A NOTE ON THE SEMANTICS OF MINIMAL INTUITIONISM

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Introduction

Georgacarakos has provided in [1] Kripke-style semantics for the minimal intuitionistic logic, **Im**. As he points out, there are two differences between these semantics and the standard semantics for the full intuitionistic logic, **I**:

(i) In the canonical model, the set of possible worlds is interpreted as the set of all negation-saturated and absolutely saturated theories. In the canonical model for **I**, however, each possible world is understood as a negation-saturated theory.

(ii) The valuation of negative formulae is, according to Georgacarakos, as follows: $\neg A$ holds in possible world a iff A does not hold in every negation-coherent world b such that Rab. In **I**, it suffices to require: $\neg A$ holds in a iff A does not hold in every b such that Rab.

The aim of this paper is twofold:

(i) We show that the models proposed by Georgacarakos are unnecesarely strong: we interpret canonically possible worlds as minimal intuitionistic theories with certain properties among which consistency (in either of both senses, negation-coherency and absolutely coherency) is not necessarely found. There is, however, at least one negation-coherent world; namely, **Im**, the actual world.

(ii) We explain, so we think, Georgacarakos' valuation of negative formulae by providing in the first place a semantics for the positive fragment of intuitionistic logic, and then by introducing negation by means of a falsity constant. Semantics for **Im** defined with negation as primitive are also provided. Georgacarakos' models are special cases of those we propose.

1. Positive intuitionistic calculus I+

Axioms:
A1. $A \to (B \to A)$
A2. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
A3. $A \to (A \vee B)$ / $B \to (A \vee B)$
A4. $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$
A5. $(A \& B) \to A$ / $(A \& B) \to B$
A6. $A \to (B \to (A \& B))$

Rules:
Modus ponens (If $\vdash A$ and $\vdash A \to B$, then $\vdash B$).

2. Models for I+

A *model structure* is a pair $<K,R>$ where K is a non-null set and R is a binary relation on K reflexive and transitive. A model is a triple $<K,R,\models>$ where $<K,R>$ is a model structure and $\models$ is a (valuation) relation from K to the sentences of **I**+ satisfying the following conditions for all wff A,B and $a,b \in K$

(i) If Rab and $a \models A$, then $b \models B$
(ii) $a \models A \& B$ iff $a \models A$ and $a \models B$
(iii) $a \models A \vee B$ iff $a \models A$ or $a \models B$
(iv) $a \models A \to B$ iff for all $b \in K$, if Rab and $b \models A$, then $b \models B$.

A formula A is valid if $a \models A$ for every $a \in K$ in all models $<K,R,\models>$. It is easy to show that **I**+ is semantically consistent, that is, if A is a theorem of **I**+, then A is valid.

3. Completeness of I+

A *theory* is a set of formulas of **I**+ closed under modus ponens; that is, a is a theory if whenever $A \to B \in A$ and $A \in a$, then $B \in a$. A theory a is *prime* if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$. Finally, a is *regular* if a contains all theorems of **I**+. Now, we define the *canonical model* as the triple $<t\mathbf{I}+, R, \models>$ where $t\mathbf{I}+$ is the set of all prime regular theories; R is defined on $t\mathbf{I}+$ as follows: for all a, b, $\in t\mathbf{I}+$, Rab iff $a \subseteq b$; and $\models$ is a relation from $t\mathbf{I}+$ to the sentences of **I**+ such that for each formula A and $a \in t\mathbf{I}+$, $a \models A$ iff $A \in a$. Then, we prove

LEMMA 1. Let $<t\mathbf{I}+, R, \models>$ be the canonical model. If $a \in t\mathbf{I}+$, then a (i) is closed under provable entailment, and (ii) is closed under &.

Proof. (i) We prove that if $A \rightarrow B$ is a theorem and $A \in a$, then $B \in a$. Suppose $A \rightarrow B$ a theorem and let $A \in a$. By the regularity of a, $A \rightarrow B \in a$. So, $B \in a$ (a is closed under modus ponens).
(ii) Suppose $A, B \in a$. By A6 and (i), $B \rightarrow (A \& B) \in a$; by closure under modus ponens of a, $A \& B \in a$.

LEMMA 2. **I+** is prime; that is, if $A \vee B$ is a theorem, then A is a theorem or B is.

Proof. By the surprisingly simple strategy of the canonical metavaluations, R.K. Meyer proves in [2] that a number of logics have the disjunctive property. Among these logics are **I+** and **I+,f**, **Im** (Cfr. infra).

LEMMA 3. The canonical model is indeed a model.

Proof. By Lemma 2 t**I+** is non-null: **I+** is a prime regular theory. It remains to prove that the canonical R and $\vDash$ satisfy the conditions of §2. Now, the canonical R is clearly reflexive and transitive. So, we prove that the canonical $\vDash$ satisfies the clauses (i)-(iv) of §2. Clause (i) is immediate; clause (ii) easily follows by A5 and A6; clause (iii) by A3 and the fact that all members of t**I+** are prime. So, the clause of interest is (iv).

Subcase (I). If $a \vDash A \rightarrow B$, then for all $b \in t$**I+**, if Rab and $b \vDash A$, then $b \vDash B$.

Proof. Suppose $a \vDash A \rightarrow B$ and (for some $b \in t$**I+**) Rab and $b \vDash A$. It suffices to prove $b \vDash B$. By definition of $\vDash$, $A \rightarrow B \in a$, $A \in b$; by Rab, $A \rightarrow B \in b$; by closure under modus ponens of b, $B \in b$. So, $b \vDash B$ by definition of $\vDash$.

Subcase (II). If $a \nvDash A \rightarrow B$, then there is some $b' \in t$**I+** such that Rab', $b' \vDash A$ and $b' \nvDash B$.

Proof. Define $b = \{C \mid A \rightarrow C \in a\}$

(i) b is closed under modus ponens
Suppose $C \rightarrow D$, $C \in b$. By definition, $A \rightarrow (C \rightarrow D)$, $A \rightarrow C \in a$.
By A2, $A \rightarrow D \in a$. Thus, $D \in b$.

(ii) b is regular
Suppose C a theorem. Then $A \rightarrow C$ is a theorem by A1. So, $A \rightarrow C \in a$ (a is regular). Thus, $C \in b$.

(iii) $b \vDash A$
By the regularity of a, $A \rightarrow A \in a$. Thus, $A \in b$ and $b \vDash A$ by definition of $\vDash$.

(iv) $b \nvDash B$
Immediate. If $b \vDash B$, then $a \vDash A \rightarrow B$ by definitions of b and of $\vDash$.

(v) Rab
Suppose $C \in a$. By A1, $A \rightarrow C \in a$. So, $C \in b$. That is, $a \subseteq b$. Thus, Rab.

Now we show how to extend b to the required b'. Define X as the set of all theories x such that $b \subseteq x$ and $B \notin x$. By Zorn's Lemma, X has a maximal element b' such that $B \notin b'$. Suppose b' is not prime. Then for some wff C, D $C \vee D \in b'$ but $C \notin b'$ and $D \notin b'$. Define $[b', C] = \{F \mid \exists E(E \in b'$ and $(C \,\&\, E) \rightarrow F \in \mathbf{I+})\}$, $[b', D] = \{F \mid \exists E(E \in b'$ and $(D \,\&\, E) \rightarrow F \in \mathbf{I+})\}$. It is easy to show that $[b', C]$ and $[b', D]$ are closed by modus ponens; on the other hand, it is clear that b' is strictly included in $[b', C]$ and in $[b', D]$. Thus $[b', C]$ and $[b', D]$ are regular theories. By the maximality of b', $B \in [b', C]$ and $B \in [b', D]$; by definition, $(C \,\&\, E) \rightarrow B$, $(D \,\&\, E') \rightarrow B \in \mathbf{I+}$, E and E' being formulas in b'.

Now, by elementary properties of & and $\vee$, $((C \vee D) \,\&\, (E \,\&\, E')) \rightarrow \rightarrow B \in \mathbf{I+}$. So, $B \in b'$ (Cfr. Lemma 1) which is impossible. Therefore, b' is a prime regular theory. Now, Rab' follows by Rab and definition of $R(b \subseteq b')$; $b' \vDash A$ by (iii) above, and, finally, $b' \nvDash B$ by $B \notin b'$ and definition of $\vDash$.

We prove

Theorem (Completeness). If A is valid, then A is a theorem of $\mathbf{I+}$.

Proof. Suppose A is not a theorem. Then, $A \notin \mathbf{I+}$. So, A is invalid in the canonical model.

4. *The logic I+, f*

The logic **I+, f** is a definitional extension of **I+**. To formulate **I+, f** we add to the sentential language of **I+** the propositional falsity constant **f**, and we define $\neg A =_{DF} A \to \mathbf{f}$. Then, we note that the following are theorems: $A \to \neg\neg A$; $(A \to B) \to \to(\neg B \to \neg A)$; $(A \to \neg B) \to (B \to \neg A)$; $(A \to \neg A) \to \neg A$; $(A \to B) \to ((A \to \neg B) \to \neg A)$, etc.

5. *Models for I+, f*

A *model* for **I+, f** is a quadruple $<K, R, S, F>$ where $<K, R, F>$ is a model for **I+** and $S \subset K$ satisfying the clause,

(v) $a \models \mathbf{f}$ iff $a \notin S$

added to the conditions (i)-(iv) of §2.

We note that $\neg A \to (A \to B)$ is invalid. Consider a model $<K, R, \models>$ with $K = \{a, b\}$, $S = \{a\}$, and let Rab, $b \models A$ and $b \not\models B$. It is clear that $a \models \neg A$ and $a \not\models A \to B$. So, $a \not\models \neg A \to (A \to B)$.

Now let us define a theory a as inconsistent iff $\neg A \in a$, A being a theorem. The *canonical model* is, then, the triple $<t\mathbf{I+}, \mathbf{f}, R, \models>$ where $t\mathbf{I+}, \mathbf{f}$, R and $\models$ are defined similarly as in §3 and S ($\subset t\mathbf{I+}, \mathbf{f}$) is interpreted as the set of all consistent theories. We note the following

LEMMA 4. Let $<t\mathbf{I+}, \mathbf{f}, R, \models>$ be the canonical model, and let $a \in t\mathbf{I+}, \mathbf{f}$. Then, a is inconsistent iff for some wff $A \,\&\, \neg A \in a$.

Proof. Suppose a inconsistent. Then, for some theorem A, $\neg A \in a$. But a is regular. Thus, $A \in a$. Therefore, $A \,\&\, \neg A \in a$ (Cfr. Lemma 1). Suppose now $A \,\&\, \neg A \in a$. As $(A \,\&\, \neg A) \to \neg B$ (B is a theorem) is derivable, $\neg B \in a$.

Now, for proving the completeness of **I+, f**, we proceed as follows. Given that **I+, f** is prime (Cfr. Lemma 2), we only have to prove that the canonical $\models$ satisfies clause (v). It is clear that it suffices to prove

for all $a \in t\mathbf{I+}, \mathbf{f}$, $a \models \mathbf{f}$ iff a is inconsistent

So, suppose $a \models \mathbf{f}$. By definition of $\models$, $\mathbf{f} \in a$. Now $\mathbf{f} \to (A \to \mathbf{f})$ (A is a theorem) is a theorem of **I+**. Thus, $A \to \mathbf{f} \in a$ (Cfr. Lemma 1). So, a is inconsistent. Suppose now a inconsistent. Then $A \to \mathbf{f} \in a$, A being a theorem. But a is regular. So, $A \in a$. By closure under modus ponens, $\mathbf{f} \in a$ and, thus, $a \models \mathbf{f}$.

6. *The minimal intuitionistic calculus Im*

The minimal intuitionistic calculus **Im** is the result of adding the axiom

A7. $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$

to **I+**.

7. *Models for Im*

A model for **Im** is a quadruple $<K, S, R, F>$, where $<K, R, F>$ is a model for **I+** and $S \subset K$ satisfying the clause,

(vi) $a \vDash \neg A$ iff for all $b \notin S$
$b \nvDash A$ or not-Rab.

We note that, given the semantic interpretation of **f** in §5, we have

$a \vDash A \rightarrow \mathbf{f}$ iff $a \vDash \neg A$

as required.

For proving the completeness of **Im**, we define the canonical model similarily as in §5, and interpret **S** again as the set of all consistent theories. Then, as **Im** is prime (Cfr. Lemma 2), it only remains to prove

For all $a \in t\mathbf{Im}$, $a \vDash \neg A$ iff for all consistent theories $b \in t\mathbf{Im}$, $b \nvDash A$ or not-Rab.

Subcase (I). If $a \vDash \neg A$, then $b \nvDash A$ or not-Rab for all consistent $b \in t\mathbf{Im}$.

Proof. Suppose $a \vDash \neg A$ and (for reductio) $b \vDash A$ and Rab (b is consistent). By definitions, $\neg A \in a$, $A \in b$ and $a \subseteq b$. So, $\neg A \in b$. Then, b is inconsistent (Cfr. Lemmas 1, 4) contradicting our hypothesis.

Subcase (II). If for all consistent $b \in t\mathbf{Im}$ $b \nvDash A$ or not-Rab, then $a \vDash \neg A$.

Proof. Suppose $a \nvDash \neg A$. We show that there is some consistent $b' \in t\mathbf{Im}$ such that $b' \vDash A$ and Rab'. Define $b = \{C \mid A \rightarrow C \in A\}$. As in subcase (ii) of Lemma 3, it is easy to show that b is a non-null regular theory such that Rab and $b \vDash A$. Now we extend b to the required b'. Define X as the set of all theories x such that $b \subseteq x$ and $\neg A \notin x$. By

Zorn's Lemma there is a maximal b' such that $\neg A \notin b'$. It is clear that b' is consistent. If not, $\neg A \in b'$ by the theorem $(B \,\&\, \neg B) \rightarrow \neg A$. Further, b' is prime. Suppose it is not. Form as in Lemma 3 the non-null regular theories $[b', B]$ and $[b', C]$. By the maximality of b', $\neg A \in [b', B], [b', C]$ whence it is easy to show $B \,\&\, \neg C \in b'$. By the theorem $(B \,\&\, \neg C) \rightarrow \neg(B \vee C)$, $\neg(B \vee C) \in b'$, and, therefore, b' is inconsistent ($B \vee C \in b'$; cfr. Lemma 4). In consequence, b' is a prime regular theory such that $b' \models A$ and Rab'.

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