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FREGE ON THE PURPOSE AND FRUITFULNESS OF DEFINITIONS

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In this paper, I want to discuss some problems in Frege's theory of definition. I shall begin by characterizing his conception of definitions in *Begriffsschrift* (*BS*) of 1879. In the second part, I shall deal with Frege's thesis about the systematic fruitfulness of "good" definitions in mathematics and logic, which he states in "Booles rechnende Logik und die Begriffsschrift" (*BRL*) of 1880-81 and in *Die Grundlagen der Arithmetik* (*GLA*) of 1884. In his mature period, Frege abandoned this thesis. In the concluding and main part of the paper, I shall investigate in detail the change of mind which he has undergone concerning the purpose and value of definitions.

I

In *BS*, Frege explains his conception of definitions by taking as an example the definition of a hereditary property in a series. He refers to it as sentence or formula (69). In contrast to a judgment asserting an identity between contents ("$\vdash A \equiv B$"), this sentence does not assert that the two sides of the equation have the same content; rather it *stipulates* that they are to have the same content. Since in (69) we do not acknowledge a judgeable content as true, but make a stipulation, this formula is, according to Frege, "not a judgment; and consequently, to use a Kantian expression, also *not a synthetic judgment*. I make this remark because Kant holds that all judgments in mathematics are synthetic" (*BS*, p. 56, *CN*, p. 167f). If (69) were a synthetic judgment in the Kantian sense — a judgment extending our knowledge — the sentences derived from it would be so too. Frege thinks, however, that from a logical point of view, the new sign introduced by the definition, hence also the definition itself, is in principle dispensable. Since the *definiendum* acquires the same content as the *definiens*, no extension in the content of the system is effected. The purpose of definitions is to introduce the defined expression as a linguistic abbreviation of the defining expression. By means of a definition, one can in fact achieve a considerable simplification in the conduct of proof; but one does not create the possibility of carrying out proofs that would have been impossible without it. Frege seems to emphasize this when he says that nothing follows from sentence (69) which could not have been derived without it. In the present context, this remark presumably is also to explain why the judgments inferred from formula (69) cannot be synthetic. Frege's restriction of the purpose and usefulness of definitions to their function as abbreviations and simplifications is in accordance with his later theory of definition in *Grundgesetze der Arithmetik* (*GGA*), "Über die Grundlagen der Geometrie" (*GLG I*, 1903) and "Logik in der Mathematik" (*LM*); but it clashes with his thesis, stated in *GLA*, that the really fruitful definitions in mathematics are indispensable for constructing gapless proofs and that in deriving other statements from them our knowledge may be extended.

Once the content of the *definiens* has been bestowed upon the *definiendum* the definition is immediately turned into an analytic judgment; for "we can only get out what was put into the new symbols in the first place" (*BS*, p. 56, *CN*, p. 168). This dual role of formula (69) is indicated by the use of a double judgment-stroke. Thus, in derivations, (69) can be treated as an ordinary judgment, i.e., it can be used as a premise of inferences. To be sure, there it appears with only one judgment-stroke (see *BS*, §§ 25 ff, *CN*, pp. 170 ff). In § 27 of *GGA*, Frege describes in more precise terms the transformation of a definition into a sentence of Begriffsschrift (for short: sentence). There he says that the double-stroke of definition, which appears as a doubled judgment-stroke combined with a horizontal, is replaced by the judgment-stroke. The resulting sentence can now be used in proofs in the same way as an axiom or an already proved theorem. The fact that the definition of a hereditary property in a series is immediately transformed into an analytic judgment means that it becomes a tautology not extending our knowledge (cf. *KS*, p. 263, *CP*, p. 274; *NS*, pp. 224 ff, *PW*, pp. 207 ff). Thus the expression "analytic judgment" is used here precisely in the Kantian sense, as an equivalent for "epistemically trivial judgment". It is true that also according to Frege's own criterion of analyticity in *GLA* definitions turned into assertoric sentences are to be construed as analytic truths. Frege does, however, hold that there are analytic statements containing valuable extensions of our knowledge.

In § 3 of *GLA*, Frege defines an analytic truth as one which can be derived exclusively from general logical laws (i.e., from primitive truths of logic neither needing nor admitting proof relative to a particular system) and definitions. He adds that we must also take account of all sentences on which the admissibility of the definitions, appealed to in the proof, depends. In a footnote, Frege emphasizes that he does not intend to confer a new sense on the term "analytic" but only to state accurately what Kant has meant by it. However, in comparing this remark with the criticism Frege levels against Kant's definition of analytic judgments in § 88 of *GLA*, one cannot help feeling that in his footnote he exercises restraint and does not quite faithfully inform us about his real intention. For it is obvious that, in the light of his logicist thesis, Frege has to redefine the term "analytic". According to Kant, analytic judgments do not extend our knowledge, they are epistemically trivial. If Frege had endorsed Kant's conception of analyticity and, at the same time, adhered to his basic tenet that the truths of arithmetic are analytic, he would have been unable to account for the informativeness of arithmetical laws and true numerical equations of the form "$a = b$" in which "a" and "b" have different senses, i.e., refer to the same number in different ways. Thus Frege's remark in the footnote ought not to be taken at face value.

The fact that the truth of a sentence "S" (or of the thought it expresses) is capable of being recognized by an immediate insight does, on Frege's view, not imply that "S" is trivially true. Thus he conceives of logical and geometrical axioms as self-evident truths but not as trivial truths. The converse implication does, however, hold: epistemic triviality implies self-evidence. The statement "Oxford = Oxford", for instance, is trivially true (i.e., uninformative) and hence self-evident; it is, of course, also analytically true. The same applies to all identity-statements of the form "$a = b$" in which "a" and "b" not only refer to the same object but also express the same sense. Such statements can be transformed into statements of the form "$a = a$" without altering their sense; and the latter are true — at least in a logically perfect language (Begriffsschrift) where every well-formed singular term has exactly one reference — even in virtue of their logical form alone. For a true statement "$a = b$" to be analytic, sameness of sense of "a" and "b" is a sufficient condition, but it is not a necessary one. Frege thinks that the self-evidence of a truth cannot serve as a general criterion of analyticity. According to him, there are sentences lacking self-evidence which nonetheless have to be acknowledged as analytically true as, for example, the equation "135664 + 37863 = 173527" or formula (133) of *BS*. The latter states that the ancestral of a many-one relation is a simple ordering when restricted to the objects to which a given object is ancestrally related.

II

In § 88 of *GLA*, Frege finds fault with what he sees as the narrowness of the Kantian definition of analyticity; for he believes it to be the source of underestimating the epistemic import and value of analytic judgments. Frege correctly observes that according to the Kantian definition of "analytic", the division of judgments into analytic and synthetic is not exhaustive. Kant in fact only considers universal affirmative judgments:

> there, we can speak of a subject concept and ask — as his definition requires — whether the predicate concept is contained in it or not. But how can we do this if the subject is an individual object? In these cases there can simply be no question of a subject concept in KANT's sense (*GLA*, *FA*, p. 100).

Even such simple sentences as "$F(a)$", "$G(a,b)$" and "$\exists xF(x)$" fall outside the scope of Kant's definitions of "analytic" and "synthetic". In the case of a universal affirmative judgment "$\forall x(F(x) \supset G(x))$" too, Frege would prefer that we speak not of a subject concept and a predicate concept, but of a subordinated and a superordinated concept. For the distinction between subject and predicate is, as he has already explained, not to appear in his logical calculus(1).

The way in which Kant distinguishes between analytic and synthetic judgments rests on an understanding of the nature of concepts formed along the lines of the traditional patterns of Aristotelian logic. Recalling his example of an analytic judgment — "Gold is a yellow metal" (symbolically: "$\forall x(G(x) \supset Y(x) \wedge M(x)$") — we easily see that Frege was justified in objecting that Kant seems to have defined the concept in terms of coordinate concepts. This, however, is one of the least fruitful kinds of concept formation. Already in *BRL*, Frege subjects the formation of concepts, that can be represented by the Boolean notation, to a critical scrutiny. Such a formation can only account for the construction of a concept through logical multiplication or addition, i.e., through conjunctive or disjunctive connection of coordinate characteristic marks or of previously given concepts.

(1) Cf. *BS*, p. XIII and § 3, *CN*, p. 107 and § 3; *NS*, pp. 130, 153, 155, *PW*, pp. 120, 141, 143; *KS*, p. 168 footnote 2, *CP*, p. 183 footnote 2; *WB*, pp. 103, 164f, *PMC*, pp. 68, 100 ff.

> In this sort of concept formation, one must, then, assume as given a system of concepts, or speaking metaphorically, a network of lines. These really already contain the new concepts: all one has to do is to use the lines already there to demarcate complete surface areas in a new way. It is the fact that attention is principally given to this sort of formation of new concepts from old ones, while other more fruitful ones are neglected which surely is responsible for the impression one easily gets in logic that for all our to-ing and fro-ing we never really leave the same spot (*NS*, p. 38, *PW*, p. 34).

This criticism of the Boolean method of concept formation is to be seen against Frege's thesis that judgments (judgeable contents) are prior to concepts. "As opposed to this, I start out from judgments and their contents, and not from concepts... I only allow for the formation of concepts to proceed from judgments" (*NS*, p. 17, *PW*, p. 16)(2). Frege contrasts to the unfruitful kind of concept formation through conjunctive or disjunctive connection of coordinate characteristic marks his own method of scientifically fruitful concept formation through analysis of a judgeable content, as illustrated by means of several examples in *BRL*. Thus by splitting, for instance, the judgeable content $\frac{\gamma}{\beta}(O_\gamma + 4 = 12_\beta)$ — in words: 12 is a multiple of 4; that is, 12 follows O in the arithmetical series with difference 4 — into a *constant* and a *variable* part we may arrive at different concepts each of which has proved its fruitfulness in mathematics(3). To this way of forming concepts corresponds in effect the method of constructing concept- and relation-expressions (more generally: function-names) as explained by Frege in the sections 26 and 30 of *GGA*, and which I in short refer to as "gap formation"(4). (1) If in the above judgeable content we regard the

(2) See the detailed examination of Frege's priority thesis in Schirn [1984].

(3) Numbers with opposed signs are here excluded from the concept of the multiple. O is considered as a multiple of itself. Concerning the definition of "$\frac{\gamma}{\beta}f(x_\gamma, y_\beta)$" — in words: "y follows x in the f-series" or "x precedes y in the f-series" — see *BS*, *CN*, § 26; *NS*, p. 24, *PW*, p. 22.

(4) Concerning the following examples see *NS*, p. 36, *PW*, pp. 32f. I retain the terminology employed by Frege in *BRL*. In § 26 of *GGA I*, Frege states three rules governing the correct formation of function-names. I call these rules "rules (principles) of gap formation". As will be obvious, their application presupposes the prior application of the rule of insertion (i.e., the insertion of admissible argument-expressions into the argument-places of function-names), and, in the first place, to *primitive* function-names. According to the first rule, we remove from a complex proper name (function-value name) a proper name that forms a part of it

number 12 as replaceable by something else — what is indicated by substituting x for 12 —, we obtain the concept of the multiple of 4: $\frac{\gamma}{\beta}(O_\gamma + 4 = x_\beta)$. (2) If we further imagine the number 4 as replaceable within this concept, we attain the concept of the relation of a number to its multiple: $\frac{\gamma}{\beta}$ 3 4^F (To this formation of a first-level concept and of a first-level relation corresponds the application of the first and the second rule of gap formation in *GGA*.) (3) If in the above judgeable content we consider the number 4 alone as replaceable, we form the concept of the common (aliquot) factor of 12: $\frac{\gamma}{\beta}(O_\gamma + x = 12_\beta)$.

In comparing the Boolean definitions with Frege's definitions of the continuity of a function(5), of a limit(6) and of following in a series(7), we find that Frege does not use boundary lines of already available concepts to form the boundaries of the new ones. Rather his definitions draw entirely new boundary lines; every element in them "is intimately, I might almost say organically, connected with the others" (*GLA*, *FA*, p. 100). And this is exactly the source of their scientific fruitfulness. Here too, already known concepts are used to construct the new ones. But now these familiar concepts are linked with each other in various ways by means of the signs for generality, negation and the conditional.

> What we shall be able to infer from them, cannot be inspected in advance; here, we are not simply taking out of the box again what we have just put into it. The conclusions we draw from them extend our knowledge, and ought therefore, on KANT's view, to be regarded as

or coincides with it, at some or all of the places where it occurs and mark the gap(s) so formed as a single argument-place of type 1, i.e., appropriate to admit proper names. We thereby obtain a one-place first-level function-name. Following the second rule, we remove from a complex one-place first-level function-name a proper name at some or all of the places where it occurs in it and render recognizable the resulting gap(s) as a single argument-place of type 1. We thereby form a two-place first-level function-name. According to the third principle, we remove from a complex proper name a monadic, respectively dyadic, first-level function-name at some or all of the places where it occurs and mark the gap(s) as a single argument-place of type 2 (appropriate to admit monadic first-level function-names), respectively of type 3 (appropriate to admit dyadic first-level function-names). We thereby construct a second-level function-name with one argument-place of type 2, respectively of type 3.

(5) See *NS*, pp. 26 ff, example 13, *PW*, pp. 24 ff.

(6) See *NS*, p. 27, example 16, *PW*, p. 25.

(7) See *BS*, *CN*, § 26.

> synthetic; and yet they can be proved by purely logical means, and are thus analytic.
> The truth is that they are contained in the definitions, but as plants are contained in their seeds, not as beams are contained in a house. Often we need several definitions for the proof of some sentence, which consequently is not contained in any one of them alone, yet does follow purely logically from all of them together (*GLA*, *FA*, pp. 100f).

In § 91 of *GLA*, Frege offers formula (133) of *BS* as an example of such a sentence; it is here rendered somewhat differently from its original form:

> If the relation of every number of a series to its successor is one- or many-one, and if *m* and *y* follow in that series after *x*, then either *y* comes in that series before *m*, or it coincides with *m*, or it follows after *m*.

Frege emphasizes that this sentence might at first sight appear to be synthetic. Its proof, however, does not rely on any axiom of intuition, as is guaranteed by the fact that in the chain of inference no link is missing. For it is precisely the lack of gaps in the conduct of proof which brings to light every axiom or every assumption upon which a proof is based; only in this way do we gain a secure basis for assessing the epistemological status of the law that is proved (cf. *GLA*, *FA*, §§ 90 ff; *GGA I*, pp. VII, 1, *BLA*, pp. 3, 29). From the proof of formula (133) we can see — so Frege claims — that sentences extending our knowledge may contain analytic judgments.

In *BRL* and *GLA*, the usefulness of definitions in mathematics and logic is not generally seen as restricted to their function as abbreviations and simplifications. The distinguishing mark of "really good" definitions lies rather in the fact that they embody a process of fruitful concept formation. This process takes place by analyzing a judgeable content into a constant and a variable part, by applying the method of gap formation respectively. In the context of *GLA*, a definition has to prove its worth by creating the possibility of giving gapless proofs. "Those that could just as well be omitted and leave no link missing in the chain of proofs should be rejected as completely worthless" (*GLA*, *FA*, § 70; see also *GLA*, *FA*, p. XXI). In *GLA*, Frege thus seems to regard his definitions of Number, of equinumerosity, of following in a series etc. as indispensable means of proving the fundamental principles of arithmetic in a gapless (and purely logical) manner. Thus, for instance, the definition of Number as the extension of a concept will prove fruitful only if it is possible to derive from it the familiar properties of the natural numbers. Already in the introductory remarks in § 4 of *GLA* we seem to encounter the idea that appropriate definitions are indispensable for a gapless conduct of proof. In trying to satisfy the inescapable demand for strict proofs of the basic laws of arithmetic, we are, according to Frege, bound sooner or later to arrive at sentences whose proof only becomes possible once we analyze their component concepts (to be more precise: the concept-expressions occurring in them) into simpler concepts or else reduce them to something more general. He further adds: "Now here it is above all Number which has to be either defined or recognized as indefinable. On the outcome of this task will depend the decision as to the nature of the laws of arithmetic" (*GLA*, *FA*, p. 5).

Only when we have, through definition, reduced the concepts and relations of arithmetic to something more general — which, in the context of the logicist programme, means to reduce them to purely logical concepts and relations — is the possibility of strict and concise proof of the fundamental principles of arithmetic secured. In the introduction of *GGA*, Frege reminds us of the fact that in *GLA* Number was reduced to the relation of equinumerosity and this in turn to many-one correspondence. He writes: "If I am right in thinking that arithmetic is a branch of pure logic, then a purely logical expression must be selected for 'correspondence'. I choose 'relation' for this purpose. Concept and relation are the foundation-stones upon which I erect my structure" (*GGA I*, p. 3, *BLA*, p. 32). When Frege talks of the "dissolution" of concepts into simpler ones in § 4 of *GLA*, he may be thinking of what he later in *LM*, though with some reservations, called an analytic definition. There he grants that a definition yielding a logical analysis may enable us to prove a truth which would otherwise have been unprovable. I shall return to this idea.

III

After the systematic development of a theory of definition in *GGA*(8), Frege abandoned his former conception of the fruitfulness of good defini-

(8) Cf. *GGA I*, *BLA*, part 2. "Definitions", especially § 33 where Frege states seven principles of definition; see also *GGA II*, § 55-67; *NS*, pp. 164-170, *PW*, pp. 152-156; *WB*, pp. 181-186, 194-198, *PMC*, pp. 112-118, 125-129.

tions, as expressed in *BRL* and in *GLA*. In what follows, I want to examine this change in how the usefulness of definitions is assessed.

In order to prevent possible misunderstandings, let me begin by listing the characteristic marks of *fruitful* definitions as encountered in *GLA*: (1) they represent a kind of concept formation in which, to use Frege's geometrical image, entirely new boundary lines are drawn; (2) they enable us to carry out gapless proofs, something that would have been impossible without them; (3) we may draw inferences from them which extend our knowledge. This, however, is not to say that a fruitful definition as such adds to our knowledge. Frege nowhere claimed that it does.

I shall now turn to the question of how Frege abandoned his former convictions on the nature of definitions. In *GLG I* (1903), we are told that a definition proper is nothing but a means for collecting a complex content (sense) into a short sign, thereby making the sign easier for us to handle. In this alone consists the usefulness of definitions in mathematics. "Never may a definition strive for more. And if it does, if it wants to engender real knowledge, to save us a proof, then it degenerates into logical sleight of hand" (*KS*, p. 263, *CP*, p. 275). In a footnote, Frege observes that one could add as a feature of the usefulness of definitions that they enable us to attain a clearer grasp of the sense of a word, which before was only "half-consciously" associated with that word. This, however, is less an aspect of the usefulness of the definition itself, as of the practice of defining. Once the definition has been put forward it is irrelevant whether the defined word or sign was introduced for the first time or whether a sense was already attached to it(9). The idea that a definition is capable of producing genuine knowledge, of sparing us the trouble of proof, is also incompatible with Frege's view of definitions in *GLA*. In his criticism of Hankel's formal theory of negative, fractional, irrational and complex numbers (*GLA*, *FA*, §§ 92 ff), we see that he denies definitions any creative potential. A clear expression of the thesis of the non-creativity of definitions is to be found in the preface to *GGA*: "Just as a geographer does not create a sea when he draws boundary lines and says: the part of the ocean's surface bounded by these lines I am going to call the Yellow Sea, so too the mathematician cannot really create anything by his defining" (*GGA I*, p. XIII, *BLA*, p. 11;

(9) See also *GLG II* (1906) in *KS*, p. 290, *CP*, pp. 302f. Frege points out that the mental activity preceding the formulation of a definition does not appear in the systematic construction of mathematics. Hence, it is irrelevant whether this activity was analytic or constructive, whether the *definiendum* had already been used before or whether it was newly invented.

see also *GGA I*, p. VI, *BLA*, p. 2 and *KS*, p. 127, *CP*, p. 139). What we need to notice is that confining the usefulness of definitions to their abbreviatory and simplificatory function in *GLG I* (1903) contradicts the conception of the fruitfulness of good definitions advocated in *GLA*.

Also in *GLG I* (1906), Frege stresses the simplification achieved by their means as the primary purpose of definitions proper, i.e., of constructive definitions. He then considers the following case. Suppose that the defined sign was not newly invented, but was already in use in ordinary discourse or in a scientific treatment that precedes the truly systematic one. As a rule, however, this use is too vague or fluctuating for strict scientific purposes. Nevertheless, let us assume that it does satisfy the highest demands for precision. It might then seem that a definition is superfluous. Yet, Frege insists that even in such a case we cannot dispense with a definition; for its real significance lies in the logical construction out of the primitive elements of the system. The insight into the logical structure which it grants is not only valuable in itself, it is also a precondition for understanding the logical relations between the truths of the system. The definition is thus a constituent of the system of science (cf. *KS*, p. 289, *CP*, pp. 301f).

According to these remarks, the value and usefulness of definitions proper consist *not only* in their practical function as abbreviations and simplifications. Rather, they prove fruitful when viewed from the perspective of the notion of a system, insofar as they provide valuable insights into the construction of the system, consequently into the network of inferential links in it. Definitions do not extend the content of a system. They do not allow for proofs that would have been impossible without them. Frege's characterization of the real significance of constructive definitions as constituents of a mathematical system at the same time touches on the issue concerning the value of mathematical knowledge. In the posthumously published paper "Logische Mängel in der Mathematik", he emphasizes that, regarding mathematical knowledge, grasping the logical connections between the truths of a mathematical system is more important than its content. He writes: "If you ask what constitutes the value of mathematical knowledge, the answer must be: not so much what is known as how it is known, not so much its subject-matter as the degree to which it is intellectually perspicuous and affords insight into its logical interrelations" (*NS*, p. 171, *PW*, p. 157; see also *NS*, p. 221, *PW*, p. 205).

In *LM* (1914), where Frege deals extensively with the role and nature of definitions and distinguishes constructive from analytic definitions, he ignores that aspect of the usefulness of definitions proper which was brought to light in *GLG I* (1906): their granting understanding of the logical structure of a system. From a logical point of view, definitions are here said to be entirely inessential and dispensable([10]). The following line of thought is intended to clarify this. A definition transformed into an assertoric sentence is only apparently used as a premise in the construction of a mathematical system. Frege says *apparently*, for what is here presented in the form of an inference is no source of new knowledge, but in fact only a means of effecting an alteration of expression; and this alteration is, theoretically speaking, dispensable (cf. *NS*, p. 225, *PW*, p. 208). A definition in the proper sense of the word cannot secure the provability of a thought which would be unprovable without it. In presenting as a definition a sentence whose role is to secure the provability of a truth, what we have is not at all a pure definition, i.e., an arbitrary stipulation introducing a new simple sign for a complex sign, whose sense results from the way it is put together. Rather, the sentence in question has to contain something which either, being regarded as a theorem stands in need of proof, or else has to be recognized as an axiom.

We need, however, to distinguish between the sentence and the thought it expresses. When we replace the *definiens* as a constituent expression of a sentence by the *definiendum* it is true that we obtain a different sentence, but we do not get a different thought. If we want to prove the thought in such a way that the *sentence* reached by the above substitution occurs in the proof, then we do need the definition. However, if the thought in question is at all provable within the mathematical system, if it does not have the status of an axiom in it, then we can always prove it in its original sentential form, without recourse to the definition.

The upshot, so far, is as follows. However considerable in practice the simplification of expression, conciseness of the chain of inference and perspicuity of proof achieved by means of constructive definitions may be, in principle such definitions do not affect the provability of a thought. They can only have an essential role in the system if the *sentence* is taken as the object of proof. Yet even here, where we view the inference drawn from a definition turned into an assertoric sentence as being an inference from a premise qua sentence (and not qua true thought) to the conclusion qua sentence, no new knowledge is attained. And, to be sure, according to Frege what we prove is not sentences, but thoughts.

([10]) Frege stresses that they are nevertheless of great *psychological* importance.

The above reasoning conflicts with the conception of the fruitfulness of good definitions, stressed, as it was, in *GLA*. When Frege comes to describe the nature of constructive definitions in *LM* he does not mention at all the first characteristic mark of fruitful definitions — their representing a process of genuine concept formation. The remaining two characteristic marks, intimately connected with the first, turn out to be incompatible with the purpose of definitions proper. Admittedly, Frege is well aware in *LM* that his disparaging attitude towards the usefulness of definitions — their being, from a logical point of view, wholly inessential — may provoke objections. Thus it might be objected that a definition does after all yield a logical analysis of the sense of a sign, and therefore does provide knowledge of the parts of that sense. The analytical activity is surely not to be ignored as irrelevant. Frege is sensitive to the above objection and regards it as to some extent justified. The possibility of a science discovering, as it develops, that the sense of an expression which it had long considered as simple is in fact capable of being split up into simpler logical constituents, is a real one. By means of such an analysis the number of axioms of that science may allow for reduction. For it may be that a true thought containing a complex constituent is provable only once this constituent is further analyzed. This is because the thought may be provable from truths containing parts arrived at by analyzing the complex constituent. Frege concedes: "This is why it seems that a proof may be possible by means of a definition, if it provides an analysis, which would not be possible without this analysis, and this seems to contradict what we said earlier. Thus what seemed to be an axiom before the analysis can appear as a theorem after the analysis" (*NS*, p. 226, *PW*, p. 209).

Frege does not explicitly address the question of whether we are here facing a real or an apparent contradiction. I believe, however, that it is possible to solve this difficulty by examining how he characterizes the status of so-called analytic definitions in contrast to constructive definitions. Let us therefore take a closer look at his characterization.

An analytic definition carves up the complex sense of a simple sign "*A*" of long-established use into simpler components. The composite expression "*Q*" formed by this method represents the analysis; the complex sense of "*Q*" must result from the familiar senses of its parts and the manner "*Q*" is put together. Since "*A*" already has a sense, the identity of the sense of "*A*" and that of "*Q*" does not rest on an arbitrary stipulation. It can be recognized only by an immediate insight. But if this is the case, then the identity of sense is an axiom(11). For this reason, Frege suggests that we altogether avoid the word "definition" in dealing with the logical analysis of the sense of a simple sign which already has an established use. The correctness of an analysis cannot be proved. The only criterion of correctness is our immediate insight into the complete coincidence of the sense of the simple expression and that of the composite expression.

Let us suppose that correct analytic definitions, being axioms, form admissible constituents of a mathematical system, that they belong to its unprovable primitive truths. This implies that they can be used as premises in the construction of the system. Yet neither they nor what is inferred from them provides new knowledge. In that respect, they are indistinguishable from constructive definitions transformed into assertoric sentences. It holds equally for both kinds of definition that the thought, which a sentence expresses, is not affected by substituting the *definiendum* for the *definiens*(12). Frege thus describes this role of analytic definitions — their making proofs possible — as follows: it is perhaps possible to prove a truth (containing a complex constituent), that was hitherto thought unprovable, by using truths in which the elements attained through analysis of the complex constituent occur. The impression that this contradicts his earlier thesis that a definition cannot secure the provability of a truth is deceptive. This is because the analytic definition resting on an immediate grasp of an identity of sense is not at all a pure definition, but something to be acknowledged as an axiom.

It now becomes a pressing question whether, in the light of the distinction between constructive and analytic definitions drawn (with some reservations) in *LM*, the definitions of Number, following in a series etc., which Frege puts forward in *GLA*, are to be classified as analytic. In favour of an affirmative answer it could be adduced that the *definiendum* is here always an expression of long-established use(13), with which we associate a sense,

(11) In *LM* (*NS*, p. 227, *PW*, p. 206), Frege does not use the word "*Lehrsatz*" but "*Theorem*", and not "*Grundsatz*" but "*Axiom*", understanding by theorems and axioms true thoughts.

(12) For the sake of convenience, I retain the expression "analytic definition". Of course one cannot, strictly speaking, speak of the *definiendum* and the *definiens* of an axiom.

(13) Admittedly, this is not true of the term "equinumerous" ("*gleichzahlig*"). Frege himself requests that this word be considered an arbitrarily chosen symbol whose meaning is to be taken, not from its linguistic composition, but from the following stipulation (definition): The concept F is equinumerous with the concept G if and only if there is a relation R which

albeit perhaps a rather vague one. Obviously, these definitions are not a matter of free, arbitrary stipulation, which is an essential feature of constructive definitions. For only a sign which as yet has no sense can have a sense arbitrarily assigned to it. In the case of a successful analytic definition qua axiom there remains, as Frege emphasizes, "no room for an arbitrary stipulation, because the sign already has a sense" (*NS*, p. 227, *PW*, p. 210). Elsewhere he does, however, admit that for "reasons of expediency"([14]) we may be well advised to choose, in a constructive definition, an expression of long-established use instead of a completely new simple sign. Yet, also here we need to treat the chosen sign as if it were entirely new and had no sense prior to the definition. Frege writes:

> We must therefore explain that the sense in which this sign was used before the new system was constructed is no longer of any concern to us, that its sense is to be understood purely from the constructive definition we have given. In constructing the new system we can take no account, logically speaking, of anything in mathematics that existed prior to the system. Everything has to be made anew from the ground up. Even anything that we may have accomplished by our analytical activities is to be regarded only as preparatory work which does not itself make any appearance in the new system itself (*NS*, p. 228, *PW*, p. 211; cf. *KS*, p. 290, *CP*, pp. 302f).

I do not find this standpoint quite convincing, especially since Frege does not specify what exactly he has in mind when speaking of "reasons of expediency" nor clarifies this by example. I shall, however, leave possible scruples aside in order to focus my attention again on the issue concerning the status of the definitions set up in *GLA*.

Both, Dummett and Resnik, are inclined to think that Frege's definitions in *GLA* are analytic([15]). According to Resnik, the explicit definition of Number "The Number which belongs to the concept F is the extension of

correlates one-to-one the objects falling under F with the objects falling under G. In symbols ("E" is to abbreviate "equinumerous"):
$E_x(F(x),G(x)) := \exists R(\forall x(F(x) \supset \exists y(R(x,y) \wedge G(y))) \wedge \forall y(G(y) \supset \exists x(R(x,y) \wedge F(x))) \wedge \forall x\forall y\forall z((R(x,y) \wedge R(x,z) \supset y = z) \wedge (R,(x,z) \wedge R(y,z) \supset x = y)))$.
Thus, this definition should probably not be regarded as an analytic one.

([14]) The word used by Frege is "*Zweckmäßigkeitsgründe*".

([15]) Dummett [1981], pp. 251 f, 339 and Reznik [1980], pp. 181, 183, 185.

the concept *equinumerous with the concept F*" (*GLA*, *FA*, § 68) is a result of an analysis of the concept of number. Frege supposes, Resnik further thinks, that anyone who takes care to follow his analysis will understand that it only renders explicit what was already implicitly present in our concept of number. The first claim is correct. Frege's definition of Number as an equivalence class of the equivalence relation of equinumerosity is in fact the result of a logical investigation of number statements and of the concept of number, despite the fact that extensions of concepts are introduced rather abruptly([16]). The second assertion is, however, open to question. I do not think it is justified to interpret Frege as intending his definition of Number as an explanation of our intuitive, pre-theoretical understanding of the concept of number. Surely, not everybody would agree that Frege's definition does capture our intuitive understanding of that concept, whatever such understanding may be. And certainly, the majority of mathematicians contemporary to Frege would not have granted his having sufficiently made out the claim that his definition provides an adequate explication of our intuitive grasp of the concept of number. So much, at least, is clear: in his repeated methodological criticism of the doctrines held by contemporary mathematicians, Frege complains that not even a minimal degree of agreement exists about the nature of the most fundamental concepts of arithmetic. This is most striking in the case of the concept of number.

> How little value is commonly placed on sense and definitions can be seen from the sharply conflicting accounts that mathematicians give of what *number* is. (We are speaking here of the natural numbers.) Weierstrass says "Number is a series of things of the same kind". Another says that certain conventional shapes produced by writing, such as 2 and 3, are numbers. A third is of the opinion: if I hear a clock strike three I see nothing in this of what three is. Therefore it cannot be anything visible. [...] Obviously each of these attaches a different sense to the word "number". So the arithmetics of these three mathematicians must be quite different. [...] Or is it not the explanation rather that we have really to do with the same science; that this man *does* attach the same sense to the word "number" as that man, only he does not manage to get hold of it properly? Perhaps the sense appears to both in such a haze that when they make to get hold of it, they miss it. (*NS*, pp. 232-4, *PW*,

([16]) See Schirn [1983], where the difficulties which Frege encounters through his introduction of extensions of concepts in *GLA* are treated extensively.

215-7; see also *NS*, pp. 239 ff, 80 ff, *PW*, pp. 221 ff, 72 ff; *GLA*, *FA*, part II)

It is thus plausible to see Frege as intending his explicit definition of Number as a precise characterization of the sense of a familiar simple expression, which usually is not clearly grasped, "but whose outlines are confused as if we saw it through a mist" (*NS*, p. 228, *PW*, p. 211)([17]). He does, however, make the questionable assumption here that the sense of the expression "extension of a concept" — which forms a part of the *definiens* — is completely known (cf. *GLA*, *FA*, pp. 80n, 117). Now, according to Frege's line of argument in *LM*, we are justified in presenting an analysis of the sense of a simple expression into simpler components as correct only if the identity of that sense with the sense of the complex expression representing the analysis is self-evident. An analytic definition can thus only be said to be successful when it is to be acknowledged as an *axiom*. In the case of Frege's definition of Number, if taken to be analytic, this condition is not satisfied. Frege himself says: "That this definition is correct will perhaps be hardly evident at first" (*GLA*, *FA*, p. 80). The identity of sense in question can only be self-evident if the sense of the simple expression (of the *definiendum*) is clearly grasped. Frege seems to confirm this when he explains: "How is it possible, one may ask, that it should be doubtful whether a simple sign has the same sense as the complex expression if we know not only the sense of the simple sign, but can recognize the sense of the complex one from the way it is put together? The fact is that if we really do have a clear grasp of the sense of the simple sign, then if cannot be doubtful whether it agrees with the sense of the complex expression" (*NS*, p. 228, *PW*, p. 211). If the identity of sense is questionable although we can derive the sense of the complex sign from its composition — and in his explicit definition of Number Frege presupposes that we do know the sense of the *definiens* in virtue of our grasp of the senses of its parts and the manner these are combined to form the *definiens* —, then this can only be due to the sense of the simple expression not being clearly grasped. In that case,

([17]) Understanding the definition in this way, is similar to regarding it as an *explication* in Carnap's sense. An explication of a concept serves to render more precise a predicate of natural language, whose meaning (sense) is not sharp or unambiguous, so that it can be used in an exact theory. This process of semantic specification and refinement is dependent on part of the original meaning of the predicate standing in need of explication. Hence, it cannot be viewed as an arbitrary stipulation of meaning.

the logical analysis will consist in working out clearly the sense of the simple expression. However, Frege does not regard the analytical activity as belonging to the construction of the system but as something preceding it. Before embarking on the construction of the system the signs to be used must have an unambiguous sense, so far as they do not acquire one in the system itself through constructive definitions.

In *GLA*, Frege does not strive for a systematic construction of arithmetic but confines himself to an informal reconstruction of the outlines of that science. The investigation is carried out within the framework of natural language and employs only a few technical devices. In the context of the intended gapless execution of the logicist programme in Begriffsschrift, it is certainly not meant to be anything but preparatory work. Also from this perspective, the definition of Number as well as the other definitions put forward in *GLA*(18) cannot be assigned the status of constructive definitions. For, on Frege's view, constructive definitions in a strict sense are only possible within the framework of a scientific system developed according to stringent rules. Strictly speaking, expressions of natural language already in possession of a sense are not, Frege thinks, to be used in a constructive definition. In *GGA*, he adheres basically to the definition of Number set up in *GLA*. However, it is now treated as an arbitrary stipulation: a new simple sign of Begriffsschrift takes the place of a complex sign of Begriffsschrift; the latter is correctly formed out of primitive names and names already defined and its sense results from its composition. By means of a constructive definition the simple expression (function-name) "Nξ" hitherto lacking a sense and a reference acquires the sense and the reference of the complex expression (cf. *GGA I*, *BLA*, §§ 38-40).

Let me end by considering Frege's proposed method of replacing the logical analysis of the sense of a sign of long-established use by a definition proper, when the correctness of the analysis is not immediately obvious (cf. *NS*, p. 227, *PW*, p. 210). Suppose that "*A*" is the simple sign already in use, whose sense we have tried to analyze by constructing the complex expression "*Q*" as a representation of that analysis. Since we are not certain whether the sense of "*Q*", which results from its composition, coincides with that of "*A*", we cannot treat "*Q*" as replaceable by "*A*". Thus the logical analysis does not provide an axiom. If we want to set up a definition proper we are not entitled to use the sign "*A*" — already in possession of

(18) Perhaps with the exception of the definition of equinumerosity.

a sense and perhaps also of blurred boundaries. Instead we must choose a new simple sign "*B*", say, which hitherto had no sense. We now confer the sense of "*Q*" on "*B*" and in this way set up a constructive definition. For "*B*" was introduced as a short-hand expression to replace "*Q*" and thus acquired a sense by arbitrary fiat. Frege maintains that by constructing the system anew from the ground up, using "*A*" but not "*B*", we can entirely bypass the question of whether "*A*" and "*B*" have the same sense.

The advantage of Frege's proposed transition from a (tentative) logical analysis of the sense of "*A*" to a constructive definition reached by substituting "*B*" for "*A*" presumably is to consist in the following: since "*Q*", which originally represented a logical analysis and later a logical construction, is now replaceable by "*B*" everywhere in the system, the practical purpose of abbreviating and simplifying is fulfilled. In this sense, the analytical activity preceding a constructive definition proves useful, although it does not appear in the system itself. For, in the constructive definition, the complex expression "*Q*" obtained through analysis occurs as *definiens*. The method suggested by Frege does, however, pose a problem. Generally, so-called analytic definitions deal with natural language expressions of long-established use. Hence, the composite expression "*Q*" formed through analysis of the sense of "*A*" — unfortunately, Frege offers not a single example — would presumably have to consist of natural language expressions. Yet, the transition from the tentative analytic definition of "*A*" to a constructive definition is possible only by substituting a new expression of a given symbolic language, which hitherto had no sense, for the natural language expression "*A*"([19]). We would then be faced with a constructive definition in which a symbolic language expression occurs as *definiendum* and a natural language expression as *definiens*. However, the natural language expression "*Q*" was not allowed to appear in the system at all, but had to be replaced by a synonymous symbolic language expression, which is formed out of primitive expressions or already defined expressions according to the formation rules of the system. Facing this predicament, it is hardly possible to avoid the impression that Frege did not attain full clarity of the method proposed in *LM* and which I described above.

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([19]) It would, I believe, make little sense to substitute, for instance, "bazet" for "Number" in a constructive definition.

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BRL: "Booles rechnende Logik und die Begriffsschrift", in *NS*, pp. 9-52, trans. in *PW*, pp. 9-46.

BS: *Begriffsschrift, eine der arithmetischen nachgebildeten Formelsprache des reinen Denkens* (Halle s.S., 1879), reprinted in G. Frege, *Begriffsschrift und andere Aufsätze* ed. I. Angelelli (Hildesheim and Darmstadt, 1964).

CN: Conceptual notation and related articles, trans. and ed. T.W. Bynum (Oxford, 1972).

CP: *Collected Papers on Mathematics, Logic, and Philosophy*, ed. B. McGuinness, trans. M. Black *et al.* (Oxford, 1984).

FA: *The Foundations of Arithmetic. A logico-mathematical enquiry into the concept of number*, trans. J.L. Austin (Oxford, 1950, 1953).

GGA: *Grundgesetze der Arithmetik. Begriffsschriftlich abgeleitet*, vol. I (Jena, 1893), vol. II (Jena, 1903), reprinted Darmstadt and Hildesheim, 1962.

GLA: *Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl* (Breslau, 1884), reprinted Darmstadt and Hildesheim, 1961.

GLG: "Über die Grundlagen der Geometrie" (1903), I, II, in *KS*, pp. 262-272, trans. in *CP*, pp. 273-284; "Über die Grundlagen der Geometrie" (1906), I-III, in *KS*, pp. 281-323, trans. in *CP*, pp. 293-340.

KS: *Kleine Schriften*, ed. I. Angelelli (Hildesheim, 1967).

LM: "Logik in der Mathematik", in *NS*, pp. 219-270, trans. in *PW*, pp. 203-250.

NS: *Nachgelassene Schriften*, ed. H. Hermes, F. Kambartel and F. Kaulbach (Hamburg, 1969).

PMC: *Philosophical and Mathematical Correspondence*, ed. B. McGuinness, trans. H. Kaal (Oxford, 1980).

PW: *Posthumous Writings*, trans. P. Long and R. White (Oxford, 1979).

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