarded as an important area of philosophical investigation from the time of Aristotle, who gave us the first systematic attempt to explore and employ these notions. Despite much attention given to this area since then, the first formal theory of the part-whole relation, called mereology, was developed only at the beginning of our century by the Polish logician and philosopher Stanislaw Leśniewski. Unfortunately (we will explain this assessment below), Leśniewski and many others after him considered mereology as an alternative to set theory, an alternative which is more natural, less abstract and, in addition, free from paradoxes. Collective classes in Leśniewski's mereology were considered as really consisting of its elements as parts, just as constellations consist of stars, and the United States - of its states. Later, a similar theory, the calculus of individuals (see Leonard & Goodman [12]), was taken as a formal basis of a general nominalistic program (cf.Goodman [9]) and, in particular, as a basis for a nominalistic reinterpretation of the language of mathematics (Goodman & Quine [10]). In both these theories the aim was essentially the same: to retain most of the functions and capacities of 'standard' set theory, while preserving the less abstract character of their basic objects. However, as is almost clear now, these theories have failed to achieve both these goals. On the one hand, they are relatively weak when compared with set theory (for example, even natural numbers cannot be reconstructed inside them, cf. Slupecki [14]). On the other hand, the unrestricted principle of forming collective classes in Leśniewski's mereology (which plays essentially the same role in mereology as the comprehension axiom in the standard set theory) and a similar postulate of forming fusions in the calculus of individuals have given rise to grave doubts about the non-abstract character of their objects and thereby about their use as nominalistic theories. If we may form concrete (non-abstract) objects from arbitrary ingredients, however diverse, it seems that our understanding of what is real will be seriously distorted relative to its natural (however vague) meaning. The same argument implies also that both these theories cannot serve as general theories of part-whole proper, because in order to be such theories, they must somehow distinguish between 'wholes' and 'non-wholes'. But if we are allowed to form 'wholes' in an almost arbitrary way, there are no means to distinguish between them.

All this suggests that in order to obtain a viable theory of part-whole we must reconsider this predestination of mereology as an alternative to set theory. As a more specific implication of this thesis we must give up the crucial 'set-theoretical' axiom of mereology. As it will be seen in what follows, in the weaker theory, which will be proposed here, it is in fact possible to achieve more with respect to the two goals, mentioned above. But first we will briefly consider two other theories, which are closely connected with the subject of our paper.

Contemporaneously with Leśniewski, Alfred North Whitehead was developing a philosophical theory, which used some means similar to that of mereology (see Whitehead [15,16,17]). One of his aims was to build a theory of space and time which would not be based on the notion of point (resp., instant) as a primitive. In order to define points in terms of extended regions, Whitehead proposed his 'method of extensive abstraction', according to which points are defined roughly as chains of infinitely converging regions, ordered by the relation of being part. But although Whitehead rejected (on philosophical grounds) the ordinary set-theoretical approach to the analysis of space and time, he didn't consider his theory to be a general alternative to set theory. Moreover, objects of his theory were supposed to be continuous and consequently the theory didn't contain principles, corresponding to the mereological 'comprehension axiom'. Nonetheless (in fact, owing to that) Whitehead's theory allowed one to describe more subtle structural features than was possible in standard mereological theories, discussed above.

It is clear that the notion of a point is crucial for the standard set-theoretical interpretation of the continuum. Having points we may (under some conditions) identify extended regions with sets of their points and define the relation of being part among them via simple set-theoretical inclusion. Here arises a general question about the nature of points and their function in the structure of the continuum. This question becomes especially important if we reject the above mentioned reduction of extended regions to points. Surprisingly enough, one of the most plausible answers may be found in

Aristotle's account of magnitudes, which we are now going to describe.

According to Aristotle, continuous magnitudes are always bounded. Boundaries are external limits (extremities) of magnitudes, and in the case of lines, just points. In addition, continuous magnitudes always have parts, which are also magnitudes. These parts also have boundaries which are at the same time internal divisions of the magnitudes, containing these parts. We may conclude that boundaries and magnitudes are entities of different kinds and, moreover, that boundaries have a derived being in the Aristotle's theory: they cannot exist independently, and presuppose those objects of which they are boundaries (cf.Phys. 6.3, 234a15-16; GC 1.2, 316b6-8; Meta 11.2, 1060b12ff). Boundaries are not parts of magnitudes and since something is made up of some components if and only if it can be divided into them, we have that magnitudes cannot be composed of boundaries (cf.GC 1.2,316a24ff). Thus, in particular, a line cannot consist of points. Note that within such an understanding of the notion of point it is a perfectly natural and grounded assertion. Aristotle also gives us the following definition of continuity and discreteness: the continuous is that of which any two adjacent parts have the same common boundary, while discrete is that of which two adjacent parts have different boundaries (cf. Phys. 5.3, 227a10-13; Meta 11.12,1069a5ff). Thus, the role of internal boundaries in continuous magnitudes is twofold. On the one hand they are divisions of the magnitude, and on the other hand they join (connect) parts of this magnitude (cf. Phys. 4.11, 220a10ff; 4.13, 222a10-20). In other words, internal boundaries are necessary ingredients in forming continuous magnitudes from its parts. They connect magnitudes and thereby participate in forming from them an integral continuous whole.

As will be clear, the theory proposed below has much in common with Aristotle's views described above and hence could be regarded in some respects as their restoration(1). However, the proposed theory is intended to cover not only continuous structures, but discrete 'wholes' as well.

(1) It must be noted that in order to give a full description of Aristotle's theory of magnitudes we need some additional components, the most important being Aristotle's theory of potentiality.

2. *Objects and their structure*

Our universe of discourse will comprise entities of two kinds — objects and connections, which will be called mereological elements(2). Objects are assumed to be integral entities, having no disjoint parts, and in order to preserve this integrality of the whole, these parts must be connected with other parts of the same object by connections, belonging to it. Thus, the role of connections consists in uniting (binding) objects together, forming from them some new integral wholes. Below we are going to give a formal description of this structure.

By a *mereological space* we will mean a triple $\langle O, C, \leq \rangle$, where O is a set of *objects* (which we will denote by lowercase letters from the beginning of the Latin alphabet: $a,b,c,\ldots$), C is a set of *connections* (small Greek letters) and $\leq$ is a (generalized) *relation of being part*, defined on the union of these sets: for two objects $a \leq b$ means that a is a part of b, while $\alpha \leq a$ (where α is a connection and a is an object) means that α is an internal connection of a. Axiom M3 below will stipulate that no object can be a part of a connection. Below, lowercase letters from the middle of the Latin alphabet ($p,q,r,\ldots$) will denote mereological elements, that is, members of $O \cup C$.

DEFINITION 1.

(i) Two mereological elements *overlap* iff they have a common part:

$$p \circ q \equiv (\exists r)(r \leq p \ \& \ r \leq q)$$

(ii) An element p overlaps a set of mereological elements M iff it overlaps some member of M:

$$p \circ M \equiv (\exists q)(q \in M \ \& \ p \circ q).$$

DEFINITION 2.

(i) A set of mereological elements M *covers* another set N iff any element, overlapping N, overlaps also M:

$$N \trianglelefteq M \equiv (p)(p \circ N \rightarrow p \circ M).$$

(ii) Two sets M and N *mereologically coincide* iff each is covered by another:

$$M \sim N \equiv M \trianglelefteq N \ \& \ N \trianglelefteq M$$

(2) Thus, the language of our theory will in fact correspond to a thing language of the form IC in Carnap's classification, cf. Carnap [6].

(iii) An element p is a *mereological sum* (or fusion) of a set M iff $\{p\}$ mereologically coincides with M.

DEFINITION 3.

(i) A set M *almost covers* another set N iff any object, overlapping N, overlaps also M:

$$N \trianglelefteq_o M \equiv (a)(a \circ N \rightarrow a \circ M)$$

(ii) Two sets *almost coincide* iff each is almost covered by another:

$$M \sim_o N \equiv M \trianglelefteq_o N \ \& \ N \trianglelefteq_o M.$$

In this paper we shall follow the practice of not distinguishing in formulas unit sets from their respective elements. Thus, for example, $p \trianglelefteq_o q$ will have the same meaning as $\{p\} \trianglelefteq_o \{q\}$. Now we have all we need in order to formulate the axioms of mereological space.

AXIOM M0. '$\leq$' is a partial order on the set of mereological elements.

AXIOM M1. $\alpha \leq \beta \rightarrow \beta \leq \alpha$.

This axiom states that connections are atomic, that is, do not have other connections as parts. Although this assumption is not essential for our theory, it greatly simplifies matters.

Objects cannot be parts of connections. Moreover, objects always contain parts (either objects or connections) which do not overlap with any given connection. All these requirements are reflected in the following axiom:

AXIOM M2. $(a)(\alpha)(\sim a \trianglelefteq \alpha)$

Any connection must connect objects and hence it must be a part of some object:

AXIOM M3. $(\alpha)(\exists a)(\alpha \leq a)$

Internal connections of an object are involved in forming parts of this object (or the object itself). Hence if a connection belongs simultaneously to two objects then it is natural to suppose that it is involved in forming their common part:

AXIOM M4. $\alpha \leq a$ & $\alpha \leq b \rightarrow (\exists c)(\alpha \leq c$ & $c \leq a$ & $c \leq b)$

We will say that a connection α is an *external connection* of an object a (notation $\alpha + a$) if it connects this object with other objects. We will suppose that this holds if and only if it does not belong to a, but any object, containing α, overlaps with a (informally, this means that any object formed with the help of this connection must involve some parts from a) :

DEFINITION 4. $\alpha + a \equiv \sim \alpha \leq a$ & $\alpha \trianglelefteq_o a$

Since objects are supposed to be connected, if an object a is a proper part of an object b, then a must be connected with other parts of b and therefore b must contain some external connection of a. A slight generalization of this idea leads us to the following axiom:

AXIOM M5. $b \circ a$ & $\sim b \leq a \rightarrow (\exists \alpha)(\alpha \leq b$ & $\alpha + a)$

And finally, instead of the unrestricted 'comprehension' axiom of standard mereology we have an axiom stipulating that a sum of a set of objects exists, if these objects have some common part:

AXIOM M6. $(\exists a)(b)(b \in \mathrm{M} \rightarrow a \leq b) \rightarrow (\exists c)(c \sim \mathrm{M}))$

It can be shown that this axiom could be replaced by two conditions stating, respectively, an existence of a sum for any pair of overlapped objects and an existence of a sum for any (possibly infinite) chain of objects, ordered by the relation of being part.

The above defined structure is very rich and hence from the vast number of theorems that may be proved in it, we shall choose now only those that will be necessary in what follows. Thus, we will show first that mereological space is atomic in some sense.

DEFINITION 5.
p is an *atom* (notation At(p)) iff it does not contain proper parts (either connections or objects).

By axioms M1 and M3 all connections are atoms. But mereological space may contain atomic objects as well.

LEMMA 1. $p \trianglelefteq q \leftrightarrow p \leq q$

Proof: The implication from right to left is obvious and hence we will consider only the opposite implication. If p or q is a connection, then by M1 and M3 this implication is obvious, whereas if both p and q are objects, p overlaps q and p is not a part of q, then by M5 there is a connection, which belongs to p, but not to q and hence p is not covered by q.

This lemma says, in effect, that the relations of being part and covering coincide on the set of mereological elements. It has also the following important consequences:

LEMMA 2. (i) $p \sim \mathrm{M}$ & $q \sim \mathrm{M} \rightarrow p = q$
(ii) If p is a sum of M then it is a supremum of M.

Proof: (i) follows from Lemma 1 and the fact that $\sim$ is an equivalence. Consider (ii). Since p covers M if and only if it covers all its elements, we have that in this case it is an upper bound for M. Suppose now that q is an another upper bound for M. Since p is covered by M and M is covered by q we have that p is covered by q and hence p is a part of q. Thus, p is the least upper bound of M.

In view of the above lemma the sum of any set of mereological elements is uniquely determined (if it exists). But it should be noted that existence of a sum is not equivalent to existence of a supremum. Some sets may have the supremum, but may not have a sum.

The next lemma states that any mereological element must contain an atom.

LEMMA 3. $(p)(\exists q)(\mathrm{At}(q)$ & $q \leq p)$

Proof: If p is not an atom itself, then it is an object which must contain some proper part, say r. If r is not a connection, then from M5 it follows that p must contain some external connection of r and therefore any non-atomic object must contain some connection.

AT(p) below will denote the set of atoms of p and AT(M) will denote the set of atoms which are covered by M. Then we have:

LEMMA 4. (i) $\mathrm{M} \trianglelefteq \mathrm{N} \leftrightarrow \mathrm{AT(M)} \subseteq \mathrm{AT(N)}$
(ii) $p \sim \mathrm{AT}(p)$

Proof: (i) If M is not covered by N, then some mereological element q overlaps M, but not N. Hence q has some part in common with some element from M. By Lemma 3 this common part must in turn contain some atom, say p, and it is obvious that p is covered by M, but not by N, q.e.d. (ii) We need to prove only that p is covered by AT(p). If it were not so, then p would contain some part q, which is not overlapped with AT(p). But q must contain some atom, which thereby belongs to AT(p) — a contradiction.

The above lemma shows that any object is a sum of its atoms. Moreover, the set of atoms of an object determines this object in the sense that different objects correspond to different sets of atoms. Therefore we have a formal possibility of defining objects via their sets of atoms. It is interesting to note, that under an assumption that there are no atomic objects, any object will be a sum of its connections[3].

Now we will introduce some technical notions, which we shall use in what follows.

DEFINITION 6.
A *component* of a set of mereological elements M is a maximal object, covered by M.

LEMMA 5. (i) Any object, covered by some set M, is a part of some component of M.
(ii) Different components of any set M are not overlapped.

Proof: (i) For an object a, covered by some set M, we will consider the set $M' = \{b: a \leq b \ \& \ b \trianglelefteq M\}$. By the axiom M6 this set has a sum, which we shall denote by c. Now, c is obviously covered by M. Suppose that $c \leq d$ for some object d, covered by M. Since $a \leq c$, we have $a \leq d$ and therefore $d \in M'$. Hence $d \leq c$ and c is a maximal object, covered by M.

(ii) Suppose that a and b are two overlapping components of M. Then their sum, say c, exists and is also covered by M. In view of maximality of a we have $c \leq a$ and hence $b \leq a$. Therefore $a = b$.

[3] That is why Aristotle's theory of potentiality is vital for preserving the thesis that magnitudes are not made up of boundaries, because in its framework we simply cannot speak about all internal boundaries being simultaneously actualizable (cf.GC 1.2, 316b20ff).

DEFINITION 7.
An object b *bounds* an object c (notation $b[c$) iff b overlaps c, but is not a part of c:

$$b[c \equiv b \circ c \;\&\; \sim b \leq c$$

From M5 it follows that b bounds c if and only if b contains some external connections of c. By a *quasiborder* of an object b we will mean the set of all objects that bound b.

If M is a set of objects, then we have the following equivalence:

LEMMA 6. $b \trianglelefteq \mathrm{M} \leftrightarrow b \circ \mathrm{M} \;\&\; (c)(b[c \rightarrow (\exists d)(d \epsilon \mathrm{M} \;\&\; d[c))$

Proof: If b is covered by M, then it obviously overlaps M. Now, if $b[c$, then b must contain some external connection of c. This connection must belong to some object d from M and hence d overlaps c and is not a part of c, that is, $d[c$. On the other hand, if b is not covered by M, but overlaps M, then b must overlap some component of M, say e. Then $b[e$ and by Lemma 5 e is not bounded by elements from M, q.e.d.

The above equivalence has an interesting feature that the right side of it does not refer to connections and hence the notion of covering could be defined using only relations between objects.

3. *Topological representation*

Here we shall show that there is a natural connection between mereological spaces and topological spaces of some kind.

LEMMA 7. The family of sets $\{\mathrm{AT}(a)\}$ for all objects $a \epsilon \mathrm{O}$ forms an open base of some topological space on the set of all atoms of a mereological space.

Proof: Axiom M2 implies that the union of all such sets coincides with the set of all atoms. Hence this family will form a base of some topological space on the set of all atoms if and only if for any atom x, if $x \epsilon (\mathrm{AT}(a) \cap \mathrm{AT}(b))$, then for some object c, $x \epsilon \mathrm{AT}(c)$ and $\mathrm{AT}(c) \subseteq \mathrm{AT}(a) \cap \mathrm{AT}(b)$ (cf. Kelley [11]). But it is easy to see that this is equivalent to the following formula:

$$x \leq a \ \& \ x \leq b \rightarrow (\exists c)(x \leq c \ \& \ c \leq a \ \& \ c \leq b),$$

which immediately follows from M4.

The following lemma shows the role of sets of the form AT(a) in such a generated topology.

LEMMA 8. In the topology generated by the open base {AT(a)}, the set of all sets of the form AT(a) coincides with the set of all open connected sets.

Proof: Note that in the generated topology any open set is a union of some sets of the form AT(a). Now, if A is an open, non-connected set in this topology, then it must be a union of two non-empty disjoint open sets M and N. Suppose now that A = AT(b) for some object b. Since M and N are open, b must be covered by the components of M and N. Let c be a component of M. Then c is a proper part of b (since c is not overlapped with N) and hence $a[c$. But c is not overlapped either with other components of M, or with components of N, and hence we have a contradiction with Lemma 6.

Suppose now that an open set A does not coincide with any set of the form AT(a). Then in the mereological space it must have more than one component and all these components must form a covering of A. But in this case all atoms of some component and all other atoms of A form two sets, satisfying all conditions of M and N above, and therefore A is not a connected set.

From the fact, that the family {AT(a)} forms a base of the generated topology we obtain now

COROLLARY. The generated topological space is locally connected.

Now we are in a position to prove the following representation theorem:

THEOREM 1. The set of areas (open connected sets) and non-open points of any locally connected topological space form a mereological space and the topology generated by this mereological space coincides with the initial topology.

Proof: The relation of being part in a mereological space is defined in an obvious way: it coincides with the relation of inclusion for areas and unit sets of non-open points. Then axioms M0, M1, and M3 are obvious. In a locally connected topological space any point belongs to some area and hence M2 holds. The intersection of two open sets is always open and hence can be decomposed in a locally connected space into open components. Therefore axiom M4 also holds.

Suppose that axiom M5 does not hold for some areas A and B. Then any point from A—B($\neq \phi$) must belong to some area, which is not intersected with B. Let C be the union of all areas that are not intersected with B. Then $A \subseteq B \cup C$, $B \cap C = \phi$, $A \cap B \neq \phi$, and $A \cap C \neq \phi$ — a contradiction with the connectedness of A.

And, finally, the proposition corresponding to axiom M6, is easily proved for all topological spaces. Thus, areas and non-open points of any locally connected topological space T form a mereological space. On the other hand, these areas constitute a base of the topological space T', generated by that mereological space. Hence, spaces T and T' have the same base, and therefore coincide.

The above theorem shows, in fact, that from the formal point of view the two notions, that of a mereological space and that of a topological space, are equivalent, the only difference being the choice of primitive objects and relations. It is clear, however, that these theories are radically different in their 'metamathematical' assumptions and give different perspectives on the same subject, the most important being the structure of the continuum.

We will conclude this section with a definition, that will play an important role in the following.

DEFINITION 8.
If x and y are two atoms of a mereological space, then x is *subordinated* to y iff $y \trianglelefteq_\sigma x$.

In other words, x is subordinated to y if any object containing y contains also x. The relation of subordination induces a hierarchy on the set of atoms, which has an important role in our theory.

The above relation is not in general a partial order, but only a pre-order. In other words, different atoms may subordinate one another. But in all spaces that we will consider, the following condition holds:

(M_0) $(\alpha)(\beta)(\alpha \sim_\sigma \beta \rightarrow \alpha = \beta)$

This condition is equivalent to the requirement that the relation of subordination is a partial order. It corresponds also to the topological T_0-axiom.

In the following sections we shall consider two concretizations of the notion of mereological space: continuous and discrete spaces. The main difference between *continuity* and *discreteness* will be linked to an existence of a primary, minimal object, that contains a connection. While in discrete objects for any connection there must exist a minimal object containing it, continuous objects will be characterized by explicit rejection of this condition for any connection[4].

4. *Continuous mereological spaces*

A mereological space will be said to be *continuous* if for any two atoms x and y

(MC) $x \trianglelefteq_0 y \rightarrow x=y$

This means that no atom can subordinate other atoms. The above condition is equivalent to the following two conditions, stating, respectively, that connections cannot subordinate other connections and that any non-atomic object cannot contain atomic object-parts:

(i) (Homogeneity) $\alpha \trianglelefteq_0 \beta \rightarrow \alpha=\beta$
(ii) (Weak non-atomicity) $a<b \rightarrow (\exists c)(c<a)$

Indeed, if $a<b$, then by M5 b must contain some external connection α of a: $\sim\alpha \leq a$ & $\alpha \trianglelefteq_0 a$. Therefore by (MC) a cannot be an atom and hence it must contain some connection, say β. But then by homogeneity some object d must contain α, but not β. This object must overlap a and hence by M4 a and d must contain a common object-part, say c. It is obvious, that c is a proper part of a (since it does not contain β) and hence (MC) implies weak non-atomicity. On the other hand, if two different atoms x and y are such that x subordinates y, then x is obviously a connection and hence by homogeneity y must be an atomic object. By axiom M2 the connection x is a part of some object b, which thereby contain y. But then by weak non-

(4) It is interesting to note that at one point Aristotle gives a similar description of continuity (cf.Phys. 6.1, 231b2-6).

atomicity y must have proper parts, which is impossible.

Condition (MC) corresponds to the topological T_1-axiom in the associated topological space and hence could be considered as a plausible candidate for explication of the notion of continuity. There may be, however, some additional attractive requirements for the notion of a continuous object. It is plausible to assume, for example, that if a continuous object a is not a part of another object b, then a must contain an object, which does not overlap b. This requirement can be expressed as follows:

(R) $a \trianglelefteq_0 b \rightarrow a \leq b$

This condition excludes, for example, such objects, which correspond to areas with a single deleted point(5). In fact, adopting (R) is equivalent to restricting the universe of objects to only those objects that correspond to *regular areas* in the associated topological space. It is interesting to note that the above condition is implicit in those mereological theories (among them all 'standard' ones) that use the relation of overlapping as a primitive and define the relation of being part as $a \trianglelefteq_0 b$.

Our theory in its present form cannot be considered, however, as a theory of such regular areas in view of axiom M6, because the union of regular sets is not in general regular. We may, however, formulate the following weakened version of M6, which will be true for regular areas:

(M6r) $(\exists a)(b)(b \epsilon \mathrm{M}\ a \leq b) \rightarrow (\exists c)(c \trianglelefteq_0 \mathrm{M}\ \&\ \mathrm{M} \trianglelefteq c)$

It can be shown (see Bochman [4]), that a theory based on axioms M0-M5, M6r and (R) is in fact a theory of regular areas in the sense that there is a one-to-one correspondence between objects of such a mereological space and regular open areas in the associated topology(6).

4.1 *Whitehead's 'connection'.*

In Whitehead [17] we find a theory of extension, which is based on the primitive relation of 'extensional connection', the meaning of which is

(5) This condition is especially plausible if points are considered as definable entities, since in this case we simply fail to distinguish objects which in our terminology almost coincide.

(6) Associated topological space will in fact be locally connected and semiregular, because its base will consist of connected regular sets.

illustrated by a set of diagrams. It seems that this meaning is adequately reflected in the following definition:

DEFINITION 9.
a touches b (notation a^*b) iff $(\exists\alpha)(\alpha \trianglelefteq_0 a \;\&\; \alpha \trianglelefteq_0 b)$

For the associated topological space this means that closures of the corresponding areas must have a common point. On the basis of this relation Whitehead defines the relation of being part in the following way:

$$a \leq b \;\equiv\; (c)(c^*a \rightarrow c^*b).$$

It can be shown that the above equivalence amounts in our theory to the condition of regularity (R) and the following condition:

(W) $(c)(c^*a \rightarrow c^*b) \rightarrow a \circ b.$

This last condition is relatively weak. The topological condition, corresponding to (W), holds, for example, in all regular topological spaces.

5. *Discrete mereological spaces*

In this section we consider objects with discrete structure. As has been said above, the distinctive feature of discreteness will be taken to be an existence of minimal ('primary') objects, containing connections. To be more specific, we will define discrete mereological spaces as mereological spaces, satisfying the following axiom:

(MD) $(\alpha)(\exists a)(\alpha \leq a \;\&\; (b)(\alpha \leq b \rightarrow a \leq b)).$

Note that this axiom entails both M2 and M4. It is clear that for any connection there is only one minimal object containing it. For any atom x we will denote by $\langle x \rangle$ the minimal object, containing x. Note, that if x is an atomic object, then $\langle x \rangle = x$.

Now we will give a characterization of topological spaces, corresponding to discrete mereological spaces.

LEMMA 9. In the topology associated with a discrete mereological space, any intersections of open sets are open.

Proof: Suppose that some point x in the associated topology belongs to the intersection of a family of open sets $\{A_i\}$. Then for any A_i there is an object a_i such that $x \leq a_i$ and $a_i \trianglelefteq A_i$. If x is not open itself, then it must be a connection and hence by (MD), $<x> \leq a_i$. Therefore $AT(<x>) \subseteq \bigcap_i A_i$ and so this intersection is open.

Topological spaces, in which any intersections of open sets are open, are known by the name (appropriate to us) *A(lexandrov)-discrete topological spaces* (cf. Alexandrov [2]). We now prove that these spaces actually represent discrete mereological spaces.

THEOREM 2. Non-open points and areas of any A-discrete topological space form a discrete mereological space.

Proof: It is easy to show that A-discrete topological spaces are locally connected (see Bochman [4]). Therefore the unique mereological space must correspond to any such space. Hence it is sufficient to prove that this mereological space is discrete. Consider the intersection of all open sets containing some common non-open point. This set is open and connected, because in the opposite case it could be decomposed on a pair of proper open subsets and the above common point would belong to one of these subsets. Hence (MD) holds, and the corresponding mereological space is discrete.

A-discrete topological spaces have also another representation (see Alexandrov op.cit.). Any pre-ordered set generates an A-discrete topological space with open sets corresponding to filters on this set. On the other hand, the relation $xRy \equiv x \in \overline{\{y\}}$ on points of any A-discrete topological space is a pre-order, and the topology generated by this pre-order coincides with the initial topology. Now the above relation R is in fact the relation of subordination in the associated mereological space (see Definition 8). Thus, discrete mereological spaces are determined, in fact, by the relation of subordination on their atoms.

We consider below some particular discrete spaces and show that they represent some well known structures.

5.1. *Systems*

Having at our disposal a theory of discrete objects, it is tempting to give yet another definition of the notion of system. However, our framework is still too broad for this aim. Systems are in certain aspects constructive entities, whether they are obtained through synthesis or analysis, and hence they involve some 'finiteness' conditions, which we must take into account. But first we will assume that the relation of subordination in systems must be a partial order, that is, a pair of connections cannot subordinate one another. This condition reflects the postulate (M0) above, which may be now rewritten in the following form:

(M_0') $<\alpha> = <\beta> \rightarrow \alpha = \beta$

If this condition holds, then we can show that for any connection α the object $<\alpha>$ is a sum of α and all objects which are maximal proper parts of $<\alpha>$ (it is obvious, that these objects do not contain α). Hence α can be seen as joining the maximal proper parts of $<\alpha>$ into a new integral object, $<\alpha>$. This reflects our informal interpretation of the role of connections.

The second plausible condition for the notion of system is a requirement that any non-atomic object must contain *independent* connections, that is connections which are not subordinated to any other connection from that object:

(S1) $(a)(\exists x)(\text{At}(x) \mathbin{\&} x \leq a \mathbin{\&} (y)(\text{At}(y) \mathbin{\&} y \leq a \mathbin{\&} y \trianglelefteq_\sigma x \rightarrow y = x))$

It can be shown that (S1) is equivalent to the following minimality condition for the relation of subordination:

(S1') Any set of atoms, ordered by the relation of subordination, must have minimal (that is independent) elements.

If this condition holds, then any object must be a sum of primary objects, corresponding to its independent connections. This follows from the fact that any atom of an object must be a part of the primary object of some independent connection (because in the opposite case we would have an infinite chain of subordination without minimal elements).

Partial orders satisfying the minimality condition are known as *inductive partial orders*. They have, in particular, the property that any non-maximal element must have immediately following elements. In our case this means that any connection α must have immediately subordinated atoms. They are actually either atomic objects or independent connections of maximal proper parts of an object $<\alpha>$. It is, however, possible that some non-independent atom does not have any connections which immediately subordinate it. We want to exclude such a possibility, and so we shall adopt also the following condition:

(S2) If an atom x is subordinated to another atom y, then x is immediately subordinated to some atom z, which is subordinated to y.

It can be shown that in the presence of (S1) the above condition is equivalent to the condition that if two atoms stand in the relation of subordination, then they can be linked by a finite chain of immediate subordination.

Thus, the conditions (M_0), (S1) and (S2) characterize systems as discrete objects in which the relation of subordination forms a *discrete inductive partial order*. It seems that these conditions represent minimal requirements for the notion of system. We will consider below two important concretizations of this notion.

As can be seen from the axiomatics of mereological space, there are in fact only two ways of forming objects from their constituents. On the one hand, overlapping objects form new objects — their sums. These sums are formed 'by themselves', without any additional connections or objects. On the other hand, we can use connections in order to join objects into some new integral wholes. Elimination (or reduction to the necessary minimum) of each of these ways of building objects will lead in the case of systems to two useful and usable structures, namely, to hierarchical systems and hypergraphs.

5.1.1. *Hierarchical systems*

Hierarchical systems are formed only with the help of connections. Hence any non-atomic object in this case must be a primary object of some connection:

(HS) $(b)(\exists x)(\mathrm{At}(x)\ \&\ b=<x>)$

It can be shown that (HS) is equivalent to (S1) together with the following condition:

(HT) $\mathrm{At}(x)\ \&\ \mathrm{At}(y)\ \&\ <x> \circ <y> \rightarrow x \leq <y>\ \vee\ y \leq <x>$

This last condition is equivalent to the requirement that the order of subordination is a *tree order* :

$$y \trianglelefteq_o z \;\&\; x \trianglelefteq_o z \rightarrow x \trianglelefteq_o y \;\vee\; y \trianglelefteq_o x$$

In other words, hierarchical systems may be characterized as discrete objects in which the order of subordination is a discrete inductive tree order. From this it follows, in particular, that for any connection α, maximal proper parts of $<\alpha>$ are not overlapped, and coincide with primary objects for atoms, immediately subordinated to α. Moreover, we have in fact an isomorphism between atoms, ordered by the relation of subordination, and objects, ordered by the relation of being part. Hence, the relation of being part among objects is also a discrete inductive tree order. In other words, hierarchical systems represent the well-known structure of *immediate constituents* (with a possible exception that objects may have an infinite number of immediate constituents on some level of decomposition, and the number of levels may also be infinite).

5.1.2. *Hypergraphs*

By a hypergraph we will mean a triple $\langle V,E,I \rangle$, where V and E are two disjoint sets, members of which are called, respectively, *vertices* and *edges*, and I is an *incidence* function, which assigns to any edge a non-empty set of (incident) vertices. Hypergraph is a natural generalization of the notion of graph, in which any edge may have no more than two incident vertices. And just as for graphs, we may define for hypergraphs the notions of sub-(hyper)graph and of connected hypergraph. Then it can be shown that connected subgraphs and edges of an arbitrary hypergraph form a discrete mereological space, in which edges play the role of connections, vertices play the role of atomic objects, while edges together with all their incident vertices correspond to primary objects.

The above defined mereological space also satisfies the condition of Homogeneity (see Section 4):

(MH) $\alpha \trianglelefteq_o \beta \rightarrow \alpha = \beta$

Thus, hypergraphs correspond to discrete mereological spaces with a minimal possible hierarchy on their atoms.

Now we will show that hypergraphs may in fact be characterized as discrete mereological spaces, satisfying the axiom (MH). Note that conditions (M_0), (S1) and (S2) are consequences of (MH) and hence such spaces form a special kind of systems.

LEMMA 10. $<\alpha> \sim \{\alpha\} \cup \{a\text{: At}(a)\ \&\ a \leq <\alpha>\}$

Proof: If $<\alpha>$ is not covered by the above set, then it must contain some mereological element p, which is not overlapped with this set. But by homogeneity p cannot be a connection and hence it is an object. Moreover, if this object is non-atomic, then it must contain some connection which again contradicts to homogeneity. Therefore p must be an atomic object. But in this case it belongs to the above set of atomic objects from $<\alpha>$, which is impossible.

In view of this lemma any primary object consists of a connection and a set of atomic objects (which are joined by it). Thus, atomic objects and connections of such a space could be considered as, respectively, vertices and edges of some hypergraph. We have, in fact:

THEOREM 3.

(i) Sets of atomic objects and connections of a discrete mereological space, satisfying the condition (MH), form a hypergraph with the incidence function, defined in the following way: $\text{I}(\alpha) = \{a\text{: At}(a)\ \&\ a \leq <\alpha>\}$.

(ii) Connected subgraphs and edges of an arbitrary hypergraph form a discrete mereological space, satisfying (MH), and the hypergraph generated by this space coincides with the initial hypergraph.

It is interesting to note that since any mereological space corresponds to some topological space, we have as a by-product of our construction that hypergraphs have a topological representation in some A-discrete topological spaces(7).

As we said above, *graph* is a special case of hypergraph, which satisfies the additional requirement that any edge has no more than two incident vertices. Corresponding mereological spaces satisfy the following additional condition:

(7) In [4] it is shown that hypergraphs correspond exactly to A-discrete topological spaces in which all points are either open or closed. There exists also a natural characterization of both hypergraphs and graphs in terms of some dimensional topological requirements.

(GR) $\alpha \trianglelefteq_0 a \mathbin{\&} \alpha \trianglelefteq_0 b \mathbin{\&} \alpha \trianglelefteq_0 c \rightarrow a \circ b \vee b \circ c \vee a \circ c$

6. *Reduction of connections*

Here we will turn to the question which was, in fact, the starting point of Whitehead's theory of extension: when and how connections could be defined in terms of properties of objects or, to be more precise, which sets (or sets of sets, etc.) of objects uniquely determine connections ? We will not consider here all possible approaches to this problem (cf., for example, Adams [1] and Clarke [8]), but will restrict ourselves to only one possible solution.

In mereological spaces satisfying the axiom M0, the set of all objects which contain a given connection uniquely determines this connection. We now introduce the following definition:

DEFINITION 10.
A set of objects is a *connection-filter* for some connection iff it is a set of all objects which contain this connection.

Our aim will be to define connection-filters only in terms of properties of objects. We will consider first the case of continuous spaces and then will give corresponding definitions for discrete spaces.

6.1. *Definition of points*

Continuous mereological spaces correspond to locally connected topological T_1-spaces, which are characterized by the condition that all their points are closed. Therefore their complements are open and hence could be decomposed on open connected components. Then it is clear that the set of all open connected sets containing a given point coincides with the quasiborder of such components (see Definition 7). On the other hand, it could be shown that these components can be characterized as open connected sets with minimal possible non-empty quasiborder. Hence, we introduce the following definition:

DEFINITION 11.
An object c is a *point-complement* iff it has a minimal non-empty quasi-

borer:

(i) $(\exists a)(a[c)$
(ii) $(b)((\exists d)(d[b)\&(e)(e[b \rightarrow e[c) \rightarrow (a)(a[c \rightarrow a[b))$.

Now we have

THEOREM 4. A set of objects is a connection-filter iff it coincides with the quasiborder of some point-complement.

COROLLARY. Connection-filters coincide with minimal non-empty quasiborders.

The proposed solution to the problem of reduction of points in continuous spaces is not quite satisfactory because it heavily depends on the existence of non-regular objects. As was said in Section 4, if we are seriously considering the possibility of point elimination, that is, if we pretend that we know nothing about points at the outset, then areas which differ only by one point, would be indistinguishable for us. Unfortunately, our point-complements are in general non-regular and hence do not give us a satisfactory solution. Note, however, that our solution is perfectly suitable for spaces where all areas are regular, the most important case being a line (see [5]).

6.2 *Definitions of connections in discrete spaces*

The problem of reduction for connections in discrete spaces can be resolved as follows:

THEOREM 5. F is a connection-filter in a discrete mereological space iff for some object a, that satisfies the formula

(*) $\sim \mathrm{At}(a) \ \& \ \sim a \trianglelefteq \{b\colon \sim a \leq b\}$,

F coincides with the set of all objects containing a.

Proof: The above condition (*) is equivalent to

$$\sim \mathrm{At}(a) \ \& \ (\exists x)(\mathrm{At}(x) \ \& \ x \leq a \ \& \ (b)(x \leq b \rightarrow a \leq b)$$

and therefore to $\sim \mathrm{At}(a)\ \&(\exists x)(\mathrm{At}(x)\ \&\ <x> = a)$.

Hence x must be a connection and F will coincide with the set of all objects containing x.

In discrete mereological spaces, satisfying M_0, connections can be identified, in fact, with their respective primary objects. The above condition (*), which, as was shown, characterizes such objects, can be rewritten (using Lemma 6) as follows:

(D) $(\exists c)(a[c\ \&\ (b)(b[c \rightarrow a \leq b))$.

It can be shown that (*) is also equivalent to the following condition:

(**) $\sim \mathrm{At}(a)\ \&\ \sim a \trianglelefteq \{b\colon b < a\}$.

In other words, primary objects could be characterized as non-atomic objects that cannot be covered by their proper object-parts. Again, (**) can be rewritten as follows:

(D') $(\exists c)(c < a\ \&\ (b)(b < a \rightarrow\ \sim b[c))$

In *hierarchical systems* all objects are primary, while in *hypergraphs* they are just those non-atomic objects which have only atomic parts:

(DH) $\sim \mathrm{At}(a)\ \&\ (b)(b < a \rightarrow \mathrm{At}(b))$.

Thus, the problem of reduction for connections in discrete spaces is completely resolved.

7. *Pure mereological spaces*

The possibility of defining connections in terms of objects creates a possibility of building 'pure' mereological spaces with domains comprising only objects, and with the sole primitive relation of being part between them. In these spaces connections could be introduced via the above mentioned definitions.

Since our definition of points in continuous spaces has some intuitive shortcomings, we will restrict ourselves here to the case of discrete objects.

A *pure discrete mereological space* will be defined simply as a partially ordered set $\langle O, \leq \rangle$, which satisfies the following two axioms[8] :

PD1. $b[a \rightarrow (\exists c)(c \leq b \ \& \ c[a \ \& \ (\exists e)(c[e \ \& \ (d)(d[e \rightarrow c \leq d)))$
PD2. $(\exists a)(b)(b \in M \rightarrow a \leq b) \rightarrow (\exists c)(c = \sup(M) \ \& \ c \trianglelefteq_o M)$

(Here sup(M) denotes the supremum of elements of M.)

Now we will introduce the following definition (cf. condition (D) above):

DEFINITION 12.
An object b is a *primary object* (notation $\Pr(b)$) iff

$$(\exists c)(b[c \ \& \ (d)(d[c \rightarrow b \leq d))$$

The axiom PD1 could be now expressed as follows:

PD1'. $b[a \rightarrow (\exists c)(c \leq b \ \& \ c[a \ \& \ \Pr(c))$

and it is easy to see that it is a 'pure' analogue of axiom M5.

Now for any primary object c we will introduce a new element of the domain — a connection $[c]$, and we will extend the relation of being part in the following way:

(i) $[c] \leq b \equiv c \leq b$;
(ii) $[c] \leq [b] \equiv c = b$;
(iii) $c \leq [b] \equiv$ false.

Then we can prove the following

THEOREM 6. The extended pure discrete space is a general discrete mereological space satisfying the condition (M_0).

Proof: Axioms M0, M1, M3, MD and M0 obviously hold, while the axiom M5 immediately follows from PD1.

(8) Defined notions, which we use below, have the same definitions as their original counterparts except for the relation of overlapping, which is now defined as an existence of a common object-part.

Any primary object a has some proper part, say c. Therefore if $[a]$ belongs to b, then $c \leq b$ and $\sim [a] \leq c$. If $[a]$ does not belong to b, then again $(\exists c)(c \leq b \;\&\; \sim [a] \leq c)$ (for $c = b$). Therefore M2 is true.

In order to prove M6 suppose that M is a set of objects that has a lower bound. Then by PD2 it has the supremum c, which is almost covered by M. It is obvious that c almost coincides with M (cf. Definition 2 from Section 2). Hence, it is sufficient to prove that for any primary object a if $a \leq c$, then a is a part of some element from M. Suppose that for some primary object a this is not so. From the definition of a primary object it follows that for some object e, $a[e \;\&(d)(d \epsilon \mathrm{M} \rightarrow \sim d[e)$. Since all elements of M are overlapped we have that either e does not overlap M or it is an upper bound of M. Now, from $a \leq c$ it follows that e overlaps c and therefore from $c \trianglelefteq_0 \mathrm{M}$ we infer $e \circ \mathrm{M}$. But if e is an upper bound of M, then $c \leq e$ and hence $a \leq e$ (because $a \leq c$). Hence we have obtained a contradiction and so M6 holds.

Thus, the extended space satisfies all axioms of a discrete space and condition M_0.

The notion of *system* can be now characterized by way of adding the requirement that the relation of being part must be discrete and inductive on the set of primary objects.

It is easy to show that pure mereological space of *hierarchical systems* is simply discrete inductive tree order and hence the introduction of connections in this case is in some sense superfluous, because the pure theory is no less convenient and understandable than the general theory.

Now consider hypergraphs. Since for them primary objects can be defined as non-atomic objects having only atomic proper parts (cf. (DH) above), a *pure mereological space of hypergraphs* could be defined as a partially ordered set (of connected hypergraphs), which satisfies the following axioms:

PH1. $a[b \rightarrow (\exists c)(c \leq a \;\&\; c[b \;\&\; (d)(d < c \rightarrow \mathrm{At}(d)))$

PH2. $(\exists a)(b)(b \epsilon \mathrm{M} \rightarrow a \leq b) \rightarrow (\exists c)((d)(d \epsilon \mathrm{M} \rightarrow d \leq c) \;\&\; (e)(e \leq c \rightarrow (\exists f)(f \epsilon \mathrm{M}$
$\&\; (e \leq f \vee (\exists g)(g \leq e \;\&\; g \leq f \;\&\; \sim \mathrm{At}(g))))$

If we now introduce connections in the same way as for general discrete spaces, we obtain:

THEOREM 7. Extended space is a discrete mereological space, satisfying (MH).

The proof proceeds exactly as for Theorem 6, except for the case of axiom M6. It is obvious that the object c from the conclusion of axiom PH2 almost coincides with M. Suppose that for some primary object a, $a \leq c$, but a is not a part of any object from M. Then by PH2 it must have a non-atomic common part g with some object f from M. But $g < a$, because a is not a part of f, and we obtain a contradiction to the assumption that a is a primary object.

Now the condition (MH) also holds, because no primary object can be a part of another primary object (in the opposite case one of them would contain non-atomic parts).

Having a pure mereological space of hypergraphs, we can define a corresponding space for ordinary *graphs* by way of adding a condition that primary objects may have no more than two proper parts.

Postscriptum. Mereology vs. set theory

All that has been said in this paper could be construed as an argument in favour of the thesis that mereology has its own subject and cannot be considered as a modified version of set theory. However, discussing the interrelations of mereology and set theory we must distinguish between two aspects of set theory usage. On the one hand, set theory, in one form or another, is now commonly used as a language of formal theories, and our theory is no exception in this respect. On the other hand, there is also something that we will call a *set-theoretical approach* to the structure of many mathematical and non-mathematical objects, which provides for them *set-theoretical interpretations*. For example, the well-known construction allows us to reconstruct the whole building of classical mathematics inside the standard set theory, beginning from the natural numbers. Thus, we have a set-theoretical interpretation of the continuum in particular, and of the notion of space in general.

This set-theoretical approach to the structure of objects, as we understand it here, may be roughly and generally described in the following way. We choose (or introduce) some elementary objects without internal structure ('urelements') and then reconstruct the source objects as sets of such elements. The structure of these objects is now determined in terms of some external properties and relations (again, in the set-theoretical sense) on elements, sets of elements, and so on. The merits of such an approach to the structure of objects are well-known and perhaps need not be mentioned here. But in some cases the costs of adopting such an approach become very high. For example, as was already noted, atomistic interpretations of space and time have constantly caused conflicts with an intuitive understanding of these objects, despite their self-consistency and usefulness(9). Moreover, the incompleteness of set theory (for example, the independence of the continuum hypothesis) makes the continuum, construed in a set-theoretical way, essentially undetermined, because different options lead, in effect, to different continua (cf. Asenjo [3]).

On the other hand, if we turn to discrete complex objects (systems), then the above approach becomes ineffective for the following reasons. First, all possible parts of such objects are in general not known in advance and hence we cannot choose a natural class of 'urelements'. Hence, if we choose some level of decomposition of such a complex system and construct a corresponding set-theoretical model, we obtain an essentially partial description of this system. What complicates the situation still further is the possibility of different independent decompositions of the same object. As a result we must assign to an object a set of its partial models, and will never be sure that this set gives a faithful description. It seems that just these deficiencies of the atomistic approach gave rise to developing the so-called system approach to the structure of objects, in which the notion of system becomes a primary notion.

It is precisely here that mereology may become an alternative (or a complement, if you like) to set theory, because the objects in mereology are primary elements of the universe of discourse and hence there exists a possibility of 'downward' analysis of the structure of objects through their decomposition into parts.

It should be noted, however, that as was shown in this paper, under some commonly accepted suppositions the two approaches turn out to be equivalent. It seems to us that the real difference between them may arise only in the framework of some 'constructive' methodology, similar in many respects to that suggested by Aristotle.

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(9) It is interesting to note that the universum of geometrical theories since Euclid and up to Hilbert's axiomatization have always comprised not only points, but also lines and planes as undefined primitive concepts.

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