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USING HILBERT'S CALCULUS

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1. *Introduction*

Hilbert's epsilon calculus can be seen as a highbrow alternative to Fregean predicate logic, useful only for proving certain advanced meta-logical theorems. In this paper I demonstrate that it also has a ready application to undergraduate logic teaching; it simplifies considerably some practical procedures there, and clarifies the associated theoretical issues.

The epsilon calculus is a conservative extension of the predicate calculus. That means no new theorems involving just the language of the predicate calculus are provable in the epsilon calculus, although all theorems in the former are provable in the latter, and further theorems as well. The further theorems essentially employ epsilon terms, i.e. terms of the form 'ϵxFx'. In moving over to using Hilbert's epsilon calculus, therefore, the primary justification lies in the ability to prove these new theorems. But the proofs of standard theorems, it turns out, can be made much more convenient, so not only necessity, but also ease and facility drive the change to using the extended calculus. And this applies not only with the proof theory but also with the metatheory. Metatheorems of the predicate calculus become much more straightforward, but further important metatheorems become available, as well.

To illustrate the need and usefulness of the epsilon calculus I shall thus first present some specific theorems employing epsilon terms, showing how they are not available with standard processes. But I shall also demonstrate how the epsilon calculus simplifies normal predicate proofs, of theorems which are standardly available. I shall then improve upon some metatheorems of the predicate calculus, and prove some of the epsilon calculus. The latter proofs are distinctive for their brevity. But the former show something quite new about the predicate calculus: that its semantics can be dispensed with in the epsilon calculus, and therefore that it is only certain special features of the predicate calculus which give rise to that conception of things.

2. *Epsilon calculus proofs*

Hilbert's epsilon calculus, of course, is a richer logic than Frege's, employing individual terms like 'ϵxFx' for every predicate 'F' in the language. Such terms are used to formulate further logical truths which need to be codified, beyond those in the predicate calculus, for example:

There is a god.
So that god is a god.

Such a logical truth employs an expression not formalisable in the predicate calculus: the referring phrase 'that god' — and the associated abbreviation, or pronoun, 'he'. Predicate logic cannot formalise referring expressions and has had to resort, since Russell's time, to using 'incomplete symbols', i.e. certain quantifier phrases, to roughly approximate to them. Part of the shift to employing, indeed perhaps even understanding the epsilon calculus requires one to be dissatisfied with such roughness. One must insist on complete symbols instead (Goddard and Routley 1973, p558). Thus the above is properly

$(\exists x)Gx.$
So $G\epsilon xGx.$

For the epsilon term symbolises the referring expression and the associated pronoun (Slater 1986, 1987(a)(b), 1988(a)(b), 1989(a)(b)(c)(d), 1992(a)(b)). Moreover it does so without bringing in any imputation of uniqueness, as with Russellian iota terms.

The only axiom schema needed to formalise the behaviour of epsilon terms is in fact

$$Fy \supset F\epsilon xFx,$$

(so long as y is free for x in Fy). This is a specialisation of the more general thesis

$$(\exists x)((\exists y)Fy \supset Fx)$$

which is all that is available, just in the predicate calculus. But the quantifiers can be now introduced by definition, via, for instance,

$$(\exists x)Fx \equiv F\epsilon xFx,$$

and it is essentially because of this that all predicate calculus theorems are included in the epsilon calculus ones.

But a choice function semantics for epsilon terms is commonly given (Leisenring 1969), and that validates a further axiom

$$(x)(Fx \equiv Gx) \supset \epsilon xFx = \epsilon xGx,$$

which has no place in intensional logic, i.e. where discriminations may need to be made between predicates determining the same set. Staying with just the first axiom, discriminations between such predicates can be made by separating the referents of the associated epsilon terms, which become the central cases around which applications of the different predicates are based (Slater 1988(a) p297, Slater 1989(d) p28, c.f. Copi 1973, p109). Semantical interpretations for predicate symbols, instead of being their denotations, are then simply full predicates in natural language, and the epsilon symbol likewise merely abbreviates a part of ordinary speech: 'ϵxFx' means, for instance, 'that x which is F', '$\epsilon y(Fy.y \neq \epsilon xFx)$' means 'that other x which is F', and '$\epsilon x(Fx.(y)(Fy \supset y = x))$' means 'the one and only x which is F'. As we shall see more fully in section 4, the referent of 'ϵxFx', in any context of use, is then simply ϵxFx — just because it is a context of *use* — so there is no choice about *that*. It is the denotation of predicate terms, in other worlds and minds, which alone may vary, so the referents of epsilon terms are fixed, and only their (contingent) properties are a matter of choice.

On this basis we can now state, and prove several theorems of the epsilon calculus which are not in the predicate calculus. Indeed, the usefulness of the epsilon calculus first came to my notice when attempting to formalise an argument which Geach took from Strawson (Geach 1962, p125)

A man has just drunk a pint of sulphuric acid.
No-one who drinks a pint of sulphuric acid lives through the day.
So he'll not live through the day.

There is, of course, no way to formalise the pronoun 'he' in standard Fregean logic, with the conclusion as stated, and Geach took the extreme view that there was no proper argument of this apparent form. But the first premise '$(\exists x)(Mx.Dx)$' is now equivalent to 'Ma.Da' for a certain 'a', allowing that same ‘a’ to figure in the conclusion. Thus the above argument is, simply:

$M\epsilon x(Mx.Dx).D\epsilon x(Mx.Dx)$,
$(x)((Mx.Dx) \supset \neg Lx)$,
so, $\neg L\epsilon x(Mx.Dx)$.

And its proof is immediate, by instantiating the second premise appropriately, and then detaching its antecedent. Apart from the transcription into epsilon terms, therefore, the proof follows quite standard processes.

Now the pronoun in such an argument has been called by Evans an ‘E-type’ pronoun (Evans 1977), and it was soon evident to me that other E-type pronouns equally could be formalised using epsilon terms. For instance,

A man has just drunk a pint of sulphuric acid.
He won’t live through the day,

is clearly

$(\exists x)(Mx.Dx).\neg L\epsilon x(Mx.Dx)$,

rather than, say

$(\exists x)(Mx.Dx.\neg Lx)$,

which would be expressed using a relative pronoun in place of the previous personal one, as in

A man has just drunk a pint of sulphuric acid who will not live through the day.

Evans had argued independently that such pairs of forms are not equivalent, and here was the formal proof of it. For the two formal expressions are not equivalent, though the first entails the second. That entailment comes again from realising that the $(\exists x)(Mx.Dx)$ in the first is equivalent to $M\epsilon x(Mx.Dx).D\epsilon x(Mx.Dx)$, hence, by existential generalisation we get the result. The lack of entailment in the reverse direction arises because there is no requirement that we identify $\epsilon x(Mx.Dx)$ with $\epsilon x(Mx.Dx.\neg Lx)$ — the two epsilon terms in the two expressions.

But the distinction between relative and personal pronouns is not always of any moment, for it is easy to prove, by contrast, that there is a total equivalence between two comparable forms which include uniqueness clauses:

$(\exists x)(Mx.Dx.(y)((My.Dy) \supset y=x)).\neg L\epsilon x(Mx.Dx.(y)((My.Dy) \supset y=x))$,
$(\exists x)(Mx.Dx.(y)((My.Dy) \supset y = x).\neg Lx)$,

i.e.

There is a man who has just drunk a pint of sulphuric acid, and who alone has done that. He will not live through the day,
There is a man who has just drunk a pint of sulphuric acid, who alone has done that, and who will not live through the day.

The entailment from first to second follows the same pattern as before, since the first conjunct in the first statement provides, after replacement with its epsilon equivalent, a series of clauses which with the last conjunct generalise to the second statement. But now there is a requirement that the two associated epsilon terms be identified, because of the uniqueness clause. We have to identify

$$\epsilon x(Mx.Dx.(y)((My.Dy) \supset y = x).\neg Lx)$$

with

$$\epsilon x(Mx.Dx.(y)((My.Dy) \supset y = x)),$$

because the second statement entails the first conjunct in the first statement, and everything which is M and D has to be the same thing. So not just an entailment, but a full equivalence is obtained.

Sometimes, however, with related forms, an equivalence is obtained without identification of the associated epsilon terms. For we can also formalise personal pronouns which in no way are related to relative ones: what are called 'A-type' pronouns (Sommers 1982). Thus the sequence

A man is at the door. Oh! It is not a man, it's a woman, is

$(\exists x)(Mx.Rx).\neg M\epsilon x(Mx.Rx).W\epsilon x(Mx.Rx).$

The A-type pronoun 'it', in this case, is 'the man at the door' who, in the second sentence, is said, paradoxically, to be not a man. Now the second sentence, as a change of mind, makes perfectly good sense, on its own. For there is no logical requirement that $M\epsilon x(Mx.Rx)$ — or that $R\epsilon x(Mx.Rx)$ — since, together, they only constitute the contingent truth $(\exists x)(Mx.Rx)$. However, there is no way that this sentence could make sense if it was conjoined to the first sentence. And neither would we make sense if we joined two similar sentences with a relative pronoun. Thus there is no sense in

$(\exists x)(Mx.Rx.\neg Mx.Wx),$

But while that makes the two forms logically equivalent, they are so independently of whether the two associated epsilon terms refer to the same thing. The two forms are logically equivalent simply because each, separately, is inconsistent.

3. *Predicate calculus proofs*

Allen Hazen (Hazen 1987) has some points to make akin to those now to be made here. For he tries to correct traditional judgements about the pedagogic usefulness of the predicate quantification rules, by showing the advantage of the epsilon calculus ones. But Hazen still seems to regard epsilon terms as mere computing devices, since he does not address the main pedagogic question of how these terms are to be read when in his premises and conclusions. Moreover he does not anticipate the reduced semantics for the epsilon calculus sketched above, and considered more fully in section 4. But the difference between the epsilon calculus and the predicate calculus is, in the first place, a difference in their locus of application, not their mode of operation. This was the point of section 2. So even on the first matter of improving predicate proofs we can better Hazen. For we do not see epsilon terms as merely improved computing devices: we give them a sense, and thereby obtain a complete rationale for using the new rules.

I shall do that by focussing on the way quantification is handled in natural deduction systems, Hilbertian natural deduction systems, as well as Fregean ones. So consider first the practical difficulties which are commonly encountered with 'flagging' in many direct Fregean predicate proofs. Virginia Klenk is typical of many text book compilers when she gives the following set of quantifier rules (Klenk 1983, p295):

Universal Instantiation

$$\frac{(x)\varphi x}{/\therefore\ \varphi a}$$

Existential Instantiation

$$\frac{(\exists x)\varphi x}{/\therefore\ \varphi a} \quad \text{provided we flag 'a'.}$$

Universal Generalisation

$$\frac{\begin{array}{l}\rightarrow \text{flag a} \\ \varphi a\end{array}}{/\therefore\ (x)\varphi x}$$

Existential Generalisation

$$\frac{\varphi a}{/\therefore\ (\exists x)\varphi x}$$

to which she must add the following flagging restrictions:

R_1 A flagged letter may not appear, either in a formula or as a letter to be flagged, previous to the step in which it gets flagged.
R_2 A flagged letter may not appear either in the premises or in the conclusion of a proof.
R_3 A flagged letter may not appear outside the subproof in which it gets flagged.

Clearly, in natural deduction treatments of Hilbert's logic the troublesome rules of Existential Instantiation and Universal Generalisation may be replaced by

$$\frac{\varphi\epsilon x\neg\varphi x}{(x)\varphi x} \qquad \frac{(\exists x)\varphi x}{\varphi\epsilon x\varphi x}$$

and then not only the flagging restrictions, but also flagging itself may be abandoned. For instance, to prove '(x)Gx' from '(x)(Fx ⊃ Gx)' and '(x)Fx' we do not need to proceed

1. (x)(Fx ⊃ Gx)	Assp
2. (x)Fx	Assp
3. → flag a	F.S. (U.G.)

4. | Fa	U.I. 2, a/x
5. | Fa ⊃ Ga	U.I. 1, a/x
6. | Ga	M.P. 4, 5
7. (x)Gx	U.G. 6, a/x,

but can omit the 'flag a' line, and also the subproof notation, simply taking 'a' to be '$\epsilon x \neg Gx$'.

So much Hazen might have said, had he gone into details. But the further point we can now add shows *why* taking 'a' to be '$\epsilon x \neg Gx$' works. This is not just a means of deriving the answer, for unlike with 'Ga', '$G\epsilon x \neg Gx$' itself could be a conclusion drawn. So it must have an interpretation in its own right. The object $\epsilon x \neg Gx$ is a *putative counterexample* to the generalisation required. Indeed if anything is going to be not G then it is, by the epsilon axiom, so it is the *strongest* putative counterexample. That is why showing it is G is sufficient to establish nothing is not G.

The flagging restrictions in Klenk's procedure, of course, prevent any conditional conclusion of the form

$$(\exists x)\varphi x \supset \varphi a$$

being drawn from the rule of Existential Instantiation, which makes her '∴' in that rule quite inappropriate. The lack of entailment other writers respect by reworking this rule into another form; for instance Benson Mates states his rule of Existential Specification as follows (Mates 1965, p117):

> Suppose that $(\exists x)\Psi$ appears on line *i* of a derivation, that $\Psi\ \alpha/\beta$ appears (as a premise) on a later line *j*, and that φ appears on a still later line *k*; and suppose further that the constant β occurs neither in φ, Ψ, nor in any premise of line *k* other than $\Psi\ \alpha/\beta$; then φ may be entered on a new line. As premise-numbers of the new line take all those of lines *i* and *k*, except the number *j*.

But given $\beta = \epsilon\alpha\Psi\alpha$ there would be an entailment, in Hilbert's calculus, between the corresponding lines *i* and *j*, making the repetition of φ needless, and the reduction of premises automatic. And again the conclusion of the entailment on line j is a conclusion in its own right, so the term used is not a mere computing device. The entailment is of the kind we first considered, with '$\epsilon\alpha\Psi\alpha$' a phrase referring to the object in the antecedent existential clause.

In total, therefore, in place of, say,

{1}	1. (∃y)(x)Fxy	P
{2}	2. (x)Fxa	P
{2}	3. Fba	2 US
{2}	4. (∃y)Fby	3 EG
{2}	5. (x)(∃y)Fxy	4 UG
{1}	6. (x)(∃y)Fxy	1, 2, 5 ES,

we get, with a = ϵy(x)Fxy, an inference to the desired conclusion which terminates at line 5, since line 2 is a straight consequence of line 1. Moreover with b = ϵx¬(∃y)Fxy the move from line 4 to line 5 is immediate, without reference to any of Mates' restrictions on Universal Generalisation. Thus he says (Mates 1965, p108):

> The sentence $(\alpha)\varphi$ may be entered on a line if $\varphi\ \alpha/\beta$ appears on an earlier line and β occurs neither in φ nor in any premise of that earlier line; as premise numbers of the new line take those of the earlier line.

The presence of his restrictions means that Mates' move is again not an inference from the earlier premises. The earlier premises must be checked to ensure that β is sufficiently arbitrary, indeed is totally variable. But arbitrariness and variability is not an issue in the epsilon case, and indeed the appropriate β may occur in a premise from which the earlier line is derived. Thus in place of {1} and {2} above we may have the same conjoined with, say 'Pϵx¬(∃y)Fxy', and the same conclusion would be available. All that matters is that this epsilon term denotes the strongest putative counterexample to the proposed universal conclusion.

The complication of such standard rules as Universal Generalisation (in all its varieties) and Existential Instantiation and Specification is perhaps one reason why indirect proofs are often preferred. In Richard Jeffrey's account of Truth Trees, for instance (Jeffrey 1967, 1981, 1991), much of the clerical fastidiousness involved in applying the above predicate calculus direct rules is avoided, since he requires (Jeffrey 1967, p113) merely Universal Instantiation, mostly with 'old' names, and Existential Instantiation, invariably with 'new' ones. But the latter rule, again, is not a rule of *inference*, despite Jeffrey's description of it as such, being, amongst other things, context dependent because of the novelty clause. So the appeal of the indirect process is enhanced, once we replace Jeffrey's rule by a proper inference rule, and not simply because of the provision of context independence. For the resulting entailment enables us, invariably, to transform the indirect procedures so that they become direct ones.

Jeffrey, by contrast, would not be able to extend his direct proof concept of 'Coupled Trees' from the propositional to the predicate case, expressly because his Existential rule does not generate consequences. Thus he says (Jeffrey 1967, p93):

> In a direct proof, we start with the premises and try to get to the conclusion via sound rules of inference. In the corresponding modification of the tree method, we start a tree with the premises, omitting the conclusion:

```
     ✓ A⊃B
     ✓ A⊃¬B
        /      \
    ¬A          B
     /  \      /  \
  ¬A   ¬B   ¬A   ¬B
                   ×
```

> The three open paths represent all possible ways in which the premises can both be true, and each path contains the conclusion '¬A'. Then every possible way in which the premises can both be true is a way in which the conclusion would be true.

The general idea is that a conclusion is directly available if it is derivable in each open branch, using, when necessary, tautological premises of the form 'A ∨ ¬A'.

Now there is often a predicate calculus example of this, for instance, in the derivation of '(∃x)Gx' from '(∃x)Fx' and '(x)(Fx ⊃ Gx)' as follows:

```
✓ (∃x)Fx
  (x)(Fx ⊃ Gx)
      Fa
✓ Fa ⊃ Ga
   /     \
¬Fa       Ga
 ×      /    \
✓  ¬(∃x)Gx  (∃x)Gx
    (x)¬Gx
       ¬Ga
        ×
```

But outside Hilbert's calculus the further derivation of the conclusions 'Fa', 'Ga', would not be justified, because they are not given an independent sense. Hence only indirect proofs, in this mode, are available for the predicate calculus, since then any individual terms will remain mere computing devices.

But we have discussed the natural interpretation of conclusions of the form 'FϵxFx' and 'GϵxFx' above. And having given such statements a sense, indirect proofs using Truth Trees may always be converted to direct proofs, for instance by adding to the premises a disjunctive fork containing the conclusion to be drawn, in one branch, and its negation in the other. If and only if the branch with the negation of the conclusion in it closes, the conclusion will remain in every open branch. Thus the above two proofs may be normalised as

```
✓ A ⊃ B                        ✓ (∃x)Fx
✓ A ⊃ ¬B                         (x)(Fx ⊃ Gx)
   /   \                            /       \
¬A      A                     (∃x)Gx      ✓¬(∃x)Gx
       / \                                    Fa
     ¬A   B                              ✓ Fa ⊃ Ga
      ×  /  \                               /     \
       ¬A   ¬B                           ¬Fa       Ga
        ×    ×                            ×      (x)¬Gx
                                                   ¬Ga
                                                    ×
```

The predicate proof on the right needs one small, but significant modification, which will be given later.

Kit Fine, in his recent book on standard quantification rules, promises a further one dealing with the epsilon calculus, but it would seem he has not yet pursued the matter very far, since he does not foreshadow in any way (c.f. Fine 1985, pp 90-93) the radical simplifications and clarifications which it is now evident are immediately available. Moreover, the expedient of using arbitrary names in natural deduction treatments of Fregean predicate logic Fine almost exalts to a virtue, since he constructs a 'provisional' ontology of arbitrary objects for such names to range over (Fine 1985, Ch. 1). In direct contrast to Hazen, he therefore does not clearly see that these terms are mere computing devices with no other logical significance, introduced merely to facilitate deductions. Fine almost takes arbitrary names to be real names referring to a previously unknown breed of thing. It is to be hoped that when Hilbert's calculus comes to be used more frequently at an everyday level such metaphysical theorising will be excluded. Likewise giving terms *no* meaning, as with Hazen, must be ruled out. Certainly the individual terms in Klenk's and Mates' predicate proofs above are mere computing devices with no external significance. But Hilbertian epsilon terms refer to objects, and the objects they refer to are still objects in *this* world, i.e. not objects out of our normal experience.

4. *Predicate calculus metatheory*

Now not only are the quantification rules and their rationale easier to apply and understand once we come to use Hilbert's calculus, the specific way it appears we should do this turns out to have many other theoretical benefits. For if we adopt the modification of Jeffrey's Truth Tree approach described above, much of the meta-theory and semantics associated with both the predicate calculus and the epsilon calculus becomes more straightforward. Indeed the Truth Tree method encapsulates Herbrand's Theorem, and the proofs of this and Skolem's Theorem together with the First and Second Epsilon Theorems become more available.

Let us contrast, therefore, the tautology calculus (TC), developed, following Jeffrey, from the rules

$$\frac{\neg\neg p}{p} \qquad \begin{array}{c} p \,.\, q \\ | \\ p \\ q \end{array} \qquad \begin{array}{c} p \vee q \\ / \quad \backslash \\ p \quad\; q \end{array} \qquad \frac{\neg(p \,.\, q)}{\neg p \vee \neg q} \qquad \frac{\neg(p \vee q)}{\neg p \,.\, \neg q}$$

with the predicate calculus (PC) developed from these with

$$\frac{(x)Fx}{Fa} \qquad \frac{(\exists x)Fx}{Fb} \qquad \frac{\neg(x)Fx}{(\exists x)\neg Fx} \qquad \frac{\neg(\exists x)Fx}{(x)\neg Fx}$$

where 'b' must be 'new', and 'a' must be 'old' unless no other terms are available, and with the epsilon calculus (EC) developed from the former rules together with

$$\frac{(x)Fx}{Fa} \qquad \frac{(\exists x)Fx}{F\epsilon xFx} \qquad \frac{\neg(x)Fx}{(\exists x)\neg Fx} \qquad \frac{\neg(\exists x)Fx}{(x)\neg Fx}$$

where 'ϵxFx' is an individual expression for all predicates 'F' in the language, and where 'a' must be first '$\epsilon x\neg Fx$', and then any 'old' term. In all of these systems, by reducing closed formulae of the form '$P.\neg C$' to absurdity ($\times$), we prove '$P \supset C$' and thereby validate '$P \therefore C$'. But we can also derive C directly from P, if the tree

```
     P
    / \
   C   ¬C
```

closes on the right. Note that the need for the first instantiation from (x)Fx to be $F\epsilon x\neg Fx$, in the EC rules, was omitted in the normalised predicate proof above. It corresponds to the possibility that 'a' is not old, in the PC rules. But it is of more considerable moment, and its full significance will be discussed later.

Now the adequacy of the Truth Tree method depends on Herbrand's Theorem (c.f. Hodges 1977, §36). Herbrand's Theorem, giving a certain understanding of the existential quantifier, comes by generalisation upon such EC results as

$$(\exists y)(x)\neg Fyx \equiv (\exists y)\neg Fy\epsilon xFyx,$$

in which the epsilon term introduced on the right is called a Herbrand function. We can in this way remove all universal quantifiers from prenex formulae, and the additional fact that, as it might seem, existential quantifiers symbolise disjunctions of unspecified length, makes all such formulae in a sense equivalent to disjunctions. The exact statement of the additional fact comes through applying Jeffrey's 'Tree Theorem' i.e. König's Lemma (Jeffrey 1967, p151): if there is no open path through a tree there is some finite stage at which there is no open path. So in the case above, if no valuation makes

$$\neg(\exists y)\neg Fy\epsilon xFyx$$

true, that means that the tree

$$\begin{array}{l}(y)\ Fy\epsilon xFyx \\ \quad Fa\epsilon xFax \\ \quad Fb\epsilon xFbx \\ \quad\quad \ldots\end{array}$$

in which $a = \epsilon y\neg Fy\epsilon xFyx$, $b = \epsilon xFax$, and so on, closes in a finite length. But that reduces to absurdity a finite conjunction of instances of the matrix Fyx. So it makes tautologous a finite disjunction of instances of $\neg$Fyx, the matrix of the original existentially quantified expression. The result is reversible because each of those instances of $\neg$Fyx entails the original existentially quantified expression.

By generalisation, any predicate calculus formula is valid just so long as a finite disjunction of instances of its matrix (when it is in prenex normal form) is tautologous, i.e. if and only if a finite conjunction of instances of the negative of that matrix reduces to absurdity, as in indirect truth trees. Jeffrey's adequacy proof of the Truth Tree method supports this conclusion by means of a semantic argument about valuations. He shows (Jeffrey 1967, p171) that there is no open path through a finished tree if and only if no valuation makes all initial sentences true.

However, Jeffrey's argument for the 'if' part of his biconditional contains a difficulty, since he shows that each open path would describe some valuation in which the initial sentences are true through showing that if an initial sentence is false in a valuation so is one of its conclusions, and repeated application of this principle until a false atomic, or denial of atomic, sentence was obtained would stop that sentence being true in the valuation described by the path. So his rules only allow Jeffrey 'limited upward correctness' (Jeffrey 1967, p167), since in order to maintain his principle he has to say, with respect to his universal quantifier elimination rule, that the quantification there be limited merely to the universe of discourse of the path. Only that way can he ensure that a false universal premise has one of its drawn conclusions false. But this makes the interpretation of that quantifier variable.

Now the EC process is an improvement upon Jeffrey at this point, since the rules give total upward correctness in his sense. If it is false that everything is F, then for sure one of the stated consequences of '(x)Fx' is false, namely the immediate one, since if (x)Fx is false then $(\exists x)\neg Fx$, and so $\neg F\epsilon x \neg Fx$. Hence, in the EC rules, the universal quantifier does, invariably, mean 'everything'. That is ensured by the special provision for novelty.

But it is the contrast between the two sorts of rules which concerns us here, for it shows there is a need for an *interpretation* in the one case, where there is no need in the other. Lack of ability to specify the prime counterexample to a universal statement leaves Jeffrey saying that if such a statement is false just one of its instances is false, but with no way of ensuring that that instance is one which has actually been drawn, without restricting the universal claim just to the universe of discourse of the path. It thus seems necessary that there be a *model* for a universal statement which restricts it so that it does not cover strictly 'everything', but only, say two things. Jeffrey gives an example of this in Jeffrey 1991, p53:

```
1      ✓  (∃x)Px
2      ✓  (∃x)Qx
3    ✓  ¬(∃x)(Px.Qx)
4              Pa
5              Qb
6       (x) ¬(Px.Qx)
7      ✓     ¬(Pa.Qa)
8      ✓     ¬(Pb.Qb)
               /   \
9            ¬Pa  ¬Qa
              ×    /  \
10              ¬Pb  ¬Qb
                      ×
```

Jeffrey is here trying to invalidate "Someone's up, someone's down, so someone's up and down" and the middle path of the tree, thus defines, for him, a certain 'interpretation', C:

Domain:	{1, 2}
Extensions of 'a', 'b'	1, 2
Extension of 'P'	{1}
Extension of 'Q'	{2}

He remarks "...line 6... draws truth up from the lines (7 and 8) that come from it by UI — not because UI is complete (it isn't), but because in C the items named by 'a' and 'b' exhaust the domain, so that what's true of both of them is true of everything".

By contrast, using epsilon terms, the extensions of 'a' and 'b' are automatic, while the full extension of 'P' and 'Q' is irrelevant, and likewise whether 'a' and 'b' exhaust the universe. We need an extra line, before line 7, with an instantiation to $\epsilon x(Px.Qx)$ to ensure completeness, and then the tree shows that someone can be in, and someone can be out, and *everyone* can be not both in and out, so long as that one who is in (ϵxPx, i.e. a) is not out, and that one who is out (ϵxQx, i.e. b) is not in. By 'everyone' here is indeed meant everyone, although how many more there are than a and b is of no concern, and likewise of no concern is the further extension of 'P' and 'Q' — although, of course, nothing further is both P and Q.

A comparable consequence holds for the understanding of existential statements. These, it might seem, are unlike individual statements in that they merely say that one of a group of distinct objects has a certain property, without specifying which. An interpretation of an existential quantifier would therefore just specify the domain, i.e. extent of the quantification. But while we know from Herbrand's Theorem that provable existential statements relate to tautological disjunctions, our semantical interpretation of *any* exitential statement must now also respect the fact that we can say, for instance,

$$(\exists y)\neg Fy\epsilon xFyx \equiv \neg Fa\epsilon xFax,$$

and so the above disjunction of instances of $\neg Fyx$, it turns out, is only true because a certain specific one of its disjuncts is true. In Herbrand's Theorem, what is ultimately provable, therefore, is not a statement about a series of objects, as lack of epsilon terms might lead us to believe, but merely a statement about one object. So the alternative terms in the disjunction come in maybe just as nominal options for referring to that thing. After all, the disjunction in Herbrand's Theorem is not an exclusive one, bringing in necessarily real options, so all its alternatives might be the same. The existential quantifier is thus misleadingly thought of if it is taken to relate to a plural universe of individuals: an existentially quantified statement is just another statement about a single individual, merely a nameless one. Remember, we can only prove by logic that there is one thing, $(\exists x)(x = \epsilon y(y = y))$. So even a provable disjunction cannot be a statement which requires a more extensive universe of objects for it to be understood. In the disjunction of instances of $\neg$Fyx above, there is thus only necessarily a more extensive 'universe of discourse' in its proper sense: there may be more than one name in the distinct formulae, but that does not reflect on how many objects there are in the world. On this understanding, when doing logic with the epsilon calculus, though not when with the predicate calculus, we can stop thinking about the objects to which our language refers, indeed perhaps stop doing semantics in its traditional sense. For the truth values of formulae will then be given simply as they arise in deductions, along with the identity of any terms, including epsilon terms, found there.

Jeffrey's argument for the 'only if' part of his biconditional above shows again how the need for semantics arises in his case. For like the 'if' part it also contains a difficulty. The difficulty now is a consequence of the point we noticed before: because his rules do not always generate entailments Jeffrey can only get 'limited downwards correctness' (Jeffrey 1967, p165). Given a valuation of the premises, the conclusions of his rules are only guaranteed to be true either in that valuation or in some nominal variant of it. The epsilon rules, of course, get round this difficulty by not employing names, only descriptions, and being thereby totally downward correct: if there is an F then *that* F is F, irrespective of what name is used to refer to it. As a result there is no need for any 'valuation'. But the point also indicates where any 'choice' or 'interpretation' would be in any semantics for epsilon terms. It would be nowhere, for there is, for instance, absolutely *no choice* or interpretation regarding what entity 'ϵxFx' refers to in arguments of the form

$$\frac{(\exists x)Fx}{F\epsilon xFx.}$$

The 'ϵxFx' here refers not to *anything* which is F, but a certain (unknown) thing which is F brought up in the quantifier antecedent. The object involved is *given* in the quantifier introduction, and so any alternatives in regard to it are merely nominal ones deriving from ignorance of that thing's name.

A modification of Jeffrey's charming story about Zeus will illustrate this, at the same time as leading us out of the traditional choice-function semantics for epsilon terms. Thus Jeffrey says in part (1st ed. p170):

> If the rule in question was one of those for the quantifiers and was not vacuous, the top list has only one descendant: a box containing a single sentence of the form ... *n* ... where *n* is some name. In this case, Zeus adds the sentence to the path and addresses himself to the question of whether he shall retain the valuation of stage 1 or replace it by a nominal variant. The answer depends on whether the name n is new to the path, as it *must* be if the rule in question was the one for (∃x), and as it may be if the rule was the one for (x). If the name is not new, he keeps the valuation of stage 1; if it is new, he replaces that valuation by a nominal variant in which the name n is assigned a referent that makes ... n ... true but which otherwise agrees with the valuation of stage 1. There is sure to be such a referent for n since the premise (x)...x... or (∃x)...x... is true in the valuation of stage 1.

Now in line with this, and the traditional semantics of epsilon terms, we might say, in the EC case, that Zeus, coming upon a sentence of the form 'FϵxFx', derived from one of the form '(∃x)Fx', should again arbitrarily assign to the epsilon term a member of the class {x | Fx}. But Jeffrey's conception of a valuation (Jeffrey 1967, p158) requires that the truth value of any atomic sentence left open by an open path through a tree be false. And the presence of epsilon terms even in proofs between epsilon-free formulae already upsets that conception, since, on it, the epsilon terms might have no identity. For no formulae like 'ϵxFx = a' need be present, and so all could, more often than not, be false. But a slackening of Jeffrey's rule to allow the referents of epsilon terms to be chosen merely in accord with their quantifier introduction (i.e. for ϵxFx to be just some F, if (∃x)Fx is given, and just anything if ¬(∃x)Fx is given), is not sufficient to cover the full case where proofs are between premises and conclusions which include epsilon terms. For these further formulae may restrict any such choice in unpredictable ways. However we are saved from this difficulty, once we take a fresh look at the matter, since there is no requirement, in the logic of the case, that Zeus makes any choice. Indeed any choice might need revision, once further premises involving epsilon terms are brought in, so the full, or fuller, identity of ϵxFx has in general to be left to emerge through the arrival of formulae involving that term.

If we do still speak of a valuation, therefore, there is no requirement that any valuation produces determinate truth values for all sentences. Indeed, we already know the *referent* of any term, since 'ϵxFx' invariably refers to ϵxFx, in any context of use. Hence all that remains to be determined are the extensions of predicate terms, including the identity symbol. But the initial sentences (if they are consistent) are what settles this, and so are what determines the identity, just as much as the other properties of ϵxFx. On that understanding we may remain in ignorance of that identity, even though a limited series of nominal or real alternatives might be left open regarding it. But any such identity is irrelevant to the logic — unless specifically it is required by the initial sentences. Separating the identity of ϵxFx from the referent of 'ϵxFx' is perhaps the most crucial point needing to be understood, in coming to use the epsilon calculus.

Certainly when no valuation satisfies the initial sentences any choice for the extensions of predicates is as good as the rest. But in that case there is no consistent way to make every choice, so all choices are as good as the rest only because none of them is any good. And when some valuation does satisfy the initial sentences, as above, then while the extensions of predicates have to be selected with more care, to preserve the consistency, still then we must let the natural generation of the truth tree determine it, i.e. we must let the 'choice' be made for us. A choice is only available for what is irrelevant.

5. *Epsilon calculus metatheory*

The point behind this, about the merely apparent choice with countermodels may be put more formally by considering an EC advance on the First Epsilon Theorem. The First Epsilon Theorem states that if C is provable in PC then ($C_1 \vee C_2 \vee \ldots \vee C_n$) is provable in EC, where 'C' is a prenex formula, and the 'C_i's are certain substitution instances of 'C's matrix. It is therefore a version of Herbrand's Theorem, which merely says that if C is provable in PC then a disjunction of instances of its matrix, in a language enriched with Herbrand functions, is provable in TC, i.e. is tautologous.

But for EC a much stronger result is available than even the First Epsilon Theorem, as we have seen. For if C is provable in PC then C_i is provable in EC, for a certain i, simply by using the epsilon quantifier equivalences. But these equivalences can be applied in the reverse way. For it is also the case that if $\neg$C is consistent then the negation of that C_i is *consistent*, and so $\neg C_i$ will provide the forced countermodel to C being provable.

But while one of the $\neg C_i$'s will be true, it will essentially employ epsilon terms, and so only in EC can we specify it. In PC we cannot do so, and the PC rules, in such a case, will only produce a set of nominal variants for what is true. If we think, from this lack of specificity, that the alternative which is true *cannot* be specified, the PC procedures may make it appear that there is a choice where in fact there is none. For the variants are only nominal, and do not offer real alternatives. And so misconceptions about the semantics of the epsilon calculus may also arise, as a result. For the fact that epsilon calculus formulae do not need an interpretation will be obscured, if the contrary fact that predicate calculus formulae do need an interpretation is taken as an invariable principle. Certainly it is not apparent *on their faces* what PC formulae are about, but EC formulae only need a *reading* to be *understood* for predicate terms — so even that translation needn't be explicitly given.

Notice that there is an even more specific result available in EC than the stronger meta-logical result above. For, unlike with the First Epsilon Theorem, Herbrand's Theorem, and the stronger result relating a proof of C to a proof of C_i, the reduction of any prenex formula C to an equivalent quantifier free instance is a straight thesis of EC. Equally we could introduce Skolem functions given, for instance, by the epsilon term, in

$$(y)(\exists x)Fyx \equiv (y)Fy\epsilon xFyx.$$

But the existence of Skolem functions being thus demonstrated by means of equivalences advances Skolem's actual historical theorem considerably. Skolem would require merely that something with the form of the right hand side here was provable when the left hand side was provable, the proofs being in different theories. He therefore presented a meta-logical theorem, like those before. But a version of Skolem's Theorem is provable non-metalogically, through theses like the above, in the one calculus.

And the detail of that same-calculus proof also reveals an important fact about the present epsilon calculus, which gives us direct access to the Second Epsilon Theorem. The Second Epsilon Theorem shows that the epsilon calculus is a conservative extension of the predicate calculus, which was our point of departure at the beginning of this paper. But now we have a very direct proof of it. For the proof of the Skolem resolution goes (with a as before, and $c = \epsilon y\neg(\exists x)Fyx$):

✓ ¬((y)(∃x)Fyx ≡ (y)FyεxFyx)

/	\
/	(y)FyεxFyx
(y)(∃x)Fyx	✓ ¬(y)(∃x)Fyx
✓ ¬(y)FyεxFyx	✓ (∃y)¬(∃x)Fyx
✓ (∃y)¬FyεxFyx	✓ ¬(∃x)Fcx
¬FaεxFax	(x)¬Fcx
(∃x)Fcx	¬FcεxFcx
✓ (∃x)Fax	FaεxFax
FaεxFax	FcεxFcx
×	×

The point to note about this procedure is that the epsilon calculus rules, while they provide us with a result about an epsilon expression 'εxFyx' which involves a variable, and which hence is a function, themselves only involve the introduction of constants: 'a' and 'c' are just *names*. Coming from closed formulae, such terms have all their variables bound. This means that if the premises and conclusion are epsilon-free there is no way that non-constant epsilon terms can enter at any place into a deduction, and so one of the major difficulties in proving the Second Epsilon Theorem is overcome.

The Second Epsilon Theorem states that any EC deduction, with epsilon-free premises and conclusion, may be replaced by a PC deduction between the same formulae. But in axiomatic treatments of the epsilon calculus, even if the premises and conclusion in a deduction are epsilon-free, other, more complex, epsilon expressions may enter, by means of substitutions into the axioms (Leisenring 1969, p65), and so the fact that any such EC deduction may be transformed into a PC one is very hard to prove. But the transformation required in the present setting is quite patent: simply change to 'new' names all epsilon terms introduced first in the quantifier elimination rules.

6. *Conclusion*

In summary, therefore, it has emerged that Hilbert's epsilon calculus has been shrouded in quite unnecessary mystery. For the history of this calculus might have made it seem a collection of complex, even disordered theorems, unstructured to any great purpose. But this jungle, if that is how it was perceived, now turns out to have been more a reflection on the disadvantaged point of view from which the epsilon calculus was traditionally observed. In fact it is Fregean predicate logic which has the obfuscating, bureaucratic structure, and the clarity and beauty of Hilbert's calculus, in a natural deduction setting, may, by contrast, seem almost beyond belief. Correctly viewed it clearly presents a straightforward but extensive theory of considerable consequence to logic. It embraces important additional theorems not available in Fregean predicate logic, and its proofs of standard Fregean theorems are both shorter and more comprehensible. The metatheorems of the epsilon calculus are also much shorter, when approached using Truth Trees; but the metatheory of such trees itself comes to be improved through their application in the new context. As a result, the traditional semantics of Fregean predicate logic is seen to be highly specific to it. Indeed, it now appears, there is no need for such a semantics of Hilbert's epsilon calculus.

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