Logique & Analyse **127-128** (1989), 241-245

ON PARADOXES IN NAIVE SET THEORY

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In 1979 Meyer, Routley and Dunn found a new form of Russell's paradox, different from that found by Curry. By means of elementary syntactic transformations it is possible to clarify the relation which these paradoxes have with each other. Also, we will find new relationships with important metatheoric results relative to Peano's arithmetic (PA). We will assume that the set theories mentioned in this paper satisfy the following conditions: A) The set of theorems is closed under substitution of equivalents. B) All instances of the principle of abstraction are theorems (the principle of abstraction states that $t \in \{x\colon Fx\} \leftrightarrow Ft$ for any term t and any open well-formed formula Fx).

From Curry (1942) to Curry (1960)

'$\vdash \alpha$' is a means of abbreviating the sentence 'α is a theorem', and we consider a set theory T_1 for which the following is fulfilled: a_1) all wffs of the form $\neg(\alpha) \rightarrow \neg(\alpha)$ are theorems (in short, $\vdash \neg(\alpha) \rightarrow \neg(\alpha)$); b_1) if $\vdash \alpha \rightarrow \neg(\alpha)$ then $\vdash \neg(\alpha)$ (in short, $\vdash \alpha \rightarrow \neg(\alpha) \Rightarrow \vdash \neg(\alpha)$); c_1) if $\vdash \alpha$ and $\vdash \neg(\alpha)$ then $\vdash \beta$ (in short, $\vdash \alpha, \vdash \neg(\alpha) \Rightarrow \vdash \beta$). This is sufficient to arrive at Russell's paradox as follows:
let $R = \{x\colon \neg x \in x\}$, by B) (1) $R \in R \leftrightarrow \neg R \in R$, by a_1) (2) $\neg R \in R \rightarrow \neg R \in R$, from (1) and (2) by A) (3) $R \in R \rightarrow \neg R \in R$, from (3) by b_1) (4) $\neg R \in R$, from (1) and (4) by A) (5) $R \in R$, finally from (4) and (5) by c_1) (6) q. The interest of this derivation lies in its possible syntactic transformations. If we uniformly substitute in it each expression of the form $\neg(\alpha)$ for $\alpha \rightarrow \beta$, the result is Curry's paradox 1942 in the T_2 system, obtained from T_1 using the same transformation. (For T_2 then: a_2) $\vdash (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)$; b_2) $\vdash \alpha \rightarrow (\alpha \rightarrow \beta) \Rightarrow \vdash \alpha \rightarrow \beta$; c_2) $\vdash \alpha, \vdash \alpha \rightarrow \beta \Rightarrow \vdash \beta$.) If in this we uniformly substitute each expression of the form $\alpha \rightarrow (\beta \rightarrow \gamma)$ for $(\beta \rightarrow \alpha) \rightarrow (\beta \rightarrow \gamma)$ the result is a new version of Curry's paradox 1942 in the T^1_2 system obtained from T_2 by applying the same transformation. For

T^I_2 then: $a^I_2)$ $\vdash (\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$; $b^I_2)$ $\vdash (\alpha \to \alpha) \to (\alpha \to \beta) \Rightarrow \vdash \alpha \to \beta$; $c^I_2)$ $\vdash \alpha$, $\vdash \alpha \to \beta \Rightarrow \vdash \beta$. Now, let us uniformly substitute in the above mentioned T^I_2 paradox each expression of the form $\alpha \to \beta$ for $\alpha \to (\alpha \to \beta)$. The result is another version of Curry's paradox in the T^{II}_2 system obtained from T^I_2 by applying the same transformation. In effect, for T^{II}_2 we have the following:

$a^{II}_2)$ $\vdash (\alpha \to (\alpha \to (\alpha \to (\alpha \to \beta)))) \to ((\alpha \to (\alpha \to (\alpha \to (\alpha \to \beta)))) \to (\alpha \to (\alpha \to \beta)))$

$b^{II}_2)$ $\vdash (\alpha \to (\alpha \to \alpha)) \to ((\alpha \to (\alpha \to \alpha)) \to (\alpha \to (\alpha \to \beta))) \Rightarrow \vdash \alpha \to (\alpha \to \beta)$

$c^{II}_2)$ $\vdash \alpha$, $\vdash \alpha \to (\alpha \to \beta) \Rightarrow \vdash \beta$

and the derivation is: $R = \{x: x \in x \to (x \in x \to q)\}$

(1) $R \in R \leftrightarrow (R \in R \to (R \in R \to q))$ by B)

(2) $(R \in R \to (R \in R \to (R \in R \to (R \in R \to q)))) \to ((R \in R \to (R \in R \to (R \in R \to (R \in R \to q)))) \to (R \in R \to (R \in R \to q)))$ by $a^{II}_2)$

(3) $R \in R \to (R \in R \to R \in R)) \to ((R \in R \to (R \in R \to R \in R)) \to (R \in R \to (R \in R \to q)))$ from (1) and (2) by A)

(4) $R \in R \to (R \in R \to q)$ from (3) by $b^{II}_2)$

(5) $R \in R$ from (1) and (4) by A)

(6) q from (4) and (5) by $c^{II}_2)$

The interest of this derivation lies in one of its possible syntactic transformations. It is notable that if we uniformly substitute in the T^{II}_2 paradox each expression of the form $(\alpha \to (\alpha \to \beta)) \to ((\alpha \to (\alpha \to \beta)) \to (\alpha \to \gamma))$ for $\beta \to \gamma$ the result is Curry's paradox in the T^{III}_2 system obtained from T^{II}_2 by applying the same transformation, it is, Curry's paradox 1960! In effect, for T^{III}_2 we have: $a^{III}_2)$ $\vdash (\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$; $b^{III}_2)$ $\vdash \alpha \to (\alpha \to \beta) \Rightarrow \vdash \alpha \to (\alpha \to \beta)$; $c^{III}_2)$ $\vdash \alpha$, $\vdash \alpha \to (\alpha \to \beta) \Rightarrow \vdash \beta$. And, obviously, $b^{III}_2)$ is dispensable with.

The notion of syntactic transformation is in any case more general than what is suggested by the previous examples. Instead of operating on the expressions of an object language, one may directly act on those of a metalanguage. In this sense, if we uniformly substitute in the T_2 paradox each expression of the form $\vdash \alpha \Rightarrow \vdash \beta$ for $\vdash \alpha \leftrightarrow \beta$ the result is a stronger version of Curry's paradox 1942 in the T^{IV}_2 system obtained from T_2 by applying the same transformation (For T^{IV}_2 then: $a^{IV}_2)$ $\vdash (\alpha \to \beta) \to (\alpha \to \beta)$; $b^{IV}_2)$ $\vdash (\alpha \to (\alpha \to \beta)) \leftrightarrow (\alpha \to \beta)$; $c^{IV}_2)$ $\vdash \alpha$, $\vdash \alpha \to \beta \Rightarrow \vdash \beta$.) The interest of this will be seen in the coming section.

On Meyer, Routley and Dunn's paradox

The above systems T_2, T^{III}_2 and T^{IV}_2 may be related in several ways to the Meyer-Routley-Dunn paradox.

I) Let us uniformly substitute each expression of the form $\alpha \rightarrow (\beta \rightarrow \gamma)$ for $\beta \rightarrow (\alpha \rightarrow \gamma)$ in the above T_2 derivation. The result is a new version of Curry's paradox 1942 in the T_3 system obtained from T_2 by applying the same transformation. For T_3 then: $a_3)$ $\vdash \alpha \rightarrow ((\alpha \rightarrow \beta) \rightarrow \beta)$; $b_3)$ $\vdash \alpha \rightarrow (\alpha \rightarrow \beta) \Rightarrow \vdash \alpha \rightarrow \beta$; $c_3)$ $\vdash \alpha$, $\vdash \alpha \rightarrow \beta \Rightarrow \vdash \beta$. T_3 is of particular interest because $a_3)$ is clearly the Modus Ponens axiomatic schema in implicative form (compare with $a_2)$). This suggests an additional syntactic transformation. It is notable that if we uniformly substitute in the anterior T_3 paradox each expression of the form $\alpha \rightarrow (\beta \rightarrow \gamma)$ for $(\alpha \,\&\, \beta) \rightarrow \gamma$ the result is Meyer, Routley and Dunn's paradox in the T_4 system obtained from T_3 by applying the same transformation. For T_4 then: $a_4)$ $\vdash (\alpha \,\&\, (\alpha \rightarrow \beta)) \rightarrow \beta$; $b_4)$ $\vdash (\alpha \,\&\, \alpha) \rightarrow \beta \Rightarrow \vdash \alpha \rightarrow \beta$; $c_4)$ $\vdash \alpha$, $\vdash \alpha \rightarrow \beta \Rightarrow \vdash \beta$. T_4 is besides a weaker system than that employed by Meyer, Routley and Dunn, in which instead of $b_4)$ appears the idempotency condition of conjunction.

II) Let us uniformly substitute each expression of the form $\alpha \rightarrow (\beta \rightarrow \gamma)$ for $\beta \rightarrow (\alpha \rightarrow \gamma)$ in the above T^{III}_2 derivation. The resulting derivation is a new version of Curry's paradox 1960 in the T^{III}_3 system, obtained from T^{III}_2 by applying the same transformation. And if in the paradox in T^{III}_3 we uniformly substitute each expression of the form $\alpha \rightarrow (\beta \rightarrow \gamma)$ for $(\alpha \,\&\, \beta) \rightarrow \gamma$ the result is a new version of the Meyer-Routley-Dunn paradox in the T^{III}_4 system, obtained from T^{III}_3 by applying the same transformation. For T^{III}_4 then: $a^{III}_4)$ $\vdash (\alpha \,\&\, ((\alpha \,\&\, \alpha) \rightarrow \beta)) \rightarrow \beta$; $c^{III}_4)$ $\vdash \alpha$, $\vdash (\alpha \,\&\, \alpha) \rightarrow \beta \Rightarrow \vdash \beta$. T^{III}_4 is weaker than the system employed by Meyer, Routley and Dunn and it only contains, in addition to A) and B) above, a variant of the Modus Ponens rule with the corresponding axiomatic scheme.

III) The above mentioned substitutions in I) and II) may be realized in a similar way on the T^{IV}_2 derivation of the previous section. It is easy to see what derivation would result and what would be the corresponding T^{IV}_4 system. If finally we uniformly substitute each expression of the form $(\alpha \rightarrow \gamma) \leftrightarrow (\beta \rightarrow \gamma)$ for $\alpha \leftrightarrow \beta$ we get the Meyer-Routley-Dunn paradox in the system used by them and that we will call T^{IV}_5. In this system: $a^{IV}_5)$ $\vdash (\alpha \,\&\, (\alpha \rightarrow \beta)) \rightarrow \beta$; $b^{IV}_5)$ $\vdash (\alpha \,\&\, \alpha) \leftrightarrow \alpha$; $c^{IV}_5)$ $\vdash \alpha$, $\vdash \alpha \rightarrow \beta \Rightarrow \vdash \beta$.

Metamathematical connections

A final observation about certain metamathematical connections of the above paradoxes. It is well known that a proof can be given of the weak theorem of the undefinability of truth (that is, only using one half of the convention T) in such a way that on suppressing in it all appearances of the arithmetical predicate of truth, we obtain Russell's paradox in T_1. Van Benthem has shown explicitly how we arrive at Curry's paradox in T_2 by suppressing all appearances of the arithmetic predicate of provability 'Bew' in Löb's theorem. What can be said with regard to Meyer, Routley and Dunn's paradox? Let $\ulcorner \alpha \urcorner$ stand for the numeral corresponding to Gödel's number of the wff α. We can then consider the following derivation: by the fixed point theorem (1) $A \leftrightarrow \text{Bew}\ (\ulcorner A \rightarrow B \urcorner)$, from $\vdash \text{Bew}\ (\ulcorner A \urcorner) \rightarrow A$ by the propositional calculus (PC) (2) $(A\ \&\ \text{Bew}\ (\ulcorner A \rightarrow B \urcorner)) \rightarrow B$, from (1) and (2) by PC (3) $(A\ \&\ A) \rightarrow B$, from (3) by PC (4) $A \rightarrow B$, from (4) by the rule which lets us go from $\vdash A$ to $\vdash \text{Bew}\ (\ulcorner A \urcorner)$ (5) $\text{Bew}\ (\ulcorner A \rightarrow B \urcorner)$, from (1) and (5) by PC (6) A, from (4) and (6) by Modus Ponens (7) B. This paradox is basically a version of Gödel's theorem of incompleteness. In effect, the rule $\vdash A \Rightarrow\ \vdash \text{Bew}\ (\ulcorner A \urcorner)$ may be proved in PA ($\vdash_{PA} A \Rightarrow\ \vdash_{PA} \text{Bew}\ (\ulcorner A \urcorner)$). In addition, given the meaning of Bew (in the standard model) $\text{Bew}\ (\ulcorner A \urcorner) \rightarrow A$ is true. Therefore if PA is consistent $\nvdash_{PA} \text{Bew}\ (\ulcorner A \urcorner) \rightarrow A$. This is, if PA is consistent then it is incomplete: there are sentences (obviously true) with the form $\text{Bew}\ (\ulcorner A \urcorner) \rightarrow A$ that are not provable. Now, if in the previous derivation we suppress all appearances of 'Bew', we obtain Meyer, Routley and Dunn's paradox in the system T_4 (or $T^{IV}{}_5$).

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ACKNOWLEDGEMENTS

I would like to thank Idoia Mardaras Aginaga for a helpful suggestion to write this paper.

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