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A MODAL EMBEDDING FOR PARTIAL INFORMATION SEMANTICS

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In [1], a modal embedding of data semantics into the modal logic S4.1 (S4 plus the McKinsey axiom $\Box \Diamond A \rightarrow \Diamond \Box A$) is presented. (The same embedding also appears in [2], within a useful overview of the main topics in partial logics). The existence of maximal points in data models plays an important role in the proposed embedding. The author says that it would be interesting to extend such a reduction to the case in which no maximal point is postulated, and conjectures that the result should be an embedding into S4. The aim of this paper is to show how the conjectured embedding can be carried out.

1. *Partial information Semantics*

We assume a formal propositional language **PIL** (partial information language) whose logical symbols are the unary connectives $\neg$, MAY and MUST, and the binary ones $\wedge$ and $\rightarrow$. Partial information models are structures $D = \langle I, \sqsubseteq, \models, =\!\mid \rangle$, where $I \neq \phi$, $\sqsubseteq$ is a partial order on I, and $\models$ and $=\!\mid$ are relations between elements of I and propositional variables which satisfy the following restrictions: for any $i \in I$ and any propositional variable p it is not the case that both $i \models p$ and $i =\!\mid p$ hold simultaneously, and for any $j \in I$ such that $i \sqsubseteq j$, $i \models p$ implies $j \models p$, and $i =\!\mid p$ implies $j =\!\mid p$. D is a data model iff every maximal chain in the partial ordered set $\langle I, \sqsubseteq \rangle$ contains a greatest element j such that for every p either $j \models p$ or $j =\!\mid p$. Such elements are called maximal elements of D.

Relations $\models$ and $=\!\mid$ are extended to every wff by the following clauses:

$i \models \neg A$ iff $i =\!\mid A$, and $i =\!\mid \neg A$ iff $i \models A$.
$i \models A \wedge B$ iff $i \models A$ and $i \models B$, and $i =\!\mid A \wedge B$ iff $i =\!\mid A$ or $i =\!\mid B$.
$i \models A \rightarrow B$ iff for all j, $i \sqsubseteq j$, $j \models A$ only if $j \models B$.
$i =\!\mid A \rightarrow B$ iff for some j, $i \sqsubseteq j$, $j \models A$ and $j =\!\mid B$.
$i \models$ MAY A iff for some j, $i \sqsubseteq j$, $j \models A$.
$i =\!\mid$ MAY A iff for all j, $i \sqsubseteq j$, not $j \models A$.

$i \models$ MUST A iff for all j, $i \sqsubseteq j$, there is some k, $j \sqsubseteq k$ such that $j \sqsubseteq k$ and $k \models A$.
$i \dashv$ MUST A iff there is some j, $i \sqsubseteq j$, such that for every k, $j \sqsubseteq k$ implies $k \dashv A$.

Notice that if D is a data model, the clauses for MUST are equivalent to those formulated in [1]:

$i \models$ MUST A iff for all maximal j, $i \sqsubseteq j$, $j \models A$.
$i \dashv$ MUST A iff for some maximal j, $i \sqsubseteq j$, $i \dashv A$.

We have formulated more general clauses because those in [1] make no sense when D is not a data model.

2. *From information models to modal models*

Given a model $D = \langle I, \sqsubseteq, \models, \dashv \rangle$ we can obtain a modal model $M_D = \langle W, R, V \rangle$ as follows:

$$W = \{u_i\colon i \in I\} \cup \{v_i\colon i \in I \text{ and for some p neither } i \models p \text{ nor } i \dashv p\}.$$

(where, for each $i, j \in I$, u_i, v_i are any two different objects and distinct from u_j, v_j).R is the reflexive and transitive closure of the set

$$\{\langle u_i, u_j \rangle\colon i \sqsubseteq j\} \cup \{\langle u_i, v_i \rangle, \langle v_i, u_i \rangle\colon v_i \in W\}.$$

The valuation V is defined by:

$V(p, u_i) = 1$ iff $i \not\dashv p$,
$V(p, v_i) = 1$ iff $i \models p$.

Notice that, given D and M_D, for every $i \in I$ and every propositional variable p, $i \models p$ iff $V(\Box p, u_i) = V(\Box p, v_i) = 1$, and $i \dashv p$ iff $V(\Box \neg p, u_i) = V(\Box \neg p, v_i) = 1$. In fact, $i \models p$ iff for all $j \in I$ such that $i \sqsubseteq j$, $j \models p$, iff $V(\Box p, u_j) = V(\Box p, v_j) = 1$, iff for every $w \in W$ such that $u_i R w$ (or $v_i R w$: recall that $u_i R w$ iff $v_i R w$) $V(p, w) = 1$, because w must be either u_j or v_j, for some j such that $i \sqsubseteq j$, iff $V(\Box p, u_i) = V(\Box p, v_i) = 1$. The case $i \dashv p$ is proven in a similar way.

3. *From modal models to partial information models*

The relation R in models M_D obtained from partial information models D is reflexive and transitive. Now we show how a partial information model D_M can be obtained from any modal model M in which R is reflexive and transitive. Let $M = \langle W, R, V \rangle$. Define $D_M = \langle I, \sqsubseteq, \models, \Dashv \rangle$ as follows:

$W = I$
$u \sqsubseteq v$ iff uRv
$u \models p$ iff $V(\Box p, u) = 1$, and $u \Dashv p$ iff $V(\Box \neg p, u) = 1$.

This definition ensures that for any $u, v \in I$ such that $u \sqsubseteq v$, $u \models p$ implies $v \models p$ and $u \Dashv p$ implies $v \Dashv p$, so that DM is a partial information model. This follows from the fact that M is a model of the modal logic S4 (because R is reflexive and transitive), and so $\Box A \rightarrow \Box\Box A$ is valid. Suppose $u \models p$ and $u \sqsubseteq v$. Then $V(\Box p, u) = 1$, and $V(\Box\Box p, u) = 1$. As $u \sqsubseteq v$, uRv, thus $V(\Box p, v) = 1$ and $v \models p$. In the same way we can prove that $u \Dashv p$ implies $v \Dashv p$.

4. *A translation from* **PIL** *to* **ML**

By **ML** we understand a standard modal propositional language whose logical symbols are $\neg$, $\wedge$, $\Box$. The translation proposed here is basically the same appearing in [1], all differences being purely notational. Notice, however, that in [1] two different translations A^+ and A^- are defined for each sentence A, where, roughly speaking, A^+ is to be interpreted as the translation for "A is supported", and A^- as the translation for "A is rejected" (see the claim on p. 234). However, that double translation can be simplified in view of the samantic clause for the negation operator ($M \models \neg A$ iff $M \Dashv A$ and $M \Dashv \neg A$ iff $M \models A$), which makes $A^- = (\neg A)^+$. So, in the translation proposed below, T(A) is equivalent to A^+ (of [1]), while $T(\neg A)$ is equivalent to A^-. By this method we can avoid the double recursion used in [1], but, also, it enforces us to include a translation clause for each kind of negated formula, instead of a single clause for every formula of the form $\neg A$.

For each **PIL**-wff A we define a **ML**-wff T(A) as follows:

$T(p) = \Box p$

$T(\neg p) = \Box\neg p$
$T(\neg\neg A) = T(A)$
$T(A \wedge B) = T(A) \wedge T(B)$
$T(\neg(A \wedge B)) = T(\neg A) \vee T(\neg B)$
$T(A \rightarrow B) = \Box(T(A) \rightarrow T(B))$
$T(\neg(A \rightarrow B)) = \Diamond(T(A) \wedge T(\neg B)$
$T(\text{MAY } A) = \Diamond T(A)$
$T(\neg\text{MAY } A) = \neg\Diamond T(A)$
$T(\text{MUST } A) = \Box\Diamond T(A)$
$T(\neg\text{MUST } A) = \Diamond\Box T(\neg A)$

In order to prove the adequacy of this translation we need the following lemma:

Lemma 1: Let $D = \langle I, \sqsubseteq, \models, =\!| \rangle$ and $M = \langle W, R, V \rangle$ be a partial information model and a modal model, respectively, and suppose there is a function f from W onto I such that for every u, $v \in W$, $f(u)=i$, $f(v)=j$,

a) $i \sqsubseteq j$ iff uRv
b) $i \models p$ iff $V(\Box p, u) = 1$
c) $i =\!| p$ iff $V(\Box\neg p, u) = 1$.

Then, for every **PIL**-wff A,

$i \models A$ iff $V(T(A), u) = 1$
$i =\!| A$ iff $V(T(\neg A), u) = 1$.

(Notice that, under the hypothesis of the lemma, $f(u) = f(v)$ implies $V(\Box p, u) = V(\Box p, v)$, and $V(\Box\neg p, u) = V(\Box\neg p, v)$. Observe as well that f is onto I, i.e., for every $i \in I$ there is some $u \in W$ such that $f(u) = i$).
Proof: Induction on the complexity of A. When A is p, the lemma is trivially satisfied. When A is either $\neg B$ or $A \wedge B$, the proof is simple. So, let's consider only the remaining cases. We shall write u_i, v_i to represent elements of W such that $f(u_i) = i$, $f(v_i) = i$.

- $i \models B \rightarrow C$ iff for all $j \in I$, if $i \sqsubseteq j$ and $j \models B$ then $j \models C$, iff (induction hypothesis plus properties of f, more briefly i.h.) for all u_j such that $u_i R u_j$ $V(T(B), u_j) = 1$ implies $V(T(C), u_j) = 1$, i.e., $V(T(B) \rightarrow T(C), u_j) = 1$, iff $V(\Box(T(B) \rightarrow T(C)), u_i) = 1$.
- $i =\!| B \rightarrow C$ iff for some j, $i \sqsubseteq j$, $j \models B$ and $j =\!| C$, iff (i.h) $V(T(B), u_j) = 1$ and $V(T(\neg C), u_j) = 1$, where $u_i R u_j$, iff $V(\Diamond(T(B) \wedge T(\neg C)), u_i) = 1$.
- $i \models \text{MAY } B$ iff for some j, $i \sqsubseteq j$, $j \models B$, iff (i.h.) $V(T(B), u_j) = 1$, for some u_j, $u_i R u_j$, iff $V(\Diamond T(B), u_i) = 1$.

- $i \dashv$ MAY B: this case follows directly from the former one, because a simple inspection to the corresponding semantic clauses shows that $i \models$ MAY B iff not $i \dashv$ MAY B.
- $i \models$ MUST B iff for all j, $i \sqsubseteq j$, there is some k, $j \sqsubseteq k$ such that $k \models B$, iff (i.h.) for all u_j, $u_i R u_j$, there is some u_k, $u_j R u_k$ such that $V(T(B), u_k) = 1$, iff $V(\Box \Diamond T(B), u_i) = 1$.
- $i \dashv$ MUST B iff there is some j, $i \sqsubseteq j$, such that for all k, $j \sqsubseteq k$, $k \dashv B$, iff (i.h.) there is some u_j, $u_i R u_j$, such that for every u_k, $u_j R u_k$, $V(T(\neg B), u_k) = 1$, iff $V(\Diamond \Box T(\neg B), u_i) = 1$. ■

Now consider any partial information model D and the corresponding modal model M_D. Define the function f as follows: $f(u_i) = i$, $f(v_i) = i$, for each u_i, $v_i \in W$. f satisfies the conditions in the hypothesis of lemma 1, so we can establish the following lemma:

Lemma 2: Given D and M_D,
$i \models A$ iff $V(T(A), u_i) = 1$
$i \dashv A$ iff $V(T(\neg A), u_i) = 1$. ■

Let's turn our attention to a modal model M and the corresponding D_M. As $W = I$, the identity function defined on W satisfies the conditions of lemma 1. Thus, we can establish a new lemma, similar to lemma 2:

Lemma 3: given M and D_M,
$u \models A$ iff $V(T(A), u) = 1$
$u \dashv A$ iff $V(T(\neg A), u) = 1$. ■

The following theorem can be easily proven from lemmas 2 and 3:

Theorem 1: A **PIL**-wff A is valid in the class of all partial information models iff T(A) is valid in the class of all reflexive and transitive modal models, i.e. if T(A) is S4-valid.
Proof: Suppose that A is not valid in a model D. By lemma 2 T(A) is not valid in M_D. Conversely, if T(A) is not S4-valid, there is a reflexive and transitive model M in which T(A) is not valid, and by lemma 3, A is not valid in D_M. ■

Let D be a data semantics model. Then, for every $i \in I$ there is some maximal $j \in I$ such that $i \sqsubseteq j$. Take M_D. Let j be a maximal element in D.

Then, as for all p, either $j \models p$ or $j \dashv p$, v_i does not exist in W, and as there is no $k \in I$ such that $j \sqsubset k$ and $j \neq k$, there is no $w \in W$ such that $u_j Rw$ and $u_j \neq w$. If we apply modal bulldozing methods (see [3]) to M_D, we obtain a model $M^*_D = \langle W^*, R^*, V^* \rangle$ such that $\langle W^*, R^* \rangle$ is a partial order such that every maximal chain contains a greatest element. On the other hand, if M is a modal model such that $\langle W, R \rangle$ is a partial order and every maximal chain in it contains a greatest element, then D_M is a data model. To check this claim, consider any maximal chain X in W, and let u be its greatest element. As W = I and vRw iff $v \sqsubseteq w$, X is a maximal chain in $\langle I, \sqsubseteq \rangle$, and u is its maximal element. Moreover, for any p, $V(\Box p, u) = V(p, u)$ and $V(\Box \neg p, u) = V(\neg p, u)$, because u is a reflexive dead end. Thus, if $V(p, u) = 1$ then $i \models p$, and otherwise (if $V(\neg p, u) = 1$) $i \dashv p$.

As it is pointed out in [1], the class of all partial orders with greatest elements for maximal chains characterizes the modal logic S4.1 (S4 + $\Box \Diamond A \rightarrow \Diamond \Box A$), so that from lemmas 2 and 3 and the remarks above we can establish the following theorem:

Theorem 2: A **PIL**-wff is valid in data semantics iff T(A) is valid in the modal logic S4.1 ■

So we have extended the reduction proposed in [1] for data semantics to the case in which no maximal point is postulated. The result is a reduction which embeds partial information semantics into S4, keeping the embedding of data semantics into S4.1 presented in [1].

5. *A note on a possible extension of the above translation to quantified languages*

Partial information logic could be extended in order to obtain a quantified predicate logic. Then, partial information models should be modified in such a way that an individual domain is assigned to each point in the model (as well as suitable interpretations for predicates). It seems reasonable to impose a nested domains condition to the resulting models (as we are dealing with increasing information stages) and to interpret individual terms as rigid terms. Then, it is not difficult to extend the method developed above to obtain a translation from such quantified **PIL** to a quantified **ML** with nested domains, rigid terms and free logic, in a similar way to the translation presented in [4]. Of course, different options concerning the conditions imposed on domains or the semantic clauses for atomic sentences and quantifiers would lead to different kinds of quantified modal logics (see [5] for a classification of quantified modal logics) and, probably, different translations. But that goes beyond the aims and scope of this paper.

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