Logique & Analyse **125-126** (1989), 29-40

DUALITIES OF SELF-NON-APPLICATION AND INFINITE REGRESS

Dale JACQUETTE and Henry W. JOHNSTONE, Jr.

I. *The Duality Thesis*

A duality holds between self-non-applications and infinite regress problems and solutions. By 'self-application' and 'self-non-application' we mean what is usually meant by 'self-reference', though we prefer the above terminology for reasons of precision. 'Self-reference' and 'self-referential' are often used in the characterization of situations in which reference does not occur and is not at issue. The generalization 'All generalizations are quantified' *applies* to itself without *referring* to itself. Generalizations of this kind are *self-applicational*, but not self-referential, where a function, principle, or proposition is said to apply to something just in case it holds true of it. The generalization 'All men are mortal' similarly *applies* to or holds true of Socrates, but does not refer to or single him out for mention. Cases of self-*non*-application by contrast involve an explicit denial of self-application. The duality thesis states that the construction of an infinitely regressive series requires the positing of a condition that is contradicted by requirements for some counterpart self-non-application, and conversely. Understanding the duality between these important argument forms sheds light on the interplay of dialectical structures underlying many classical and contemporary philosophical arguments. It provides a heuristic for the identification, development, and refutation of both kinds of philosophical thesis.

II. *Structures of Duality*

To illustrate the duality relationship, consider (without scholarly apparatus) these well-known arguments from a number of different areas in the history of philosophy.

Plato's Theory of Forms is subject to the infinite regress objection often known as the Third Man. If Redness is red, then there is yet another property of redness which the two share or participate in, by virtue of which both are said to be red, and so on indefinitely. Aristotle blocks the regress by positing an alternative conception of Forms, according to which Forms, definitions, or secondary substances, are not ideal abstract eternal entities, but exist only insofar as they are exemplified in the primary substances in which they inhere. The infinite regress of Plato's Third Man is avoided by Aristotle's view that Forms exist only in the things to which they belong, that properties are self-non-applicable, so that it is false from the outset to say that Redness is red, Wisdom wise.

Aristotle again in proving the existence of an unmoved mover, argues in essence that since causation cannot be circular if cause is to precede effect, then in order to prevent an infinite regress of causes extending forever backward in time (a possibility he disallows by virtue of his prior distinction between actual and potential infinities, deployed to solve Zeno's paradoxes of motion and extension), something, the unmoved mover, must be self-caused. The infinite regress of causes is not generated merely by the assumption that every event is caused, but by the assumption that every event has a cause prior to and distinct from or other than it. The self-non-application of event causation in this sense is invoked to forestall the infinite causal regress. The regress is blocked by maintaining that there is at least one occurrence in the history of the universe to which the property of having a distinct prior origin or cause is self-non-applicable.

Russell's paradox in set theory, to approach the duality from the opposite direction, threatens to expose a diagonal self-non-application that on ordinary assumptions implies outright logical inconsistency. The paradox is prevented by postulating an infinite regress of ordered types, according to which self-application and self-non-application alike are outlawed as syntactically improperly formed, and an infinite regress of predicate types is accepted instead. The very same structure preventing diagonal self-non-application by infinite regress is introduced by Tarski's hierarchy of object languages and metalanguages, in which Epimenides' Liar-style semantic paradoxes involving truth judgment self-non-applications are disallowed by the restriction that truth or falsehood cannot be self-applicable or self-non-applicable within any language of the hierarchy, but must always be given in a higher-order metalanguage.

Gödel's first theorem can be interpreted along these lines as the arithmetized formulation of a provability self-non-application (or denial of a provability self-application) to challenge Russell's infinite hierarchy of predicate types. The reflexive arithmetization of logical syntax cleverly avoids Russell's type-theoretical restrictions on self-non-applications by attaching to unprovability predicates not other predicates, but constants or numbers coding the unprovability assertion, producing by mathematical diagonalization uncomputable numbers translatable as formally undecidable propositions.

Similar problems and solutions exhibiting the self-non-application/ infinite-regress duality can be described for Bradley's regress against the existence of relations; Ryle's regress against mental representationalism in semantic theory and the philosophy of mind; paradoxes like Grelling's and Richard's; and self-evidence in foundationalist epistemologies versus infinitely regressive sufficient reason models. In every case, a problem or paradox is proposed, or theory advanced, by appeal to self-non-applicational constructions, only to be defeated by postulating a complementary infinite hierarchy; or alternatively an infinite hierarchy is presented as a problem, paradox, solution, or consequence of theory, refutable by appropriate self-non-applicational construction.

The movement back and forth from one approach to the other as data, theory, and problems are advanced, is of philosophical importance. Yet there is scant recognition of the duality in the methodological metaphilosophical literature.(1) Is it possible for self-non-applications to supplant infinite regresses and conversely without end? What does it mean for such a relationship to hold within the warp and woof of philosophical debate? Is there any prospect of arriving at the truth about philosophical questions if it is always possible by such a method to refute either position by its semantic dual? Does the rhythmic tide of problems and solutions like those in Russell's self-non-applicational set theory paradox, infinitely regressive ramified type theory solution, and Gödel's arithmetized self-non-application limitations, permit us to project yet another infinite regress strategy in defense of the completeness of logic?

In what follows we use standard logical formalisms to identify the source of duality in the presuppositions of self-non-application and regress arguments, explain precisely how the dialectical interplay between them arises because of these presuppositions, and offer general methodological reflections on the nature and limitations of these styles of philosophical argument.

(1) See for intimations of the thesis Frederic B. Fitch, "Self-Reference in Philosophy", *Mind*, Vol. LV, 1946, pp. 64-73; revised version in *Symbolic Logic: An Introduction* (New York: The Ronald Press Company, 1952), pp. 217-225. Gilbert H. Harman, "Review of Stephen R. Schiffer, *Meaning*", *The Journal of Philosophy*, Vol. LI, 1974, pp. 224-225.

III. *Logical Mechanisms and Metaphilosophical Explanations*

The self-non-application/infinite-regress duality is formalized when five procedures are specified: (1) The formulation of a basis from which the generation of both self-non-application and infinite regress can be described; (2) Characterization of self-non-application as a particular kind of operation producing the basis; (3) Characterization of infinite regress as a particular kind of operation producing the basis; (4) A method for blocking infinite regress via self-non-application; (5) A method for dissolving self-non-application via infinite regress.

We introduce a function f which takes any object O_i as argument and yields object O_{i+1}, $f(O_i) = O_{i+1}$. A successive f-ordering of objects to which the function is inductively applied by the application of function f to any object producing a series of objects O_i, O_{i+1}, not necessarily distinct:

$$\ldots O_i,\ O_{i+1}\ldots$$

There are two possibilities in characterizing applications of function f in terms of its generation of f-ordered series of objects. A function f is regressive (R) or self-non-applicable (S), depending on whether it satisfies the first or second of these conditions:

(R) $R(f) =_{\text{df}} \forall O_i(f(O_i) = O_{i+1} \equiv O_{i+1} \neq O_{i-n})\ \&\ \exists O_i f(f) = O_i \quad (i,n \geq 0)$
(S) $S(f) =_{\text{df}} \exists O_i(f(O_i) = O_{i+1} \equiv O_{i+1} = O_{i-n})\ \&\ \forall O_i f(f) \neq O_i \quad (i,n \geq 0)$

When the application of function f produces distinct objects for any argument in the basis, as in (R), we say the function is regressive, since the application of f to any element in the series adds a new distinct object to it. When as in (S) the application of function f produces as value at least one object identical to an object taken as argument, and there is no object in the series identical to the self-application of function f to itself, then we say the application of the function is literally self-non-applicable or a self-non-application.

The definition in (R) assures that the series of objects produced by inductive application of the function is infinite, and that f is infinitely regressive. The second conjunct defines an object for the self-application of f to f, which under the induction is subject to successive applications of f. Function f applied to the object where the object is identical to $f(f)$ is really $f(f(f))$, identical to yet another object to which function f is applied under the induction, and so on indefinitely. There are deeper metaphysical soundings to be taken of this syntactic evidence for literal self-application when condition (R) obtains. It is clear from the symbolism not only that $f(O_i) = f(f(O_i))$, but also that $f(O_i) = f(f(f(O_i)))$, and indeed that $f(O_i) = \ldots f(f(f(f(O_i))))\ldots$. This shows that the seeds of infinite regress are already contained in the definition of self-application, as a further manifestation of the duality between infinite regress and self-non-application.

There is now a straightforward mechanical explanation of the duality between infinite regress and self-non-application problems and solutions. To produce an infinite regress, it is necessary and sufficient to stipulate that the regressive function satisfy the equivalent of condition (R). If it is not true that every argument to which the function or operation is applied gives rise to another, different or distinct value, then a true regress is not entailed. At some point for some object in the basis given over to the function, the very same output will be produced as value, regardless of how many times the function is applied; and there is nothing regressive about that. The regress is blocked by enforcing the contrary self-non-application characterization of the function according to which it satisfies (S). When this is done the regress stalls because its necessary precondition in (R) is flatly contradicted by (S). The logical mechanism for preventing regress by self-non-application is thus nothing more than negation, contradiction. The contrary relationship holds in the opposite direction for dissolving self-non-applications under condition (S) by enforcing infinite regress categorizations of supposedly self-non-application functions, supporting condition (R) rather than (S) for the function, thereby contradicting a necessary precondition for self-non-application.

IV. *Blocking a Regress, Dissolving a Self-Non-Application*

It goes without saying that not every philosophical argument involves either infinite regress or self-non-application. There is a sufficiently large and intrinsically interesting portion that does exhibit the duality, however, and our purpose in the present context is to try illuminating this pervasive if not universal feature.

Many infinite regresses and self-non-applications are innocuous, and the question of blocking or dissolving these does not arise. It is a result of the duality thesis on the other hand that in principle any infinite regress can be thwarted by self-non-application, and conversely. The problem of identifying rhetorical or dialectical circumstances in which it would be desirable to wield the duality against one argument form or the other is outside the scope of this investigation, and we do not propose to deal systematically or reductively with the wide range of motivations inspiring every argumentative use of the duality by philosophical opponents.

Arguments in which one argument form is invoked against its dual for critical advantage typically occur when the target argument's position is deemed inconsistent with background theoretical or metatheoretical commitments. The explanation for Aristotle's self-non-applicational treatment of inherent or immanent 'formal' definitions against Plato's theory of transcendent Forms in wake of the Third Man is that the regress runs afoul of metatheoretical quasi-aesthetic desiderata for theoretical economy, formulated in the prescient injunction of Ockham's Razor not to multiply entities beyond necessity. The reason why Russell's infinite regress of simple types is marshalled against the self-non-application of the diagonalized Russell set theoretical paradox and its semantic Liar-counterparts is that the paradoxes themselves contradict naive set theory and propositional logic, within which they are constructible. Applications of the duality also occur when theories are experimentally tested at their limits, as when Gödel's arithmetized self-non-application is leveled against the infinite regress of Russell's type theory to show that the theory despite its hierarchy of ordered syntactical predication types cannot avoid all paradoxical metatheoretical self-non-applications.

Here is an elementary example in epistemology. Suppose it is claimed that no proposition can be known to be true unless a proof or justification exists for it. 'Proposition P_i is known or known to be true' then entails 'Proposition P_i is implied by a proposition P_{i+1} known or known to be true'. (If knowledge of the truth of P_i requires knowledge of the truth of several propositions $P_{i+k},\ldots,P_{i+n}$, then consider P_{i+1} to be their conjunction or to abbreviate the conjoined knowledge of each.) We now have a choice of two options: (i) Proposition P_i is always distinct from P_{i+1}, in which case, definition (R) is satisfied and there follows an infinitely regressive chain of justifications; (ii) Proposition P_i is at least sometimes identical to P_{i+1}, in which case definition (S) is satisfied, and the thesis that everything known has a proof distinct from it is self-non-applicable.

The conflict of positions (i)-(ii) presents a paradigm confrontation of an infinitely regressive sufficient reason model and foundationalist self-justification or self-evidence model. We can block the regress of sufficient reasons by supposing that there are ultimately self-justifying bedrock foundations to knowledge; we can dissolve the self-non-application thesis that relaxes the requirement of distinct grounds for belief at the heart of foundationalist theory of knowledge by arguing for the need to reintroduce the infinite regress of 'genuine' distinct proofs for every item of knowledge. The oscillation from one position to the other predicted by the duality thesis describes an alternation of accepted justification models in several chapters of the history of epistemology.

V. *Royce's Map*

Josiah Royce, in the 'Supplementary Essay' to *The World and the Individual*, exemplifies the duality between infinite regress and self-non-application by describing a fictional map, exact in its representation of a topography to the finest detail, and therefore including a representation of the map itself, which represents itself representing itself infinitely, like a parallel juxtaposition of mirrors. The map displays every feature of the terrain, and since the map itself by hypothesis is a feature of the terrain, it must also represent itself.

The mapping representation relation postulated by Royce's example is a particular case of self-application, since the mapping applies directly to itself, mapping the mapping of the geography, and so on indefinitely, each map contained within the map containing a map of itself.

> That such an endless variety of maps within maps could not physically be constructed by men, and that ideally such a map, if viewed as a finished construction, would involve us in all the problems about the infinite divisibility of matter and of space, I freely recognize. What I point out is that if my supposed observer, looking down upon the map, saw anywhere in the series of maps within maps, a last map, such that it contained within itself *no* further representation of the original object, he would know at once...that the resources of the map-maker had failed...(2)

What is interesting about Royce's map is that it crystallizes exactly the intimate connection between self-application and infinite regress. The hypothesis according to which the mapping relation is self-applicational is suf-

(2) Josiah Royce, *The World and the Individual* (Gifford Lectures) (New York: The Macmillan Company, 1923), 'Supplementary Essay', p. 505.

ficient to generate infinite regress in a vivid though imaginary way. Self-representative systems like a map that maps itself in Royce's view participate in a mutual interimplication relation under which any self-representation system implies infinite regress, and any infinite regress implies self-representation. The infinite regress of self-representation in complete exact mappings of mappings and unending self-reflections in an ideal house of mirrors is obvious enough. But the further thesis that any infinite regress implies self-representation as a particular kind of self-application is less straightforward.

Royce holds that wherever there is an infinite series, any infinite selection of the items in the series can be placed in one-one correspondence with the items of the entire series itself. The argument is most convincing and clear-cut in the case of an infinite series of numbers, where for example, the even numbers can be put in one-one relation with the entire series of natural numbers. Wherever this is possible, as for any infinite series, the subset correlate items can be said to represent the items of the set, and since the set literally contains the subset, the representation or correlation is literally self-correlation and reflexive self-representation.

> Now the numbers form, in infinitely numerous ways, a self-representative system of the type here in question. That is, as has repeatedly been remarked, by all the recent authors who have dealt with this aspect of the matter, the number-system, taken in its conceived totality, can be put in a one-to-one correspondence with one of its own constituent portions in any one of an endless number of ways.(3)

> The relational systems of the type of the number-system especially exemplify — of course in a highly abstract fashion — the sort of unity in contrast, and of exact self-representation, which we are to learn to comprehend. Hence the stress here to be laid upon one type of self-representative system.(4)

> Yet, mathematically regarded, this is indeed only one of several possible types of self-representation.(5)

(3) Ibid., p. 515.

(4) Ibid., p. 520.

(5) Ibid.

The importance of Royce's interimplication thesis for infinite regress and self-representation for the duality thesis linking infinite regress and self-non-application is that Royce has pinpointed the essential link between self-representation as a species of self-application and infinite regress, showing that self-representation necessarily implies infinite regress and conversely. This establishes in an especially graphic way why a particular self-non-application must always be sufficient to block infinite regress, and why some other particular infinite regress must always be sufficient to dissolve self-non-application. Infinite regress implies self-application, so self-non-application blocks it, and in turn is dissolved by infinite regress.

The duality can be applied to the ideal construction of Royce's map. Royce instructs us to imagine a self-representing map in which the mapping relation is self-applicational, generating an infinite regress of maps within maps. To block the regress it is sufficient only to argue that maps by their very nature are self-non-applicational. A plausible argument to this effect is suggested by Royce himself, in commenting on the possible existence of a self-applicational map, refuted by Bradley's argument against infinite series of existents as self-contradictory.

> The whole infinite series, possessing no last member, would be asserted as a fact of existence. I need not observe that Mr. Bradley would at once reject such an assertion as a self-contradiction. It would be a typical instance of the sort of endlessness of structure that makes him reject Space, Time, and the rest, as mere Appearance.([6])

Royce is content to hold that even if Bradley's argument is accepted, it only serves further to illustrate the fact that self-representation implies infinite regress, and that indeed the implication provides the basis for any Bradley-style objection to the existence of a self-representational map.([7])

To complete the argument it is necessary only to supplement Royce's own admission about the force of Bradley's criticism by adding that maps in the

([6]) Ibid., p. 507. F.H. Bradley, *Appearance and Reality: A Metaphysical Essay* (Oxford: Clarendon Press, 1930), Ninth Impression, Corrected, pp. 154-156, 257-258; *The Principles of Logic*, Second Edition (London: Oxford University Press, 1922), Vol. I, pp. 228-233.

([7]) *The World and the Individual*, p. 507. Charles Sanders Peirce, *The Collected Papers of Charles Sanders Peirce*, edited by Charles Hartshorne and Paul Weiss, Vol. V, 'Pragmatism and Pragmaticism' (Cambridge: Harvard University Press, 1934), Lecture III, §1 "Degenerate Thirdness", pp. 49-50.

true, legitimate sense of the word are concrete spatio-temporal objects that cannot by definition be self-applicational precisely because they would then be infinitely regressive. It is only by equivocation on the word 'map' that Royce's argument has any initial plausibility, since ideal maps are not really maps at all. Self-representational or self-applicational mapping relations might imply infinite regress, but mapping relations are not maps, and self-applicational mappings themselves can be dismissed as 'mappings' by equivocation only, on the grounds that genuine mappings result in genuine maps. The conclusion then is not there exist any truly self-applicational or infinitely regressive maps or mapping relations, ideal or concrete, except in the trivial sense in which a 'mapping' just is a self-representation or self-application. But to say that self-representations or self-applications are self-representational or self-applicational is to stutter out a tautology, and to say that self-representations or self-applications imply infinite regress is to repeat the admitted basis for Bradleyan criticism of the thesis that there could ever be genuine self-representational, self-applicational, and infinitely regressive mappings or maps.

VI. *Methodological Coda*

What does the existence of the duality mean, in the most general, metatheoretical, metaphilosophical terms? There is a temptation to regard philosophical argument that involves the dialectical ebb and flow of self-non-application and infinite regress as depriving such investigations of real merit, a parlor trick easy to perform once we know the secret.

That this is not the case is evident as soon as we examine any of the philosophically interesting historical interactions of the duality involving regress and self-non-application. It is by no means trivial to have proposed the self-non-applicational Russell paradox in set theory, nor the infinite regress of ordered types as its solution, nor again the Gödel incompleteness provability self-non-applicational theorems limiting first order predicate logic stratified into Russellian simple types. The Third Man infinite regress objection to Plato's theory of Forms appears trite only thanks to the insight and intellectual labor of others. Aristotle's self-non-applicational theory of inherent secondary substances or definitions as a solution to the difficulty and method of blocking the regress is also by no means lightly suggested to casual observation even when the regress/self-non-application duality is understood and the movement from one argument form to its dual is anticipated.

At most the duality teaches us to look for counterarguments having a certain complementary form when their duals appear in specific contexts of philosophical investigation. The duality does not guarantee that arguments of the required kind will be available, let alone that they will be effortless to produce. It is one thing to acknowledge from a facile metatheoretical perspective that Russell's infinitely regressive hierarchy of syntax types can be overcome as a general solution to self-non-application paradoxes by denying its underlying self-application presupposition and reinstating yet another paradoxical self-non-application construction, and quite another to identify as Gödel did the exact and by no means trivial logical machinery for accomplishing this. Moreover, the development of thought and discovery of new principles and techniques that unfolds as the dialectic of dual argument forms is pursued is so intrinsically important, and contributes so directly to intellectual advances, that there is no cause for cynical despair in contemplating the potentially endless undulation of regressive or self-non-applicational conclusions in the history of philosophy. It is satisfying to think that human ingenuity need never be exhausted in the elaboration of problems and solutions governed by the duality of self-non-application and infinite regress. The dialectical movement from infinite regress to self-non-application and back again is not a mere back and forth repitition of the very same presuppositions for infinite regress following on the heels of the very same presuppositions for self-non-application or the reverse. Rather, subtle and wonderful changes are introduced, each refined and made more sophisticated precisely because of challenges and constraints posed by previous moments in the dialectical interplay of these more general dual categories. There is no more reason to deny philosophical significance to the movement from infinite regress to self-non-application and back again than to the fluctuations in the acceptance or rejection of any other presuppositions of thought, as in periodic transitions from empiricist to rationalist or the opposite methodologies.

Finally, there are natural though undoubtedly evolving limitations on what can count as satisfactory problems and solutions forthcoming within the duality at any moment of its development. This is true by virtue of the changeable background of assumptions against which the rhetorical propriety of invoking particular kinds of self-non-application arguments against infinite regresses and conversely is determined. Even if in principle it is always possible to block a regress by self-non-application, there may be no or no immediately evident self-non-applications that could be put forward that would not at the same time contradict other perhaps equally or more cherished presuppositions or philosophical conclusions, so that movement is precluded as too costly unless the background assumptions are critically reexamined. This may well be the case in the widespread contemporary acceptance of Gödel's impressive refutation of the Hilbertian program in mathematics, and may explain why countercriticisms of his incompleteness results, though not in principle impossible, are virtually nonexistent in the subsequent history of mathematical logic, why his metatheory has acquired the status of received truth about the limits of proof. There is a kind of progress when the waters of duality subside, and the ripples from an origin of dispute settle into equilibrium. Within a particular cultural context of fixed beliefs and overriding theoretical or metatheoretical desiderata the dialectic can come to a complete standstill, in which the victorious final stage emerges as a new almost unquestionable cornerstone of thought.

We leave these metatheoretical considerations with a final problem, a difficulty which we shall not try to answer, but for which the duality thesis has important implications. Does the dialectical interplay of infinite regress and self-non-application itself continue indefinitely in an infinite regress, or is the duality thesis self-non-applicational? The question is not inevitable, since we have acknowledged from the outset that not every philosophical or metaphilosophical argument falls under the duality thesis. But while it may appear that the duality thesis lends itself to one category or the other, if we describe it as implying an infinite regress, the thesis meta-implies the possibility of blocking the regress by redescribing it as self-non-applicational. This is supposed to block the regress, but duality entails that in principle we can block regresses and dissolve self-non-applications one by the other indefinitely, so the metatheoretical infinite regress is not blocked. To categorize the thesis as self-non-applicational, if the thesis is true, opens up the possibility of dissolving the self-non-application of the thesis by postulating a complementary infinite regress, so that the thesis turns out to be self-applicational rather than self-non-applicational.(8)

The Pennsylvania State University

(8) This has been a fully collaborative effort, and order of authorship is alphabetical only. An earlier informal version of the essay was presented before the Second International Conference of the International Society for the Study of Argumentation (ISSA), University of Amsterdam, Amsterdam, The Netherlands, June 22, 1990, under the title, "Self-Reference and Infinite Regress in Philosophical Argumentation".