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A REPRESENTATION OF INTUITIONISTIC LOGIC IN PARTIAL INFORMATION LANGUAGE

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In [1], a generalization of Data Semantics (see [2]), called partial information logic (*PIL*) is presented, together with an embedding into modal logic S4 which generalizes the embedding of Data Semantics in the modal system S4.1 appearing in [2]. On the other hand, a well known result of Gödel shows that propositional intuitionistic logic can be embedded into the modal system S4 (see [3]). Our purpose here is to show, using both modal embeddings into S4 mentioned above, that intuitionistic propositional logic can be represented in *PIL* in a very simple and natural way: just replace every intuitionistic negation "¬" by the expression "¬*MAY*".

1. *The modal embeddings*

First of all, let's recall the modal embeddings for *PIL* and intuitionistic logic. For each *PIL*-wff A we define a modal formula T(A) as follows:

$T(p)=\Box p$
$T(\neg p)=\Box\neg p$
$T(\neg\neg A)=T(A)$
$T(A\wedge B)=T(A)\wedge T(B)$
$T(\neg(A\wedge B))=T(\neg A)\vee T(\neg B)$
$T(A\rightarrow B)=\Box(T(A)\rightarrow T(B))$
$T(\neg(A\rightarrow B))=\Diamond(T(A)\wedge T(\neg B))$
$T(\mathit{MAY}\ A)=\Diamond T(A)$
$T(\neg\mathit{MAY}\ A)=\neg\Diamond T(A)$
$T(\mathit{MUST}\ A)=\Box\Diamond T(A)$
$T(\neg\mathit{MUST}\ A)=\Diamond\Box T(\neg A)$

It should be noticed that in *PIL*, $A\vee B$ is defined as $\neg(\neg A\vee\neg B)$, and from that definition it follows that $T(A\vee B)=T(A)\vee T(B)$, and $T(\neg(A\vee B))=T(\neg A)\wedge T(\neg B)$.

Now, the following result is proven in [1]:

A *PIL*-wff A is valid in the class of all partial information models iff T(A) is S4-valid.

A similar result for propositional intuitionistic logic appears in [3]: for every intuitionistic formula A, a modal formula A^m is defined and then proven that A is intuitionistically valid iff A^m is S4-valid. A^m is defined as follows:

$$p^m = \Box p$$
$$(A \vee B)^m = A^m \vee B^m$$
$$(A \wedge B)^m = A^m \wedge B^m$$
$$(A \to B)^m = \neg(A^m \to B^m)$$
$$(\neg A)^m = \Box \neg A^m$$

2. *Encoding intuitionistic logic in PIL*

We now propose a simple translation P of intuitionistic formulas to *PIL*, which is defined as follows:

$$P(p) = p$$
$$P(A \wedge B) = P(A) \wedge P(B)$$
$$P(A \vee B) = P(A) \vee P(B)$$
$$P(A \to B) = P(A) \to P(B)$$
$$P(\neg A) = \neg \mathit{MAY}\, P(A)$$

From this definition, it easily follows:

Lemma: for every intuitionistic formula A, $T(P(A)) = A^m$.
Proof: an easy and simple induction. For the basis case, $T(P(p)) = T(p) = \Box p = p^m$. The induction step is simple routine:

$$T(P(A \wedge B) = T(P(A) \wedge P(B)) = T(P(A)) \wedge T(P(B)) = A^m \wedge B^m = (A \wedge B)^m.$$
$$T(P(A \vee B) = T(P(A) \vee P(B)) = T(P(A)) \vee T(P(B)) = A^m \vee B^m = (A \vee B)^m.$$
$$T(P(A \to B) = T(P(A) \to P(B)) = \Box(T(P(A)) \to T(P(B))) = A^m \to B^m = (A \to B)^m.$$
$$T(P(\neg A)) = T(\neg \mathit{MAY} P(A)) = \neg \Diamond T(P(A) = \neg \Diamond A^m = \Box \neg A^m = (\neg A)^m.$$

$\blacksquare$

From this lemma and the results in [1] and [3] quoted above, it readily follows:

Theorem: A propositional formula A is intuitionistically valid iff P(A) is valid in *PIL*.
Proof: A is intuitionistically valid iff A^m is S4-valid, as proved in [3]. By the lemma above, $A^m = T(P(A))$. But, according to [1], T(P(A)) is S4 valid iff P(A) is valid in *PIL*. ■

Some remarks about the translation given above

The theorem established above should not surprise us at all. Kripke's semantics for intuitionistic logic is usually accompanied by heuristical motivations (see [3] and [4]) which consider the activity of an idealized mathematician who extends his knowledge along the time. Kripke models reflect the possible ways in which such knowledge may grow. Possible patterns of knowledge growth is just what Data models try to encode (see [2]), and Data language and its generalized version *PIL* are intended to represent facts about knowledge growth. So, our representation of intuitionistic logic within *PIL* is a very natural result.

It seems not difficult to extend the same result to quantified intuitionistic and partial information languages, provided that a suitable quantified *PIL* is developed. As pointed out in [1], a nested domains condition seems senible for quantified *PIL* (not only known facts should increase, but also the set of known objects should grow). This condition agrees with Kripke's semantics for quantified modal logic, which supports our claim that no difficulty should arise in the extension of our result to the quantificational case.

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Bibliography

[1] Barba Escriba, J.: "A modal embedding for partial information semantics", *Logique et Analyse*, 1989, 125-126, 131-137.
[2] van Benthem, J.: "Partiality and nonmonotonicity in classical logic", *Logique et Analyse*, 1986, 114, 225-247.
[3] van Dalen, D.: "Intuitionistic logic", in Gabbay and Guenthner (eds.) *Handbook of Philosophical Logic*, vol. III, Reidel Publishing Company, Dordrecht, 1986.

[4] Kripke, S.: "Semantical analysis of intuitionistic logic", in Crossley and Dummet (eds.) Formal systems and recursive functions, North-Holland, Amsterdam, 1965.