Logique & Analyse 129-130 (1990), 69-74

INTERPOLATION FOR CHURCH'S PURE FUNCTIONAL CALCULUS OF SECOND ORDER UNDER SECONDARY INTERPRETATIONS

Cristina CORREDOR-LANAS

Abstract

A proof is given of Craig's Interpolation Theorem for Church's Pure Functional Calculus of 2nd-order, which is provably complete w. r. t. the class of all secondary interpretations (a subclass of Henkin's general interpretations). The search for the proof allows for a deeper analysis of interpolation within the framework of 2nd-order formalisms.

1. *Introduction*

The result stated above seems not to be in harmony with other well-established facts. For, on the one hand, the class of all 2nd-order standard systems verifies interpolation, but no axiomatization exists which characterizes this class. On the other hand, Henkin proved in [1950] that it is possible to define a wider class of 2nd-order formalisms (called *general systems*) which satisfy certain closure conditions, in such a way as to be characterized by a calculus that extends the usual classical 1st-order one. Nevertheless a counterexample can be found for definability and *a fortiori* for interpolation.

In his [1956] Church defined some 2nd-order calculi and proved them to be complete w. r. t. the class of all secondary systems. In fact, he relativizes the notion of validity in Henkin's general sense to that of truth in a particular class of *normal systems*, these latter defined as those in which all axioms of $F2^{2p}$ (the Pure Functional Calculus of 2nd-order) are valid and all the rules of $F2^{2p}$ preserve validity. It straightforwardly follows that every secondarily valid wff in the calculus is a theorem of it.

As Church points out, it is not possible to extend $F2^{2p}$ by adding rules and axioms, in such a way that the theorems come to coincide with the wffs which have the value *true* for all systems of values of their free variables, under all standard interpretations. This follows from Gödel's incompleteness theorems. In particular this incompleteness result applies to any 2nd-order system containing 2nd-order Peano arithmetic, provided that at least one predicate constant has been added to the vocabulary. Then it is possible to define a categorial finite set of axioms for the set of positive integers. When general systems are defined (by means of making sure that to each way of compounding formulas in the calculus is associated an operation defined on the domains of the systems, w. r. t. which these must be closed), it is also the presence of a predicate constant what leads to the failure of Beth's Definability Theorem, henceforth of Craig's Interpolation Lemma (since both results are equivalent within the framework of regular logics).

This last remark makes Church's calculus $F2^{2p}$ particularly interesting. For it is defined as including among its primitive symbols all the individual, propositional, and functional variables, but no (individual or functional) constants. Then the calculus turns out to be closed under definability and the strategy for showing the failure of interpolation is no longer available. This fact strongly suggests that interpolation might hold for this 2nd-order formalism, complete under the class of all secondary interpretations.

2. *The Calculus $F2^{2p}$*

From now on the notations and definitions are taken to be those to be found in Church [1956]. In particular, the Simple Functional Calculus of 2nd-order has, in addition to notations of the Functional Calculus of 1st-order, quantifiers with propositional or functional variables as operators variables. The Pure Functional Calculus of 2nd- order includes (as it has just been said) all the individual, propositional and functional variables, but no individual or functional constants. The primitive vocabulary includes individual variables $a_1, a_2, \ldots, b_1, b_2, \ldots$, propositional variables p, q, r,..., predicate variables P, P^n,..., (instead of original Church's functional ones, for reasons that will be explained below) and sentence letters A, B, C,.... $S_B{}^P A$ (or A[P/B]) stands for the result of substituting P for all occurrences of B in A. Moreover the following two definitions are introduced: $f =_{df} (s)s$, $t =_{df} (\exists s)s$, where s is any variable. The syntactical and semantical notions are also those to be found in Church [1956]; we only specify here what is fundamental for the proof.

Thus the rules of inference and axiom schemata for $F2^{2p}$ are the following.

Rules of Inference

R1. Modus Ponens: From A and $A \supset B$ to infer B.

R2. Generalization: From A, if a is any variable, to infer (a)A.

R3. Alphabetic Change of Bound Individual Variables: From A, if a is an individual variable not free in C, and b is an individual variable which does not occur in C, if B results from A by substituting C[a/b] for a particular occurrence of C in A, to infer B.

R4. Substitution for Individual Variables: [This rule requires an explanation. In Church's original work, this rule permits substitution of individual variables which are not bound; since the calculus is functional, it is necessary to carefully explicit the cases in which such a substitution is possible. Nevertheless, we have assumed a vocabulary that only contains predicate variables and no functional ones, by performing the usual transformation from functions into their corresponding diagrams. Furthermore, it will be seen that the set of valid wffs in the calculus coincides with the set of *closed* secondarily valid wffs. Thus it is easy to see that under these conditions rule R4 reduces to R3].

Axiom Schemata

a1. $p \supset . q \supset p$

a2. $s \supset (p \supset q) \supset . s \supset p \supset . s \supset q$

a3. $\neg p \supset \neg q \supset . q \supset p$

a4. $(x)(p \supset F(x)) \supset . p \supset (x)F(x)$

$a4_0$. $(p)(A \supset B) \supset . A \supset (p)B$ (p is not free in A)

$a4_n$. $(P^n)(A \supset B) \supset . A \supset (P^n)B$ (P^n is not free in A)

a5. $(x)F(x) \supset F(y)$

$a5_0$. $(p)A \supset S_B{}^pA$

$a5_n$. $(P)A \supset S_B{}^{P(x1\ldots xn)}A$ ($xi \neq xj$ if $i \neq j$).

As for the semantic notions, just remember that a *system* is an ordered pair $\langle F, I\rangle$ where $F = F_0, F_1, F_2, \ldots$; F_0 is a non-empty set (the domain of individuals) and $F_i \subseteq P(F_0{}^i)$; and I is an interpretation on F, defined in the usual manner in 2nd-order. In particular $I(s) \in F_0$ for any individual symbol; and $I(Ps_1 \ldots s_n) = I(P)(I(s_1) \ldots I(s_n))$, where $I(P) \in F_n$; moreover $I(A) \in \{t, f\}$ for any closed wff A of $F2^{2p}$. A wff A in the calculus is said to be *valid w.r.t.* $\langle F, I\rangle$ iff $I(A) = t$.$\langle F, I\rangle$ is said to be *normal* if all the axioms of $F2^{2p}$ are valid w.r.t. it, and every rule of $F2^{2p}$ preserves validity w.r.t. it. A wff A is said to be *secondarily* valid iff it is valid w.r.t. every normal system.

It can now be stated the completeness result referred to above.

Theorem (Church [1956])

Any wff A of $F2^{2p}$ is secondarily valid iff it is a theorem of the calculus.

3. *Interpolation lemma for $F2^{2p}$ under secondary interpretations*

There are two possible strategies for proving interpolation: a model-theoretical one and a proof-theoretical one. The latter can be characterized as consisting of two steps. Firstly, to specify an interpolant for the axiom schemata (e.g. for A ⇒ A in a Gentzen calculus), and secondly, to find an interpolant for the conclusion of each rule, assuming it exists for the premise(s). The only difficulty arises within classical 1st-order when the introduction rule for the universal quantor is considered. For the critical condition establishes that the proper parameter (*eigenvariable*) cannot occur in the conclusion in

$$\Gamma \Rightarrow \Theta,\ A(a)\ /\ \Gamma \Rightarrow \Theta,\ (x)A(x) \qquad [\Rightarrow\forall].$$

Yet it is possible to overcome this difficulty by means of defining a notion of deducibility apparently more restrictive than, but provably equivalent to, the usual classical one (cf. Smullyan [1968]). Thus, a *deduction of A from* Γ is to be a finite tree of wffs, starting in A, finishing in axioms or members of Γ, and such that each node which is not a final node is derived from one of its predecessors by application of one of the rules. Furthermore (and here the crucial point lies), whenever [⇒∀] is applied, the subtree starting in the conclusion must be such that the proper parameter occurs in the final nodes only in axioms.

The underlying idea is that the *eigenvariables* be seen as they were implicitly general (i.e. universally quantifiable). This is feasible by making sure that they are introduced only *via* the logical axioms. And this is precisely what happens in the case of $F2^{2p}$. For none of the rules authorizes a replacement of individual or predicate variables with constants — they do not appear in the vocabulary. Only such substitutions are available as those involving alphabetic changes of individual or predicate variables. Moreover these can be restricted to the case of closed wffs without loss of generality, as said before.

Consequently, any not bound variable apparently playing the role of a proper parameter can be generalized (i. e. universally quantified), provided that conditions stated in rules R3 (and R4) be satisfied. Namely, that no other free variable would turn out to be bound, or conversely that no variable originally bound be freed by accomplishing a substitution. Generalization over all free variables of a valid wff A of $F2^{2p}$ yields a closed wff, which constitutes a provable valid sentence:

Theorem (Church [1956])
A wff A of $F2^{2p}$ is valid in all standard systems [alternatively, in all secondary systems] iff its universal closure is valid in all standard systems [in all secondary systems resp.].

This leads the path to a direct proof of the Interpolation Lemma, given the following definition. If A, B are two valid wffs of $F2^{2p}$ and $A \supset B$ is secondarily valid, let $w(A) \cap w(B)$ denote the set of their common variables, and let $L_{w(A) \cap w(B)}$ be the sublanguage in $w(A) \cap w(B)$ (i.e. , the set of wffs of the calculus whose variables belong to that set). A wff C is said to be an *interpolant* for $A \supset B$ iff (i) $C \in L_{w(A) \cap w(B)}$ and (ii) $A \supset C$, $C \supset B$ are both theorems of $F2^{2p}$.

Now the announced result can be proven.

Theorem Interpolation holds for $F2^{2p}$.
Proof Let $A \supset B$ be any secondarily valid wff of $F2^{2p}$. By hypothesis (since $F2^{2p}$ is complete w.r.t. the class of all secondary interpretations), $A \supset B$ is a theorem of the calculus. Let $a_1, \ldots, a_n$, $P_1, \ldots, P_m$ be the free individual and predicate variables resp. which occur in B but not in A. Replace each of them with a new variable, b_i instead of a_i, Q_j instead of P_j, which do not occur in B, and let B' be the result of effecting this replacement. Take then the universal closure of B', $(b_1)\ldots(b_n)(Q_1)\ldots(Q_m)B'$, denoted by B''. It is immediate to see that the following holds:

1) $B'' \in L_{w(A) \cap w(B)}$;
2) $A \supset B''$ is a theorem of $F2^{2p}$ (by rule R2 the closure of $A \supset B$ is so, and by rule R4 [R3] it holds also for $A \supset B''$;
3) $B'' \supset B$ is a theorem of $F2^{2p}$ (by axiom $a5_n$);
4) $A \supset B''$, $B'' \supset B$ are valid wffs in $F2^{2p}$ (by Church preceding theorem).

Thus B'' is the looked-for interpolant. □

University of Valladolid.

REFERENCES

[1] CHURCH, A. [1956]: *Introduction to Mathematical Logic*. Princeton Univers. Press.

[2] HENKIN, L. [1950]: *Completeness in the theory of types*; Journal of Symbolic Logic 15 (1950), pp. 81-91.

[3] SMULLYAN, R. M. [1968]: *First-order Logic*. Berlin: Springer.