A TABLEAU CALCULUS FOR DRT

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Abstract

The concern of the following paper is to present a theoretical basis for a deductive theory for Discourse Respresentation Theory (DRT). DRT languages have a well-defined syntax and semantics and can be manipulated deductively. Thus we present hereafter a tableaux calculus method called box-tableaux method for DRT; furthermore we give a proof of the consistency and completeness of the method ([1]).

0. Introduction

Discourse Representation Theory (henceforth DRT) is a theory for natural language discourse proposed by Kamp. The theory intends to cope with a number of well-known phenomena such as anaphora and events. DRT assigns to a sentence or a bit of discourse a well-defined object K called Discourse Representation Structure (DRS). A DRS can be seen as a representational language for natural language semantics; therefore we have to face in some way the inferential problem raised by any representation language – i.e. given DRS's K and K′, how can we prove the following statement: K logically entails K′. The way this statement has been formulated indicated that the notion of logical consequence can be applied to DRS's. This is in fact true since a DRS can be regarded as a variant of first-order logic; therefore a DRS has a well-defined syntax and semantics, and can be manipulated deductively in appropriate models.

In this paper we shall provide a deductive theory for DRS's based on the tableau calculus method. We follow carefully the analytic method of Smullyan [68] and (analogically) proposed the box-tableau method for DRS's. A box-tableau is a tree whose nodes are labeled by atomic for-

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mulae or their negation and DRS's (seen as intermediate results). We prove completeness and consistency of the box-tableaux and provide a variant of the liberalization rule.

In Guenther & al [86] a deductive theory for DRS's based on the tableau calculus has been investigated. The tableau method used there is a slightly different version of the analytic tableaux of Smullyan in the sense that only atomic or negated atomic formulae are inserted into the tableau. Moreover there is no detailed and complete proof of consistency and completeness of the method.

1. DRT and semantic interpretation

Discourse Representation Theory (henceforth DRT, cf. Kamp [81] assigns to a sentence or a fragment of discourse **D** a level of representation (Discourse Representation Structures, DRS's) distinct from the usual logical forms (cf. Guenthner [86]). Thus, instead of defining the truth-conditions of sentences in appropriate models, the DRT account defines them as DRS's associated with these sentences. The discourse representation construction rules work as follows: given a DRS K (associated with a discourse **D**) and a fragment **S** of discourse or a sentence, we generate a new DRS K′ which corresponds to the continuation of **D** by **S**. It turns out that DRS's can be evaluated in a model on one hand and handled deductively on the other hand. Instead of giving the syntax of a fragment of language, we shall illustrate by a few examples how DRT assigns representations to discourses.

To (1) A farmer owns a donkey

we assign the following DRS (displayed as a 2-dimensional structure):

(2)

x	y
farmer (x) donkey (y) own (x,y)	

At the top level of this structure are objects called *Discourse Markers* (namely *x* and *y*) which form the Universe of the DRS, and inside the box are three objects called *Conditions* (which are conditions on the discourse markers).

The semantics of (2) is defined (as in model-theoretic semantics) as follows:

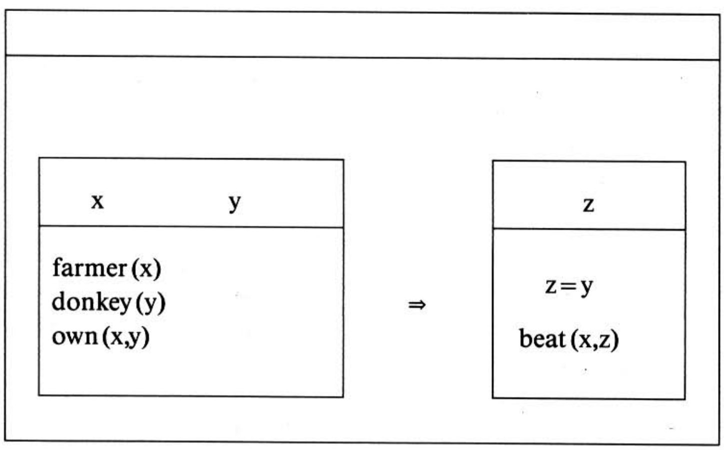
There is an embedding function *f* such that to a formal representation corresponds some situation of the world. That is to say, (2) is true in a model M if there is an embedding function *f* which maps the universe of (2) into the universe of M, and if the objects in the range of *f* satisfy every condition in (2) i.e. DRS (2) is true if there is a function *f* which assigns to *x*, *y* the extensions *a*, *b* such that *a* is a *farmer*, *b* is a *donkey* and *a owns b*.

It should be noted that in (2), *a farmer* is treated as follows: we introduce a discourse marker *x* and a condition *farmer*(*x*)
DRT assigns to universally quantified NP's a rather different representation, e.g. to

(3) Every farmer who owns a donkey beats it

is assigned the DRS

(4)



(4) is a DRS whose universe is empty and whose condition is a complex condition of the form K1 ⇒ K2 where K1 and K2 are DRS's (this

complex condition is also called a splitted DRS, or a universal condition). K1 is called the antecedent DRS and K2 the consequent DRS.
The truth condition of (4) can be illustrated as follows:

Given the model:

(5) I(farmer) = {John, Peter, Max}
I (donkey) = {Tom, Sam, Junior, Senior}
I (own) = {<John, Tom>, <John, Sam>, <Peter, Junior>, <Max, Senior>}
I (beat) = {<John, Tom>, <Peter, Junior>, <Max, Senior>}

where I an *interpretation*, the DRS (4) is *not* true in this model since John doesn't beat the donkey Sam that he owns. So the quantificational interpretaion that the DRT assigns to the indefinite NP *a donkey* is the universal quantification. Thus the truth condition for (4) is given along the following lines:
(4) is true if every embedding function that verifies the antecedent can be extended to an embedding function that verifies the consequent.

Finally, a DRS is a pair [U,CON] where U is a (pssibly empty) set of discourse markers, and CON a set of conditions. We shall first describe the syntax of the DRT language and then give its semantics.

1.1. Syntax of DRT-language

We define below the symbols of the DRT-language; we shall use the following *logical connectives*:
TRUE, FALSE
$\neg$ [for "not"]
$\vee$ [for "or"]
$\Rightarrow$ [for "implies"]

The three last symbols are called binary connectives and the one before it is called a unary connective. The first two are 0-ary connectives (or "propositional constants").

Well-formed pure DRS's

K(= [U,CON]) is a *well-formed pure DRS* iff

(i) U = {x1, x2,... xn} where *xi* is a Discourse Marker
(ii) CON a finite set of conditions C1,...,Cn each of which is either

- an atomic condition of the form P^n $(x1, x2, ..., xn)$ or $xi = xj$ where xi ∈ U and P^n is an n-ary relational symbol.
- a universal condition of the form $K1 \Rightarrow K2$ where K1 and K2 are well-formed DRS's
- a negative condition of the form $\neg K$ where K is a well-formed DRS
- a disjunctive condition of the form $K1 \vee K2$ where K1 and K2 are well-formed DRS's
- a well-formed DRS K′

It will be seen to turn out that the above syntax of DRT is a variant of the language of LPC (lower predicate calculus, without function symbols): to each DRS corresponds a formula in LPC and conversely (cf section 1.2.3 below). The language given here is a one-sorted language in the sense that there is only one type of discourse marker, while in the original DRT-language or its extensions (e.g. Kaspar & al [87] DRT-language is a many-sorted language with various types of discourse markers for *event, time, locative, plural* etc... This could easily be made up for, carrying along the sorts in every rule; we shall not, however, burden the reader with these obvious details.

DRS's with parameters

Although they can be dispensed with (using only variables, since models can be defined in Herbrand universes cf Manna [74] – alias canoncial realizations cf Kreisel & Krivine [71], we shall nevertheless describe DRS's with parameters, since the latter are standard stuff in Analytic Tableaux methods.
K (= [U, CON]) is a *DRS with parameters* iff CON is defined as in the pure case, except that the *atomic* conditions may contain *terms* (variables or parameters) instead of variables only. Henceforth all DRS's are with parameters, since the pure ones are a particular case of the ones with parameter.

1.2. Semantics of DRT-languages

The truth-definition of a DRS can be defined in models, as for the formulae of LPC.

1.2.1. Pure DRS's and models

Definition 1
A *model M for a pure DRS* K (= [U, CON]) is a pair <D, I> where D is a (non empty) domain of discourse, and I an interpretation function which (as in the LPC case) associates extensions in D with every n-ary relational symbol. Let f be a partial function whose domain includes U and whose range is in D; f is called an *admissible embedding* of U in D.

The truth definition for a pure DRS is given as follows:

Definition 2
A pure DRS is *true with respect to M* iff there is an admissible embedding of U in D which satisfies every condition in CON.

The following few lines give the satisfaction conditions for a pure DRS:

Definition 3
f *satisfies a pure condition of the form* $P^n(x1, x2,..., xn)$ iff $< f(x1), f(x2),..., f(xn) > \in I(P^n)$

Definition 4
f *satisfies a pure condition of the form* $xi = xj$ iff $f(xi) = f(xj)$

We need a new definition before giving the satisfaction definition of complex conditions:

Definition 5
Let X be a set of Discourse Markers ($U \supseteq X$), and g and f be partial functions on U. g is an *X-extension of f* if g assigns the same values to all the markers in the domain of f as f does, and in addition g assigns values to all the discourse markers in X (and more precisely the domain of g is the union of X and the domain of f).

Definition 6
f *satisfies a pure condition of the form* $K1 \Rightarrow K2$ iff for every U_{K1} extension g of f which satisfies K1 there is an U_{K2} extension h which satisfies K2.

Definition 7
f *satisfies a pure condition of the form* $\neg K1$ iff there is no U_{K1} extension g of f which satisfies K1

Definition 8
f *satisfies a pure condition of the form K1* ∨ *K2* iff there is an extension g of f which satifies K1 or K2.

Definition 9
f *satisfies a pure condition of the form K1* iff there is an extension g of f which satisfies K1.

We proceed now to show that there is a direct mapping from DRS's to LPC formulae and conversely in the following sense: to any DRS K corresponds an LPC formula F, and to any well-formed formula F in LPC corresponds a DRS K. This will give ultimately the semantics of DRT-languages.

1.2.2. Translation of DRS's into LPC formulae

Given a DRS K = [U, CON] (where U={x1,...xn}) we associate with it a formula F, denoted Pr(K), as follows:

> F=∃x1...∃xn ∧ Pr(C), where ∧ Pr(C) is the conjunction of the translations of all conditions Ck in CON.

The translations of the conditions are obtained as follows:

(i) Given a condition of the form P^n(x1,x2,...xn) its translation is P^n(x1,x2,...,xn).
(ii) The translation of a condition of the form xi = xj is xi = xj.
(iii) Given a condition of the form K1 ⇒ K2 where K1 = [U1, CON1] and K2 = [U2, CON2] (such that U1 = {x1,x2,...,xn} and U2 = {y1,y2,...,yp}), its translation is ∀x1 ∀x2 ... ∀xn (∧ Pr(C1) ⇒ ∃ y1 ∃ y2 ... ∃ yp ∧ Pr(C2)), where ∧ Pr(C1) is the conjunction of the translations of all conditions in CON1 and ∧ Pr(C2) is the conjunction of the translations of all conditions in CON2.
(iv) The translation of ¬ K1 is ¬ Pr(K1); and finally the translation of K1 ∨ K2 is Pr(K1) ∨ Pr(K2).

Thus, the application of the translation rules above will produce for DRS (4) the following LPC formula:

(1) (∀x)(∀y)[(farmer(x) ∧ donkey (y) ∧ beat(x,y)) ⇒ (∃z)(z=x ∧ beat(x,z))]

(1) is obtained stepwise as follows:

step 0 K = [{ },CON] is translated into a formula given by the translation of CON (since the universe of K is empty).

step 1 CON=[K1 ⇒ K2] with K1 = [{x,y},CON1] and K2=[{z},CON2] is translated into

$$(\forall x)(\forall y)[farmer(x) \wedge donkey(y) \wedge own(x,y)) \Rightarrow (\exists z)(z=x \wedge beat(x,y))]$$

1.2.3. Translation of LPC formulae into DRS's

We want now to show that any LPC formula F can be mapped into a DRS K, denoted Dr(F):

The Mapping Procedure

(i) We begin the mapping procedure by creating a DRS K′ = [U′, CON′] where U′ and CON′ are empty, and we obtain K by embedding F into K′ as follows:

Embedding rules

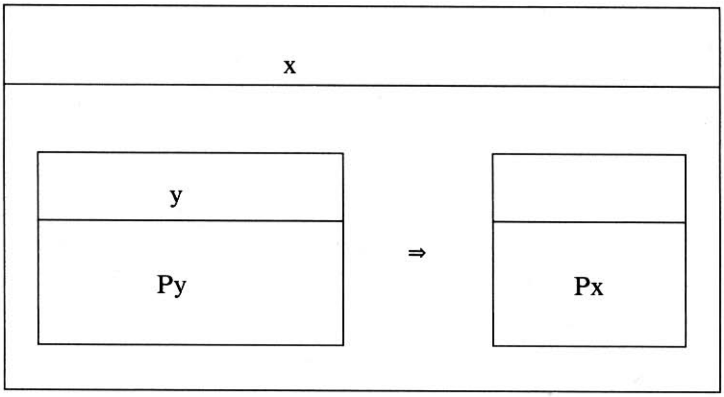
(ii) for F = (∃ x1)P, we introduce x1 into U′ and embed the translation of P into K
(iii) for F = (∀ x1) P we add to CON′ the condition (K1 ⇒ K2); where K1=[{x1},TRUE], and we embed P into K2; TRUE is true under any interpretation.
(iv) If P is a n-ary relational symbol, we add P into CON′
(v) for P = Q1 ∧ Q2, we embed successively Q1 and Q2 into K′
(vi) for P = Q1 ∨ Q2 we map Q1 into a new DRS K1, map Q2 into the new DRS K2 and add the condition K1 ∨ K2 to CON′
(vii) for P = (F1 ⇒ F2) we create a complex condition K1 ⇒ K2 that we add to CON′; we map respectively F1 into K1 and F2 into K2.

The following mapping of formulae into DRS's will illustrate the rules above:

(2) $(\exists x)[\exists y)Py \Rightarrow Px]$

is translated into the DRS

(3)

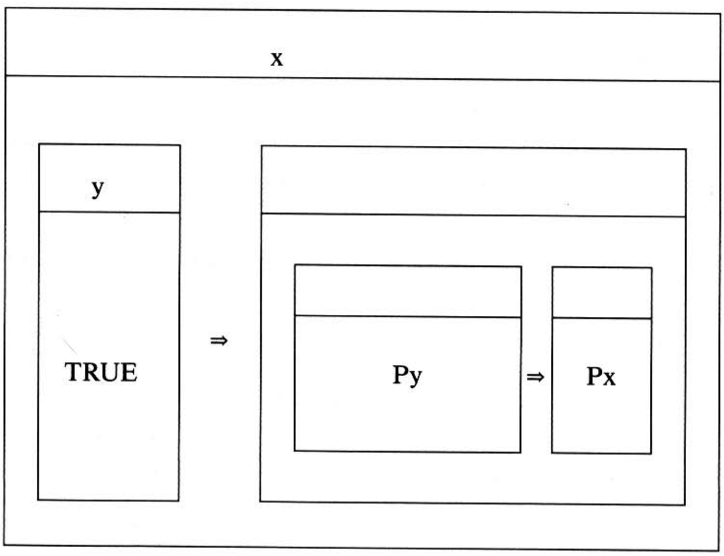


The formula

(4) $(\exists x)[(\forall y)(Py \Rightarrow Px)]$

is translated into the DRS

(5)

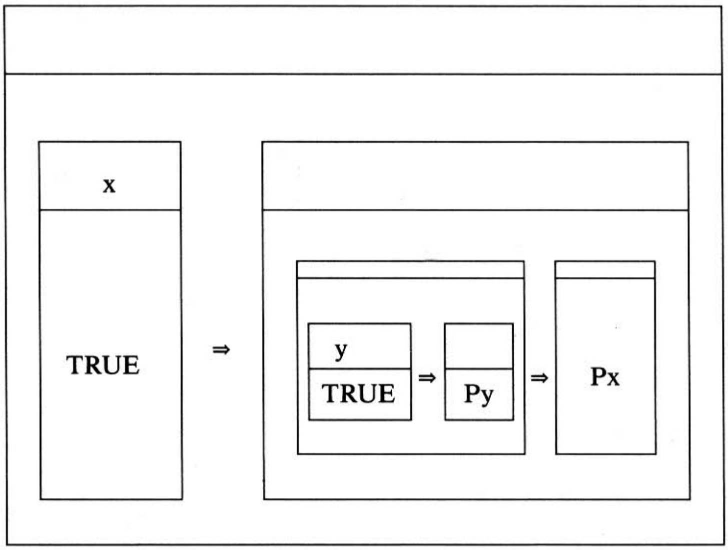


We assign to

(6) $(\forall x)[(\forall y)Py \Rightarrow Px]$

the translation

(7)



1.2.4. DRS's with parameters and models

DRS's with constants in D

Let D be any domain (non-empty set). We want to define the notion of DRS's *with constants* in D – briefly "D-conditions".

Definition 10
By an *atomic* D-condition we mean either an atomic condition of the form P^n (s1,...,sn) or si = sj, where each si is either a variable or an element of D (note that we *do not* allow any si to be a paramater) and P is an n-ary relation symbol.
Having defined the atomic D-conditions, we can then define the set of all D-conditions by the formation rules given in Section 1.1. Thus a D-condition is like a DRS with parameters except that it contains elements of D instead of parameters ([2]). This includes the "pure" conditions – those with no parameters and no constants in D – as special cases.

([2]) The notion of D-conditions is close to the notion of indeterminates of situation types in Situation Semantics theory (Cf Barwise & Perry [83]).

Definition 11
For any sequence **k** $\in D^*$, we define the D-condition $CON_{u/k}$ as the set of all conditions obtained from those in CON by substituting the sequence **k** $\in D^*$ for the sequence **u** of all elements of U. The reader will readily supply an inductive definition of this notion. The set of all *closed D-conditions* ([3]) will henceforth be denoted by $\mathbf{B}^D$.

Definition 12
Let σ be a mapping from the set of parameters ([4]) of a set **B** (of D-conditions with parameters) into D. For any condition C in **B**, let C^σ mean the result of simultaneously substituting for each parameter ai of C its image σ(ai) under σ. We shall say that **B** is (simultaneously) *satisfiable in D* if there exists an interpretation I of the predicates of **B** and there exists a "substitution" σ mapping the parameters of **B** into elements of D such that for any condition C of **B**, C^σ is true under I. Hence a *model* is now a triple M = <D, I, σ>, instead of a pair M = <D, I> in the pure case. The rest proceeds as above for pure DRS's.

2. *Analytic box-tableaux*

2.1. Preliminary definitions

We introduce now an important notion, that of a valid DRS:

Definition 13
A DRS K is *valid* ([5]) iff K is true under any interpretation I.

Lemma 1: Let Pr(K) the formula obtained by translating the DRS K into LPC; given a formula F, denote its tanslation into a DRS by Dr(F). If a DRS K is true in a model, then Pr(K) is also true in that model; and conversely. If an LPC formula F is true in a model, then Dr(F) is also

([3]) Recall that a D-condition is a condition with constants in D; hence a closed D-condition is a condition with constants in D, in which no occurrence of variable is free.

([4]) The status of parameters, with respect to DRT, is not easy to define. DRT uses only discourse markers for which we can assign extensions in a model; therefore it should be convenient to consider parameters as particular discourse markers that never appear in top of DRS's (this view has been suggested to us by H. Kamp).

([5]) Cf the Conclusion for discussion on the relevance of this notion for the current formulation of DRT.

true in that model; and conversely. Moreover Dr(Pr(K)) and K are logically equivalent.
Proof: induction on the definitions. ●

Corollary: If a DRS K is valid, so is Pr(K), the LPC formula associated to K. Conversely given F, a valid formula in LPC, its translation Dr(F) is a valid DRS in the language of DRT. ●

2.2. The method of box-tableaux

Definition 14
A *box* is either a DRS K = [U, CON] or an element of CON, for some DRS of that form, or TRUE or FALSE.
A *signed box* is a box of the form *T* C, *F* C (ie C is preceeded by the symbol T (TRUE), respectively F (FALSE)).

Definition 15
Under any interpretation a signed box *T* C is called *true* if C is true, and *false* if C is false. And a signed box *F* C is *true* if C is false and *false* if C is true.
By the *conjugate* of a signed box we mean the result of changing the symbol *T* to *F*, or *F* to *T* (e.g. the conjugate of *F* C is *T* C and conversely). Moreover *F* TRUE and *T* FALSE, resp *F* FALSE and *T* TRUE are conjugate.
BOX-TABLEAUX are a variant of the analytic tableaux of Smullyan. The idea underlying the box-tableaux system is the following: to prove a DRS K, we try to falsify K i.e. to find a model for *F* K.
We shall illustrate by an example:

Suppose we want to prove the DRS K:

K = [{ }, K1 ⇒ K2]
K1 = [{ }, K11 ⇒ K12]
K2 = [{ }, K21 ⇒ K22]

K11 = [{x}, {P(x)}]
K12 = [{y}, {Q(y), A(x,y)}]

K21 = [{x}, {P(x), R(x)}]
K22 = [{y}, {Q(y), A(x,y)}]

the following lines give the bax-tableau for K:

(1) *F* K
(2) *F* (K1 ⇒ K2)
(3) *T* (K11 ⇒ K12)
(4) *F* ([{ }, {K21 ⇒ K22}]
(5) *F* (K21 ⇒ K22)
(6) *T* P(a)
(7) *T* R(a)
(8) *F* [{y}, {Q(y), A(a,y)}]

(9) *F* P(a) (10) *T* [{y}, {Q(y), A(a,Y)}]

X X

comment:

Let's explain how is contructed the box-tableau, above, for K. At the top of the tableau is the box *F* K i.e. we are trying to derive a contradiction from the assumption that K is false. A box of the form *F*[{ }, CON] is true if one of the Conditions contained in CON at least is false. So that in (2) we write that the unique box (K1 ⇒ K2) contained in CON is false.

A box *F* (K1 ⇒ K2) is true if $(K1)_{\exists/U1}$ is true and $(K2)_{S/U1}$ is false. We denote by $(K)_{\exists/U}$, for K = [U, CON] with U = {x1,x2,...,xn}, the result of simultaneously substituting in CON for each occurrence of xi the individual parameters ai (where ai, for 1 ≤ i ≤ n, is a new parameter, and each ai is distinct from all other aj, j ≤ i). We denote by $(K)_{S/U}$, for K = [U′, CON], the box K′ = [U′, CON_U] where CON_U is the result of simultaneously substituting in CON for each occurrence of xi (which belongs to U) their substitution parameters *ai*. K1 = [U1, K11 ⇒ K12}], with U1 = { }; therefore $(K1)_{\exists/U1}$ consists in writing the CON of K1 in the tableau. In this way we obtain line (3); in line (4) we see that $(K2)_{S/U2}$ yields *F* K2. Lines (5), (6), and (7) are obtained from (4). Now look at (3); we have two ways of proving that (3) holds, so we add two new branches to the box-tableau. In the first one we add *F* $(K11)_{\forall/U1}$ and in the second *T* $(K12)_{S/U1}$. We denote by $(K)_{\forall/U}$, for K = [{x1,x2,...xn}, CON], the result of simultaneously substituting in CON for each occurrence of xi the individual parameters ai (where ai may be new or not). The universe of K11 is {x}; so we substitute for x the existing parameter a.

Now if we look at the first branch we see that (9) is a direct contradiction of (6) (i.e. (9) is the conjugate of (6)), so we put a corss after (9) to say that this branch leads to a contradiction. In the second branch we see that (8) is the conjugate of (13), so this branch leads also to a contradiction and we put after (13) a cross to "close" the branch. Thus (1) is untenable and therefore K is a true DRS under any interpretation, that is a valid DRS.

Here is the corresponding analytic tableau:

The translation of the DRS K into LPC yields the logical formula

$$Pr(K) = [(\forall x)((P(x) \Rightarrow (\exists y)(Q(y) \wedge A(x,y)))] \Rightarrow [(\forall x)[(P(x) \wedge R(x)) \Rightarrow (\exists y)(Q(y) \wedge A(x,y))]$$

the analytic tableau for Pr(K) is given along the following lines:

(1) $F\ [(\forall x)((P(x) \Rightarrow (\exists y)(Q(y) \wedge A(x,y)))] \Rightarrow [(\forall x)[(P(x) \wedge R(x)) \Rightarrow (\exists y)(Q(y) \wedge A(x,y))]$

(2)	$T\ (\forall x)((P(x) \Rightarrow (\exists y)(Q(y) \wedge A(x,y)))$	from 1
(3)	$F\ (\forall x)(P(x) \wedge R(x)) \Rightarrow (\exists y)(Q(y) \wedge A(x,y))$	from 1
(4)	$F\ P(a) \wedge R(a) \Rightarrow (\exists y)(Q(y) \wedge A(x,y))$	from 3
(5)	$P(a) \wedge R(a)$	from 4
(6)	$F\ (\exists y)(Q(y) \wedge A(a,y))$	from 4
(7)	$P(a)$	from 5
(8)	$R(a)$	from 5
(9)	$P(a) \Rightarrow (\exists y)(Q(y) \wedge A(a,y))$	from 2

(10) $F\ P(a)$	from 9	(11) $(\exists y)(Q(y) \wedge A(a,y))$	from 9
X		(12) $Q(b) \wedge A(a,b)$	from 11
		(13) $Q(b)$	from 12
		(14) $A(a,b)$	from 12
		(15) $F\ (Q(b) \wedge A(a,b)$	from 6
		(16) $F\ Q(b)$ from 15	(17) $F\ A(a,b)$ from 15
		X	X

Each branch of the tableau is closed; thus, Pr(K) is a tautology.

RULES FOR THE BOX-TABLEAUX

Here are the rules for the box-tableaux:

(1) $\dfrac{T\ \neg K}{F\ K}$ $\qquad$ $\dfrac{F\ \neg K}{T\ K}$

(2) $\dfrac{T\ (K1 \vee K2)}{T\ K1 \mid T\ K2}$ $\qquad$ $\dfrac{F\ (K1 \vee K2)}{\begin{array}{c}F\ K1\\ F\ K2\end{array}}$

(3) $\dfrac{T\ (K1 \Rightarrow K2)}{F\ (K1)_{\forall/U1} \mid T\ (K2)_{S/U1}}$ $\qquad$ $\dfrac{F\ (K1 \Rightarrow K2)}{\begin{array}{c}T\ (K1)_{\exists/U1}\\ F\ (K2)_{s/U1}\end{array}}$ $\qquad$ where K1 = [U1, CON1]

(4) $\dfrac{T\ K}{T\ CON_{\exists/U}}$ $\qquad$ $\dfrac{F\ K}{F\ CON_{\forall/U}}$

where K is not (1), (2) or (3) and K = [U, CON].

We explain now some notations in the rules above:
For K = [U, CON] we define:

$\dfrac{F\ (K)_{\forall/U}}{F\ CON_{\mathbf{u}/\mathbf{a}}}$ $\qquad$ $\dfrac{T\ (K)_{\forall/U}}{T\ CON_{\mathbf{u}/\mathbf{a}}}$ $\qquad$ with **a** any sequence of parameters

$\dfrac{F\ (K)_{\exists/U}}{F\ CON_{\mathbf{u}/\mathbf{a}}}$ $\qquad$ $\dfrac{T\ (K)_{\exists/U}}{T\ CON_{\mathbf{u}/\mathbf{a}}}$ $\qquad$ with proviso

We denote by **u/a** the substitution for the sequence **u** of the sequence of parameters **a**, where **u** is the sequence of all elements of U. By $CON_{\mathbf{u}/\mathbf{a}}$ we denote the substitution of **u** by **a** in every condition contained in CON.

Provisio:
The sequence a must be a sequence of new distinct parameters

And finally we define:

$\dfrac{T\ CON_{\xi/U}}{\begin{array}{c}T\ C1_{\xi/U}\\ \vdots\\ T\ Cn_{\xi/U}\end{array}}$ $\qquad$ $\dfrac{F\ CON_{\xi/U}}{F\ C1_{\xi/U} \mid \ldots \mid F\ Cn_{\xi/U}}$ $\qquad$ where CON = {C1, C2,...,Cn}

(where ξ is one of $\exists$, S, $\forall$)

$$\frac{F\ (\mathrm{K})_{S/U}}{F\ \mathrm{K}'} \qquad \frac{T\ (\mathrm{K})_{S/U}}{T\ \mathrm{K}'}$$

Where $\mathrm{K}' = [\mathrm{U}', \mathrm{CON}_{\mathbf{u}/\mathbf{a}}]$
Given a universe U and its substitution set S (i.e. the set consisting of s such that s is substituted for u in U), we denote by $\mathrm{K}_{S/U}$ the result of substituting s for u in every condition of CON which contains an occurence of u.

UNIFIED NOTATION:

We shall now introduce a unified notation as for analytic tableaux; for every box α of the form
$T \neg \mathrm{K}$, $F \neg \mathrm{K}$, $F\ (\mathrm{K1} \vee \mathrm{K2})$, $F\ (\mathrm{K1} \Rightarrow \mathrm{K2})$ we define the two sub-boxes $\alpha 1$ and $\alpha 2$ as follows:

if $\alpha = T \neg \mathrm{K}$ then $\alpha 1 = F\ \mathrm{K}$ and $\alpha 2 = F\ \mathrm{K}$
if $\alpha = F \neg \mathrm{K}$ then $\alpha 1 = T\ \mathrm{K}$ and $\alpha 2 = T\ \mathrm{K}$
if $\alpha = F\ (\mathrm{K1} \vee \mathrm{K2})$ then $\alpha 1 = F\ \mathrm{K1}$ and $\alpha 2 = F\ \mathrm{K2}$
if $\alpha = F\ (\mathrm{K1} \Rightarrow \mathrm{K2})$ then $\alpha 1 = T\ \mathrm{K1}$ and $\alpha 2 = F\ \mathrm{K2}$

For every box β of the form $T\ (\mathrm{K1} \vee \mathrm{K2})$, $T\ (\mathrm{K1} \Rightarrow \mathrm{K2})$ we define the two sub-boxes $\beta 1$ and $\beta 2$ as follows:

if $\beta = T\ (\mathrm{K1} \vee \mathrm{K2})$ then $\beta 1 = T\ \mathrm{K1}$ and $\beta 2 = T\ \mathrm{K2}$
if $\beta = T\ (\mathrm{K1} \Rightarrow \mathrm{K2})$ then $\beta 1 = F\ \mathrm{K1}$ and $\beta 2 = T\ \mathrm{K2}$

We call γ-box a box of the form F K and δ-box a box of the form T K (K not of the form (1), (2) or (3) above); we denote by $\gamma(\mathbf{a})$ and $\delta(\mathbf{a})$, respectively $F\ \mathrm{CON}_{\forall/U}$ and $T\ \mathrm{CON}_{\exists/U}$.

We observe that under any interpretation α is true iff $\alpha 1$ and $\alpha 2$ are both true; that β is true under any interpretation iff $\beta 1$ is true or $\beta 2$ is true. γ is true if $\gamma(\mathbf{a}) = F\ \mathrm{CON}_{\forall/U}$ is true, ie if for the sequence $\mathbf{u}$ of all elements in U, $F\ \mathrm{CON}_{\mathbf{u}/\mathbf{a}}$ is true for $\mathbf{u}$ and $\mathbf{a}$ (with proviso).

δ is true if $\delta(\mathbf{a}) = T\ \mathrm{CON}_{\exists/U}$ is true, ie if for the sequence $\mathbf{u}$ of all elements in U, $T\ \mathrm{CON}_{\mathbf{u}/\mathbf{a}}$ is true for $\mathbf{u}$ and $\mathbf{a}$ (any sequence of parameters). This observation can be extended easily (induction on the formation rules for DRS's) to a consistency proof for the box-tableau method.

Our rules for the box-tableaux can be reformulated as:

rule A: $\dfrac{\alpha}{\begin{array}{c}\alpha 1\\ \alpha 2\end{array}}$

rule B: $\dfrac{\beta}{\beta 1 \mid \beta 2}$

rule C: $\dfrac{\gamma}{\gamma(\mathbf{a})}$ where **a** is a sequence of arbitray parameters

rule D: $\dfrac{\delta}{\delta(\mathbf{a})}$ where **a** is a sequence of new distinct parameters

Definition 16

A *box-tableau* for K is an ordered dyadic tree whose points are (occurences of) boxes which is constructed as follows: we start by placing *F* K at the origin. This concludes the first stage. Now suppose we have already concluded the n-th stage; then we proceed as follows: if the box-tableau thus obtained is closed then we stop. Also, if every non-atomic (in the sense of Section 1.1) box on every open branch of the box-tableau thus obtained has been used up, we stop. If neither, then we pick a point *P* of *minimal* level (ie as high up on the three as possible) which has not yet been used and which appears on at least one open branch and we extend the box-tableau thus obtained as follows: take *every* open branch Θ passing through the box *P* and

1) If *P* is an α, we extend Θ to the branch (Θ, $\alpha 1$, $\alpha 2$).
2) If *P* is a β, then we simultaneously extend Θ to the two branches (Θ, $\beta 1$), (Θ, $\beta 2$).
3) If *P* is a δ, then we take the sequence **a** of the first parameters ai that are distinct and do not appear on the tree and we extend Θ to the branch (Θ, $\delta(\mathbf{a})$) for this **a**.
4) If *P* is a γ, then we take the sequence **a** of the first parameters ai such that $\gamma(\mathbf{a})$ does not occur on Θ and we extend Θ to the branch (Θ, $\gamma(\mathbf{a})$, γ); ie we add $\gamma(\mathbf{a})$ as an endpoint to Θ and then we repeat, at each node, an occurrence of γ.

Having performed acts 1-4 resp, we then declare *P* to be used, and this concludes stage n+1 of our procedure. Call a box-tableau obtained in this way a *systematic box-tableau.*

A box-tableau is *atomically closed* whenever each branch contains an atomic condition and its conjugate; a tableau is *closed* when each branch contains both a box and its conjugate.

3. *Consistency and completeness of the box-tableaux method*

We want to prove the consistency and completeness of the method: any box B proved by the box-tableau method must be valid and conversely. We shall give first an indirect proof of this statement.

3.1. Intuitive proof

Consistency: Any box-tableau provable box B is valid. Suppose this were not true, ie there would be a box B that were box-tableau provable but not valid. Then by the Corollary to Lemma 1, the LPC formula Pr(B) would not be valid, hence the analytic tableau with origin *F* Pr(B) would not close. Therefore there would exist an open branch on which every formula is simultaneously satisfiable, in particular *F* Pr(B); hence *F* Dr(Pr(B)) = *F* B.

Completeness: Any valid box B is box-tableau provable. Consider the box tableau **BT**(B) for B, ie the closed box-tableau starting with *F* B constructed according to the rules given above. Call the nodes of this box-tableau π Bj, with π the prefix (*T* or *F*) of the box Bj. Forgetting for a moment about the prefixes, let Pr(Bj) be the formula of LPC corresponding to the box Bj, as explained in section 1.2.3 above. Call nox $\mathbf{BT}^{\#}$ (B) the tree (it is no more a box-tableau) we obtain by everywhere substituting π Pr(Bj) for the π Bj. This tree can be extended (an easy induction on the box-tableau rules) to an analytic tableau **T**(Pr(B)) for the formula Pr(B). Since **BT**(B) is closed, so is **T**(Pr(B)) – by an obvious induction again. By the completeness theorem for analytic tableaux, Pr(B) is a valid formula, hence B is a valid DRS (and conversely).

To the hapless reader not very familiar with tableau methods all this might look like some sleight-of-hand trick, so we prove formally completeness for box-tableaux – à la Smullyan.

3.2 Formal proof

Definition 17

A *Hintikka box-set* (for an arbitrary domain D) is a set H of conditions such that the following holf for every α, β, γ, δ in the set $\mathbf{B}^{D}$ of all closed D-boxes (boxes all of whose discourse markers haeve been substituted for by elements from D):

H0: No box of $\mathbf{B}^D$ and its conjugate are both in H.
H1: If $\alpha \in$ H, then α1, α2 are both in H.
H2: If $\beta \in$ H, then $\beta 1 \in$ H or $\beta 2 \in$ H.
H3: If $\gamma \in$ H, then for every sequence $\mathbf{k} \in$ D*, $\gamma(\mathbf{k}) \in$ H.
H4: If $\delta \in$ H, then for some sequence of distinct elements $\mathbf{k} \in$ D*, $\delta(\mathbf{K}) \in$ H.

Lemma 2: (Hintikka lemma for box-tableaux): Every Hintikka box-set H for a domain D is satisfiable in the domain D.
Proof: it is enough to show this for an *atomic* valuation of the set $\mathbf{B}^D$ (as in Smullyan [68], p 47 – mutatis mutandis).

For every *atomic* box B (a set of conditions either of the form P^n(x1,x2,...,xn) or xi = xj), give it the value "true" if *T* B is an element of H and the value "false" if *F* B is an element of H; and any one of these otherwise. Now we show that each element *P* of H is true under this atomic valuation, by induction on the degree of *P*.

If *P* is of degree 0, it is immediate that *P* is true under this valuation. Now suppose *P* is of positive degree and that every element of H of lower degree is true. We must then show that *P* is also true. Since *P* is not of degree 0, it is either some α, β, γ, or δ.

If it is an α then α1, α2 are both in H (by H1), hence both true (by induction hypothesis), hence α is true.

If it is a β, then at least one of β1, β2 is in H (by H2) and hence true, so β is true.

If it is a γ, then for every sequence $\mathbf{k} \in$ D*, $\gamma(\mathbf{k}) \in$ H (by H3); but every $\gamma(\mathbf{k})$ is of lower degree than γ, hence true by inductive hypothesis. Hence γ must be true.

If it is a δ, then for some sequence of distinct elements $\mathbf{k} \in$ D*, $\delta(\mathbf{k}) \in$ H (by H4. Then $\delta(\mathbf{k})$ is true by inductive hypothesis, hence δ is true. ●

Definition 18
For any *P* on a branch Θ of degree > 0, define *P* to be *fulfilled* on Θ if either:

(i) *P* is an α, and α1, α2 are both on Θ;
(ii) *P* is a β, and at least one of β1, β2 is on Θ;

(iii) P is a γ, and for every sequence of parameters **a**, $\gamma(\mathbf{a})$ is on Θ;
(iv) P is a δ, and for some sequence of distinct parameters **a**, $\delta(\mathbf{a})$ is on Θ.

Definition 19
A *finished* systematic box-tableau is a systematic box-tableau is a systematic box-tableau that is either infinite or else finite but that cannot be further extended by continuing the systematic procedure of Definition 16 (in other words, for each open branch all non-atomic boxes have already been used).

Theorem 1: For any finished systematic box-tableau, every open branch is a Hintikka box-set (for the denumerable universe of parameters).
Proof: It is enough to show that every open branch of such a box-tableau is a Hintikka box-set. The procedure given in Definition 16 is a systematic one for automatically fulfilling all α, β and δ formulae that come our way. As for the γ-formulas, when we use an occurrence of a γ on a branch Θ to subjoin an instance $\gamma(\mathbf{a})$, the purpose of repeating an occurrence of γ is that we must sooner or later come down the branch Θ and use this repeated occurrence, from which we adjoin another instance $\gamma(\mathbf{b})$ and repeat an occurrence of γ again, which we in turn use again, and so on... In this way we are sure to fulfill all γ formulae (as well as the α, β and δ formulae).
Consider now the possibility that in systematically constructing a finished box-tableau, we may arrive – after a finite number of steps – at a stage in which the box-tableau is not closed, and yet every non-atomic point of every open branch has been used. This can obviously happen only if no γ-formulae occur on any open branch. In this case all the γformulae are vacuously fulfilled. Hence in this case every open branch is still a Hintikka box-set. ●

Theorem 2: In any finished systematic box-tableau **BT**, every open branch is simultaneously satisfiable.
Proof: this is an immediate Corollary of Theorem 1 and the Hintikka lemma for box-tableaux. ●

Theorem 3: (Completeness Theorem for Box-tableaux): If a box B is valid, then B is box-tableau provable – ie there exists a closed box-tableau for F B. Indeed, if the box B is valid, then the systematic box-tableau for F B must close after a finite number of steps.

Proof: Suppose the box B is valid. Let **BT**(B) be the finished systematic box-tableau starting with *F* B. If **BT**(B) contained an open branch Θthen by Theorem 2 Θ would be satisfiable; hence *F* B, being an element of Θ would be satisfiable, contradicting the hypothesis that B is valid. Thus B is box-tableau provable. Concerning the second statement, by König's lemma (cf Smullyan [68], p 32), a closed infinite box-tableau is impossible, because if **BT**(B) is closed then every branch of **BT**(B) is of finite length, hence **BT**(B) must itself be finite. ●

Let us come back, for a moment to the unified notation and rules A-D of Section 2.2. Under any interpretation in the domain D, the following facts obviously hold:

F1: α is true iff $\alpha 1$, $\alpha 2$ are both true.
F2: β is true iff at least one of $\beta 1$, $\beta 2$ is true.
F3: γ is true iff $\gamma(\mathbf{k})$ is true for every sequence $\mathbf{k} \in D^*$.
F4: δ is true iff $\delta(\mathbf{k})$ is true for some sequence **k** of distinct elements of D.

Lemma 3: The following facts hold:

G1: If a set **B** of D-conditions is satisfiable and $\alpha \in$ B, then $\{\mathbf{B}, \alpha 1, \alpha 2\}$ is satisfiable.
G2: If a set **B** of D-conditions is satisfiable and $\beta \in \mathbf{B}$, then at least one of the two sets $\{\mathbf{B}, \beta 1\}$, $\{\mathbf{B}, \beta 2\}$ is satisfiable.
G3: If a set **B** of D-conditions is satisfiable and $\gamma \in \mathbf{B}$, then for every sequence **a** of parameters, the set $\{\mathbf{B}, \gamma(\mathbf{a})\}$ is satisfiable.
G4: If a set **B** of D-conditions is satisfiable and $\delta \in \mathbf{B}$, and if **a** is a sequence of distinct parameters none of which occur in any element of **B**, then $\{\mathbf{B}, \delta(\mathbf{a})\}$ is satisfiable.

Proof: Verification of G4 (the non-obvious case): There is an interpretation I of all predicates of **B** in some domain D and a substitution σ of parameters of **B** into elements of D, such that for every DRS K = [U, CON] in **B**, $CON_{\mathbf{u}/\mathbf{k}}$ is true under I, where **k** is a sequence of parameters. In particular, δ^{σ} is true under I. This is a box with no parameters but with constants in the domain D and it is of existential type, call it $\delta 1$. Since $\delta 1$ is true under I, there must be at least one sequence **k** of elements of D such that $\delta 1(\mathbf{k})$ is true under I. Now σ is defined on all parameters of $\{\mathbf{B}, \delta(\mathbf{a})\}$, except for the sequence of parameters **a**. Extend σ by defining $\sigma(\mathbf{a}) = \mathbf{k}$ – call this entension σ^*. Then σ^* is defined on all parameters of $\{\mathbf{B}, \delta(\mathbf{a})\}$. Clearly, for every box C in **B**, $C^{\sigma*}$ is the same expression as C^{σ}, so $C^{\sigma*}$ is true under I. And $\delta(\mathbf{a})^{\sigma*}$ is the same set of sentences as $\delta_1(\mathbf{k})$, hence $\delta(\mathbf{a})^{\sigma*}$ is true under I. Hence, for every box C in $\{\mathbf{B}, \delta(\mathbf{a})\}$, $C^{\sigma*}$ is true under I. Thus the set $\{\mathbf{B}, \delta(\mathbf{a})\}$ is satisfiable. ●

Theorem 4: (Consistency Theorem for Box-tableaux): Every box-tableau provable DRS K is valid.
Proof: Suppose Θ is a branch of a box-tableau and that Θ is satisfiable. If we extend Θ by rule A, C, or D then the resulting extension is again satisfiable (by G1, G3, G4 above resp). If we simultaneously extend Θ to the two branches Θ1, Θ2 by one application of rule B, then at least one of Θ1, Θ2 is again satisfiable (by G2). Hence any immediate extension of a box-tableau which is satisfiable (meaning at least one of its branches is satisfiable) is again satisfiable. Therefore (induction on formation rules) if the origin of a box-tableau is satisfiable then at least one branch of the box-tableau is satisfiable and hence open. Hence if a box-tableau closes then the origin is indeed unsatisfiable; whence the Theorem by contraposition. ●

Theorem 5: Any closed box-tableau is atomically closed (in the sense of Definition 16). ●

4. Conclusion

The proof procedure for DRS's we give here is along the lines of Smullyan. The idea was to exploit the similarity of DRT languages and logic predicate languages; in particular we show that any DRS K can be converted into a first-order logic formula P, and conversely.

There is another way of providing a deductive theory for DRT that could be investigated; it is the Gentzen sequents, since a box in our tableau approach is a particular case of a block in the Gentzen systems. However Smullyan gives the proof of completeness for the block tableaux by showing that any block tableau can be converted into an analytic tableau, and conversely; therefore the completeness proof provided here for the box tableaux is a more direct proof.

However the limitation of our approach with respect to DRT concerns the notion of validity, that is not the most prominent notion within the current formulation of DRT. DRS's with discourse markers in top are hardly ever valid, and then only in the extreme case where CON (the set of conditions) is tautological. More relevant is the notion of logical consequence for DRT, even though logical consequence for DRT (noted $\vdash_{DRT}$) is not exactly equivalent to logical consequence (noted $\vdash_{LOG}$) for predicate logic. Given a DRS K′, the extension of K′ with a bit of discourse D can be seen as deductive process in the sense that, the resulting DRS K of such an extension, is obtained by proving the following statement: $K' \vdash_{DRT} K$. So that, when we state $K' \vdash_{DRT} K$, K in not independent of K′, e.g. *A man loves Mary. Mary is a girl. Every man estimates a girl he knows.* $\vdash_{DRT}$ *The man who kowns Mary smokes.* If we can define $\vdash_{DRT}$ in terms of $\vdash_{LOG}$, then, our approach which makes use of logical consequence or more precisely of semantic consequence, can, via this equivalence, apply strictly to DRT as the theory stands.

Definition:
$K \vdash_{DRT} K' \;=\; K+K' \vdash_{LOG} K'$
where K+K′ is the result of embedding K′ into K. Thus, if we cannot falsify $K+K' \vdash_{Log} K'$, then $K+K' \vdash_{Log} K'$ is valid, and in consequence, $K \vdash_{DRT} K'$ holds.

Therefore, the box-tableau method can be seen as motivated by the goal of providing a *complete positive algorithm* for the test of validity (and therefore, of satisfaction ie truth in some model) that applies to DRS's, even though the deductive theory provided here is not directly related to DRT (an alternative approach is sketched in Sedogbo [86], where we use a naive deduction technique. Nevetheless, a full deductive theory for DRT based on logical consequence $\vdash_{DRT}$ is still to be explored).

The primary goal of DRT is to deal with complex phenomena such as tense and anaphora. Needless to say that the DRT language we provide here a deductive theory for, is a one-sorted language; in other words, much more work is needed to extend this deductive theory to tense (and more precisely to event), time and generalized quantifiers.

An important issue we don't adress in this paper concerns the implementation of the box tableaux procedure. The advantage of box tableaux is that a branch can be non-atomically closed. Since the conditions in a box (or DRS) are linked by an implicit *and* (the binary connective) which is commutative, it is hard, in practice, to implement an efficient box tableaux prover which closes non-atomically. Thus it is more convenient to design a box tableaux prover for which branches close atomically (cf Ommani & Sedogbo [87]).

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