A NOTE ON GROUNDED SENTENCES

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Kripke develops the idea of grounded sentences in his paper "Outline of a Theory of Truth" ([1]) (for example, if R is the sentence "The statement 'snow is white' is true.", and S is the sentence "This sentence is true.", then R is grounded and S is not grounded). In this note, two informal/intuitive definitions of grounded will be offered for Peano Arithmetic (PA), and then a formal syntactical definition that somewhat captures the stated intuition will be given for each. Also, a definition that is based on a restriction of Kripke's work will be given, and it will be shown that this definition is equivalent to the two syntactical definitions. The syntactical definitions do not (and more generally such definitions cannot) fully capture the intuitive idea of grounded sentences, but they do provide a class of sentences for a formal language of PA whose groundedness is syntactically determined. One of the values of a syntactically determined class of grounded sentences is that in any "definition of truth" for PA one can use the Tarski schema: "$T(\ulcorner S \urcorner) \leftrightarrow S$" for grounded sentences, even though this schema cannot be used in general.

Let L′ denote a first order formal language for PA whose only function symbols are the constant symbols $\bar{0}, \bar{1}, \bar{2}, \ldots$, and let L be the language obtained from L′ by adding a relation symbol T (to denote "truth"). Note that the only terms of L or L′ are variables and the constant symbols $\bar{0}, \bar{1}, \bar{2}, \ldots$ For any formula σ of L, let $\ulcorner \sigma \urcorner$ denote the numeral of the Gödel number of σ (so $\ulcorner \sigma \urcorner = \bar{n}$ for some $n \in \omega$). A formula α is a subformula of σ if $\sigma = e\alpha f$ where e and f are expressions (possibly empty) of L (note that σ is not a subformula of $T(\ulcorner \sigma \urcorner)$).

One informal way of defining "grounded" is to describe a sentence as grounded if its truth value can be obtained by "unwrapping" the layers/occurrences of T until a sentence of L′ is obtained.

Another informal way of describing "grounded" is to define a sentence as grounded if its truth value can be determined from PA in L′.

For example, $T(\ulcorner 0 = 0 \urcorner)$ is grounded by the first informal definition since "unwrapping" the occurrence of "T" gives a sentence of L′. $T(\ulcorner 0 = 0 \urcorner)$ is grounded by the second informal definition since its truth value can be determined from knowing whether or not $0 = 0$ – that is, its truth value can be determined from PA in L′.

The definitions that follow will be used to partially capture, in a formal setting, these informal definitions.

Definition 1: The depth of a formula σ of L, denoted dp(σ), is defined by induction as follows:

(i) $dp(\sigma) = 0$ iff σ is a formula of L′.

(ii) $dp(T(\ulcorner\sigma\urcorner)) = dp(\sigma) + 1$

$dp(Tc) = 1$ iff c is a constant symbol, but $c \neq \ulcorner\alpha\urcorner$ for any formula α of L.

$dp(Tx) = \omega$ for any variable, x.

(iii) $dp(\sim\alpha) = dp(\alpha)$

(iv) $dp(\alpha \rightarrow \sigma) = \max(dp(\alpha), dp(\sigma))$

(v) $dp(\forall x\sigma) = dp(\sigma)$

(For any formula σ, dp(σ) is an ordinal)

Note that although this definition is not well defined for a natural or informal language, it is well-defined and inductive for the formal language L (there are languages for which this definition would not be well-defined – for example, if $\alpha = T(\ulcorner\alpha\urcorner)$ then dp(α) would not be well-defined – but such an α cannot be defined in the language L). Note also that if the Gödel numbering is known then the depth of a formula can be obtained in a finite number of steps from the Gödel number of the formula (this is easier in L than it might be in general since the only terms of L are variables and the constant symbols $\bar{0}, \bar{1}, \bar{2}, \ldots$).

If a formula has finite depth then the depth is an expression of the number of "nested" occurrences of T. For example, $T(\ulcorner 0 = 0\urcorner)$ has depth 1, $T(\ulcorner T(\ulcorner 0 = 0\urcorner)\urcorner)$ has depth 2,...

In terms of depth the first informal definition can be expressed by saying that a sentence is grounded if it has finite depth. Let GS_1 be the set of sentences of L whose depth is finite. GS_1 is then a set of grounded sentences whose groundedness is syntactically determined.

Definition 2: Define $\leq$ on $\{\sigma: \sigma$ is a formula of L$\}$ by $\alpha \leq \sigma$ iff any of the following hold:

(i) $\alpha = \sigma$

(ii) $\sigma = T(\ulcorner\mu\urcorner)$ and $\alpha \leq \mu$

(iii) α is a subformula of μ and $\mu \leq \sigma$

(iv) There exists a variable x of σ, a constant symbol c, and a formula μ obtained from σ by replacing one or more occurrences of x by c, such that $\alpha \leq \mu$.

The underlying idea of $\leq$ is that if R and S are sentences then $R \leq S$ if, in some sense, the truth value of R is needed to determine the truth value of S. The definition is given for formulas (this will allow, for example, $T(\ulcorner S \urcorner) \leq \forall x T(x)$ – since $T(\ulcorner S \urcorner) \leq T(x) \leq \forall x T(x)$ (and it will be shown that $\leq$ is transitive)) so for parts of the definition the idea of truth value has no meaning, but for sentences the underlying idea holds ((i) and (iii) fit this intuitive idea since in a model the truth of a statement is determined/defined by the truth value of its subformulas; similarly, the example of a model shows that $\forall x\sigma(x) \geq \sigma(c)$ fits this intuitive idea; (ii) fits this idea since the truth value of R is needed to determine the truth value of $T(\ulcorner R \urcorner)$). However, the intuitive idea breaks down in that there are sentences that satisfy the intuitive idea but do not satisfy the definition of $\leq$ – for example, $R \wedge S \not\leq R \wedge (S \wedge Q)$.

The definition of $\leq$ will be used to give a formal definition that partially captures the idea of the second informal definition.

Let GS_2 be the set of sentences S such $S \not\geq T(\ulcorner S \urcorner)$.

GS_2 is another set of grounded sentences whose groundedness is syntactically determined.

If $S \not\geq T(\ulcorner S \urcorner)$ then $S \not\geq R$ for any sentence R of depth greater than the depth of S (this can be proven by induction on formulas) (and conversely, if $S \not\geq R$ for any sentence of depth greater than the depth of S, then certainly $S \not\geq T(\ulcorner S \urcorner)$). Therefore, if $S \not\geq T(\ulcorner S \urcorner)$ then the truth value of S does not depend on the truth value of any sentence of depth greater than that of S. Therefore, if $S \not\geq T(\ulcorner S \urcorner)$ then the truth value of S is determined by the truth value of sentences of depth $\leq dp(S)$, and hence (by induction), the truth value of S can, in some sense, be determined from PA in L$'$. Therefore GS_2 is a set of grounded sentences that satisfy the second informal definition of grounded.

For example, let R be "The statement 'snow is white' is true." and let S be "This sentence is true.". To determine whether or not R is true one only needs to know whether or not snow is white (so the truth value of R can be determined from the truth value of a sentence of smaller "depth") and thus R is grounded. To determine whether or not S is true one needs to know whether or not 'S is true' is true – that is, one needs to know

"T(S)" – which is (informally) a sentence of "depth' larger than the "depth" of S. Therefore S is not grounded. (Neither R nor S have depth as defined for L, but to the extent that the idea can be applied to these sentences, the idea does fit).

Proposition: $\leq$ is transitive.

Proof: Assume that δ, α, σ are formulas such that $\delta \leq \alpha$ and $\alpha \leq \sigma$. It will be shown that $\delta \leq \sigma$.

If $\alpha = \sigma$ then clearly $\delta \leq \sigma$. If $\alpha \leq \mu$ and $\sigma = T(\ulcorner\mu\urcorner)$ then $\delta \leq \sigma$ by induction on the depth of σ. If $\alpha \leq \mu$ and μ is a subformula of σ then $\delta \leq \sigma$ by induction on σ (induction on formulas). If $\alpha \leq \mu$ and μ is a substitution instance of σ then $\delta \leq \sigma$ by induction on the number of variables in σ.

As a result of his theorem on the undefinability of truth ([2]), Tarski argued that a formal definition of truth should be given in a language that is not semantically closed (that is, that does not contain semantic predicates such as "true", "false"). Given a language L_0, Tarski argued that truth in L_0 can be discussed in a metalanguage L_1. In turn, truth in L_1 can be discussed in a metalanguage L_2,... This gives a hierarchy of languages $\{L_n : n \in \omega\}$ such that truth in L_n can be completely and formally discussed in L_{n+1}. This definition of truth does not give a single truth predicate, but instead a truth predicate at each level. Kripke ([1]) suggested that "true" does not mean "true at the level n" and he has given an informal theory of truth that has a single, partially defined truth predicate. Kripke's construction also contains a hierarchy of languages, but at each level the definition of truth is an extension of the previous definition, not a new truth predicate. The definition that follows is based on a restriction of Kripke's work. It is an attempt to formalize some of Kripke's ideas for the formal language L – and to do so in a way that is compatible with the ideas and structure of classical two-valued logic.

Definition 3: Let $\mathfrak{A}$ be an L-structure. For each ordinal α, define L_α and $Exp(L_\alpha)$ (the expressions of L_α) by induction as follows:

(i) $L_0 = (L(S_{10},S_{10}), \mathfrak{A}_0) = (L',\mathfrak{A})$

$E \in Exp(L_0)$ iff E is an expression of L'.

(ii) Assume that $\alpha = \beta + 1$ and that L_β, and $Exp(L_\beta)$ are defined.

$L_\alpha = (L(S_{1\alpha},S_{2\alpha}), \mathfrak{A}_\alpha)$ where

$S_{1\alpha}$ = {c ∈ L: there exists a sentence S ∈ Exp(L_β) with c = ⌜S⌝ such that $\mathfrak{A}_\beta \models$ S}.

$S_{2\alpha}$ = {c ∈ L: there exists a sentence S ∈ Exp(L_β) with c = ⌜S⌝ such that $\mathfrak{A}_\beta \not\models$ S} ∪ N =
{c ∈ L: there exists E ∈ Exp(L_β) − Sent(L_β), with c = ⌜E⌝}

E ∈ Exp(L_α) iff (by induction)

(a) E ∈ Exp(L_β)

(b) E = Tc for some c ∈ $S_{1\alpha} \cup S_{2\alpha}$

(c) E = σ(x) and σ(c) ∈ Exp(L_α) for all constant symbols c of L.

(d) E is any finite sequence of elements of Exp(L_α).

$\mathfrak{A}_\alpha \models$ S iff (by induction)

(a) ⌜S⌝ ∈ $S_{1\alpha}$

(b) S = Tc and c ∈ $S_{1\alpha}$
S = ~Tc and c ∈ $S_{2\alpha}$

(c) S = ~R for some R ∈ Exp(L_α) such that $\mathfrak{A}_\alpha \not\models$ R

(d) S = R → W for some R,W ∈ Exp(L_α) and $\mathfrak{A}_\alpha \models$ ~R, or $\mathfrak{A}_\alpha \models$ W

(e) S = ∀xσ(x) and $\mathfrak{A}_\alpha \models \sigma$[a] for all a ∈ |$\mathfrak{A}$|

(iii) If α is a limit ordinal then L_α = (L($S_{1\alpha}$,$S_{2\alpha}$), $\mathfrak{A}_\alpha$) where

$$S_{1\alpha} = \bigcup_{\tau<\alpha} S_{1\tau},\ S_{2\alpha} = \bigcup_{\tau<\alpha} S_{2\tau}.$$

E ∈ Exp(L_α) iff E ∈ Exp(L_τ) for some $\tau < \alpha$, or E = ∀xσ(x) and σ(c) ∈ Exp(L_α for all constant symbols c of L, or E is a finite sequence of elements of Exp(L_α).

$\mathfrak{A}_\alpha \models$ S iff $\mathfrak{A}_\tau \models$ S for some $\tau < \alpha$, or S = ∀xσ(x) and $\mathfrak{A}_\alpha \models \sigma$[a] for all a ∈ |$\mathfrak{A}$|.

Let GS_3 be the set of sentences S such that S ∈ Exp(L_ω).

It will be shown that $GS_1 = GS_2 = GS_3$. The following lemma is needed for the proof.

Lemma: If σ ∈ Exp(L_α) and $\mu \geq \sigma$ then μ ∈ Exp(L_α).

Proof: If $\sigma = \mu$ then clearly μ ∈ Exp(L_α). If $\sigma \leq \beta$ and μ = T(⌜β⌝) then β ∈ Exp(L_α) and thus (by induction on formulas) σ ∈ Exp(L_α). If $\sigma \leq \beta$ and β is a subformula of μ then β ∈ Exp(L_α) and thus (by induction on the length of μ),σ ∈ Exp(L_α).

Assume that $\sigma \leq \beta$ and that β is obtained from μ by replacing one or more occurrences of x by c. Then β ∈ Exp(L_α). To show that σ ∈ Exp(L_α) the argument will be by induction on m, the number of occurrences of variables in μ. If $m = 1$ then β contains no variables and $\sigma \leq \beta$ on the basis of (i)-(iii) of the definition of $\leq$, and thus by the previous paragraph and the definition of $Exp(L_\alpha)$, $\sigma \in Exp(L_\alpha)$. Assume true for $m \leq k$ and assume that β has $k + 1$ occurrences of variables. If $\sigma \leq \beta$ on the basis of (i)-(iii) of the definition of $\leq$ then by the previous paragraph and by the definition of $Exp(L_\alpha)$, $\sigma \in Exp(L_\alpha)$. If $\sigma \leq \beta$ on the basis of (iv) of the definition of $\leq$ then $\sigma \in Exp(L_\alpha)$ by the inductive assumption.

Proposition: $GS_1 = GS_2 = GS_3$.

Proof: $GS_1 \subseteq GS_3$ by induction on the depth of a sentence.

To see that $GS_2 \subseteq GS_1$, assume $S \in GS_2$ and suppose that $dp(S) \geq \omega$ – say $dp(S) = \omega + k$, $k \geq 0$. Then there exist $\tau_0,\ldots,\tau_k$ such that for each i, $\tau_{i+1} = T(\ulcorner\tau_i\urcorner)$, $T(\ulcorner\tau_k\urcorner)$ is a subformula of S, and $dp(\tau_0) = \omega$. Then $S \geq T(\ulcorner\tau_k\urcorner) \geq \tau_k \geq \ldots \geq \tau_0 \geq T(x) \geq T(\ulcorner S\urcorner)$. $\geq$ is transitive, thus $S \geq T(\ulcorner S\urcorner)$ – which contradicts that $S \in GS_2$. Therefore $dp(S) < \omega$, and thus $S \in GS_1$. Therefore $GS_2 \subseteq GS_1$.

Assume $S \in GS_3$, and suppose that $S \geq T(\ulcorner S\urcorner)$. $S \in Exp(L_\omega)$ thus there exists $n \geq 1$ such that $S \in Exp(L_n)$, but $S \notin Exp(L_{n-1})$. Since $S \geq T(\ulcorner S\urcorner)$ it follows from Lemma 1 that $T(\ulcorner S\urcorner) \in Exp(L_n)$. But this implies that $S \in Exp(L_{n-1})$ – a contradiction. Therefore $S \not\geq T(\ulcorner S\urcorner)$ and thus $S \in GS_2$. Therefore $GS_3 \subseteq GS_2$.

Therefore $GS_1 = GS_2 = GS_3$.

The following shows that a formalization of the Liar's Paradox is not an element of any of GS_1, GS_2, or GS_3. Let S be the sentence $\forall x(\sigma(\bar{k},\bar{k},x) \rightarrow \sim T(x))$ – where $\bar{k} = \ulcorner\forall x(\sigma(y,y,x) \rightarrow \sim T(x))\urcorner$, and $\sigma(v_1,v_2,v_3)$ represents the function whose value at $(\ulcorner\tau(x)\urcorner,n)$ is $\tau(n)$ (that is, informally, $\sigma[a,b,c]$ is a theorem of PA iff c is the Gödel number of the formula obtained by substituting b for the free variable in the formula whose Gödel number is a). (S is a sentence such that $S \leftrightarrow \sim T(\ulcorner S\urcorner)$ is a theorem of PA – that is, a formalization of the Liar's paradox, and S is not grounded under Kripke's definition.) $S \notin GS_1$ since $dp(S) = \omega$, and $S \notin GS_2$ since $S \geq T(\ulcorner S\urcorner)$. To see that $S \notin GS_3$, suppose that $S \in GS_3$ – then there exists n such that $S \in Exp(L_n)$ but $S \notin Exp(L_{n-1})$. Then $\sigma(k,k,m) \rightarrow \sim Tm \in Exp(L_n)$ for all $m \in \omega$ – in particular $\sigma(k,k,T(\ulcorner S\urcorner))$

$\rightarrow \sim T(\ulcorner T(\ulcorner S \urcorner)\urcorner) \in Exp(L_n)$. Therefore (by Lemma 1), $T(\ulcorner S \urcorner) \in Exp(L_n)$, and thus $S \in Exp(L_{n-1})$ – a contradiction. Therefore $S \notin GS_3$.

Let $GS = GS_1$ ($= GS_2 = GS_3$). Then GS is a set of grounded sentences whose groundedness can be syntactically determined, and whose groundedness can be obtained under Kripke's definition.

It was noted above that $S \not\geq T(\ulcorner S \urcorner)$ iff $S \not\geq R$ for any sentence R of depth greater than the depth of S, and that in this case the truth value of S can, in some sense, be determined from PA in L′. The idea behind $S \geq R$ is that the truth value of R is not needed to determine the truth value of S. If the idea of grounded is extended to include all sentences S such that knowing the truth value of S does not require knowing the truth value of any sentence of depth greater than dp(S), then there are sentences which satisfy this definition, yet are not elements of GS. For example, every tautology is true, therefore knowing the truth value of $\forall x(T(x) \leftrightarrow T(x))$ does not require knowing the truth value of any sentence of depth greater than the depth of $\forall x(T(x) \leftrightarrow T(x))$ and hence this ought to be a grounded sentence. More generally, if the truth value of a sentence is answered by the first order theory of pure logic, then such a sentence should be taken to be grounded – since knowing its truth value does not require knowing the truth value of any sentence of larger depth. Whether or not $\forall x T(x)$ and $\forall x \sim T(x)$ satisfy this extended definition can be argued either way. On the one hand, $\forall x T(x)$ does not satisfy the extended definition, since in order to answer whether or not $\forall x T(x)$ is true one needs to know whether or not Tc is true for each constant symbol c, and in particular, one needs to know whether or not $T(\ulcorner \forall x T(x) \urcorner)$ is true – but this sentence has depth greater than the depth of $\forall x T(x)$. On the other hand, $\forall x T(x)$ is not true since $T(\ulcorner 0 = 1 \urcorner)$ is not true, and thus it is not necessary to know the truth value of any sentence of depth greater than the depth of $\forall x T(x)$ in order to know the truth value of $\forall x T(x)$, and therefore $\forall x T(x)$ does satisfy the extended definition. Similarly, it can be argued that $\forall x \sim T(x)$ satisfies the extended definition and that $\forall x \sim T(x)$ does not satisfy the extended definition of grounded. (Or, $\exists x T(x)$ and $\exists x \sim T(x)$ satisfy the extended definition, therefore $\sim \exists x T(x)$ and $\sim \exists x \sim T(x)$ satisfy this definition – and hence $\forall x T(x)$ and $\forall x \sim T(x)$ satisfy it also.)

Another example is given by means of the following: the set of Gödel numbers of sentences in GS is recursive, and hence there exists a formula Gr(x) such that for any $m \in \omega$, $PA \vdash Gr(m)$ if m is the Gödel number of a sentence in GS, and $PA \vdash \sim Gr(m)$ otherwise. Let $R = \forall x(Gr(x) \rightarrow$

T(x)) – then R satisfies the extended definition of grounded (since the truth value of R depends only on sentences of finite depth, yet $dp(R) = \omega$) but $R \notin GS$.

The following definition extends GS to a larger set of grounded sentences, by including some of the sentences indicated above.

Definition 4: GF_4 is defined by induction as follows:
If σ is a formula of L then $\sigma \in GF_4$ if, and only if, any of the following hold:

(i) σ is a formula of L′.
(ii) σ is an instance of a tautology.
(iii) $\vdash_{L(T)}\sigma$ (σ is a theorem of the first order theory of pure logic for L(T) – where L(T) is the reduct of L whose only relation symbol is T)
(iv) $\sigma = T(\ulcorner\tau\urcorner)$, for some $\tau \in GF_4$
$\sigma = Tc$ for some constant symbol c such that $c \neq \ulcorner\tau\urcorner$ for any formula τ of L.
(v) σ is a sentential combination of formulas of GF_4.
(vi) $\sigma = \forall x\beta(x)$ for some $\beta(x) \in GF_4$
(vii) $\vdash_{L(T)}\sigma \leftrightarrow \beta$ for some $\beta \in GF_4$
(viii) σ is any of: $\forall xT(x)$, $\forall x \sim T(x)$, $\forall x(Gr(x) \rightarrow T(x))$, $\forall x(Gr(x) \rightarrow \sim T(x))$

Let GS_4 be the set of sentences S of L such $S \in GF_4$. (Any sentence of finite depth is in GS_4, so GS_4 is an extension of GS).

Whether or not an arbitrary sentence is in GS_4 is decidable, and $\forall x(T(x) \leftrightarrow T(x)) \in GS_4$ – but intuitively, any theorem of the first order theory of pure logic for L is grounded, and GS_4 does not contain all such sentences (if GS_4 did contain all such sentences it would not be decidable whether or not an arbitrary sentence was in GS_4). $\forall x(Gr(x) \rightarrow T(x)) \in GS_4$, but the set of Gödel numbers of sentences in GS_4 is recursive, thus (as for GS) there exists a formula $Gr_4(x)$ such that for any $m \in \omega$, $PA \vdash Gr_4(m)$ if m is the Gödel number of a sentence in GS_4, and $PA \vdash \sim Gr_4(m)$ otherwise. But then $\forall x(Gr_4(x) \rightarrow T(x)) \notin GS_4$.

The idea of Kripke's construction (that is, the idea used for definition 3) can be used to show that the Tarski schema "$T(\ulcorner S\urcorner) \leftrightarrow S$" works to define truth for sentences of GS_4. (Instead of starting only with sentences of L′, include $\forall xT(x)$, $\forall x \sim T(x)$, $\forall x(Gr(x) \rightarrow T(x))$, $\forall x(Gr(x) \rightarrow \sim T(x))$, all instances in L of tautologies, and all sentences S such that $\vdash_{L(T)}$S. Build as done in definition 3 except that at each level, include all sentences R such that $\vdash_{L(T)}$R $\leftrightarrow$ S for some grounded sentence S).

GS_4 can be extended to a larger class of syntactically determined grounded sentences, but the extensions are increasingly more complicated, they are less intuitive, and they cannot contain all grounded sentences.

GS, GS_4, and other such extensions give classes of sentences of L whose groundedness is syntactically determined, and for which "truth" can be given by Tarski's schema. To some extent these ideas carry over to languages that have function symbols other than constant symbols; but the intuitive ideas are less clear, and the definitions are more complicated. I do not know to what extent these definitions and results hold for informal or natural languages.

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REFERENCES

[1] Kripke, Saul. "Outline of a Theory of Truth." *Journal of Philosophy*, 72 (1975), 690-716.
[2] Tarski, Alfred. "The Semantic Conception of Truth." *Readings in Philosophical Analysis*. Ed. Herbert Feigl and Wilfrid Sellars. New York: Appleton-Century-Crofts, 1949.