CORRECTIONS TO SOME RESULTS FOR **BCK** LOGICS AND ALGEBRAS

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Introduction

A **BCK** logic is an implicational logic based on modus ponens and the following axiom schemes.

$\underline{\mathbf{B}}$ $A \supset B \,.\supset.\, (C \supset A) \supset (C \supset B)$
$\underline{\mathbf{C}}$ $A \supset (B \supset C) \,.\supset.\, B \supset (A \supset C)$
$\underline{\mathbf{K}}$ $A \supset (B \supset A)$.

In [1] R.K. Meyer and the author claimed the following result for this logic:

(I) If $A \supset B$ and C are theorems of **BCK** logic, then there is a substitution instance $A_1 \supset B_1$ of $A \supset B$ and one C_1 of C such that $A_1 = C_1 \,.\, B_1$ is therefore a theorem of **BCK** logic.

The claim was based on a theorem concerning **BCK** combinators from which the authors also drew similar "results" for **BCK** algebra and Meyer's "Fool's Model for combinatory logic".

However, A. Wronski has pointed out, in private correspondence, that the "results" for **BCK** logic and algebra fail.

It is easy to show, as he points out, that $((p \supset p) \supset p) \supset p$ and $q \supset q$ are theorems of **BCK** logic to which the "result" does not apply.

Wronski also points out that the "result" must fail in any logic in which $A \supset A$ is provable for arbitrary A as then $(p \supset p) \supset p \,.\supset.\, (p \supset p) \supset p$ and $(q \supset q) \supset (q \supset q)$ must be theorems.

In this note, we give a weaker version of **BCK** logic which does satisfy the above property. The system is in fact one considered by C.A. Meredith and the result confirms a conjecture of his dating back to 1954.

The theorem for **BCK**-combinators, on which the result is based, states that every **BCK** combinator has a principal functional character or principal type scheme (P.T.S.). This theorem (and the result for the Fool's Model) is correct, but Hindley has pointed out that its proof in [1] was not complete. He presents a new proof using lambda calculus in [5]. Below we give a simpler direct (and new) proof of the theorem.

A New Proof of Theorem A of [1]

The theorem concerns combinators made up by application from **B, C** and **K** . **BCK** terms may also include the variables x_1, ..., x_n ... . *Linear* BCK terms may contain each of these variables no more than one each.

B, C and **K** satisfy the equations:

$$\begin{array}{ll} (\mathbf{B}) & \mathbf{B}XYZ = X(YZ) \\ (\mathbf{C}) & \mathbf{C}XYZ = XZY \\ (\mathbf{K}) & \mathbf{K}XY = X. \end{array}$$

It is clear that replacing the left hand side of one of these equations by the right, in a term, shortens the term. If a term can no longer be shortened in this way, it is in *normal form.*

We will be interested in a generalised length or "glength" which is also decreased by the above process.

The *glength* of a term is obtained by counting 1 for each variable and 3 for each occurrence of **B, C** and **K**.

We will show, as part of the proof below, that each linear **BCK** term has a *type scheme* (T.S.).

Type variables *a, b, c,* ... (and type constants – if any) are T.S.s and if α and β are T.S.s so is $\mathbf{F}\alpha\beta$.

BCK - T.S.s are generated by the schemes:

$$\begin{array}{ll} (\mathbf{B}) & \vdash \mathbf{F}(\mathbf{F}\alpha\beta)(\mathbf{F}(\mathbf{F}\gamma\alpha)(\mathbf{F}\gamma\beta))\mathbf{B} \\ (\mathbf{C}) & \vdash \mathbf{F}(\mathbf{F}\alpha(\mathbf{F}\beta\gamma))(\mathbf{F}\beta(\mathbf{F}\alpha\gamma))\mathbf{C} \\ (\mathbf{K}) & \vdash \mathbf{F}\alpha(\mathbf{F}\beta\alpha)\mathbf{K} \end{array}$$

and the rule:

(**F***e*) $$\mathbf{F}\alpha\beta X,\ \alpha Y \vdash \beta(XY)$$

Lemma 1 In the presence of (**F***e*); (**B**), (**C**) and (**K**) are equivalent to the rule:

(**F***i*) If $\underline{\mathbf{B}}$, $\alpha x \vdash \beta(Yx)$, where $\underline{\mathbf{B}}$ is a set of statements of the form $\alpha_1\, x_1, \ldots, \alpha_n\, x_n$ where αx is used at most once in the deduction and $x \neq x_i$, then $\underline{\mathbf{B}} \vdash \mathbf{F}\alpha\beta Y$.

Proof (i) By (**F***e*), $\mathbf{F}\alpha\beta x$, $\mathbf{F}\gamma\alpha y$, $\gamma z \vdash \beta(x(yz))$

so $\mathbf{F}\alpha\beta x,\ \mathbf{F}\gamma\alpha y,\ \gamma z \vdash \beta(Bxyz)$
and 3 uses of (**F***i*) gives (**B**).

Similarly the T.S.s for **C** and **K** can be derived using (**F***e*) and (**F***i*).
(ii) For each step in the proof of $\underline{\mathbf{B}},\ \alpha x \vdash \beta(Yx)$

of the form $\underline{\mathbf{B}}_i,\ \alpha x \vdash \beta_i(Y_i x)$ $-(a)$
we prove $\underline{\mathbf{B}}_i \vdash \mathbf{F}\alpha\beta_i Y_i.$ $-(b)$

The initial step of this form must be $\alpha\chi \vdash \alpha x$; $\vdash \mathbf{F}\alpha\alpha\mathbf{I}$ is easily obtained as $\mathbf{I} = \mathbf{CKK}$.

If (a) is obtained by (**F***e*) there are two cases.
(I) The previous steps are

$$\mathbf{B}_j \vdash \mathbf{F}\beta_k\beta_i Y_j \qquad -(c)$$

and $\underline{\mathbf{B}}_k,\ \alpha x \vdash \beta_k(Y_k x)$
where $\underline{\mathbf{B}}_k \cup \underline{\mathbf{B}}_j = \underline{\mathbf{B}}_i$ and $\mathbf{B}Y_jY_k = Y_i$.

By the inductive step we have

$$\underline{\mathbf{B}}_k \vdash \mathbf{F}\alpha\beta_k Y_k$$

so by (**F***e*), (c) and ($\underline{\mathbf{B}}$) we have (b).
(II) The previous steps are:

$$\underline{\mathbf{B}}_j,\ \alpha x \vdash \mathbf{F}\beta_k\beta_i(Y_j x)$$

and $\underline{\mathbf{B}}_k \vdash \beta_k Y_k$ $-(d)$
where $\underline{\mathbf{B}}_k \cup \underline{\mathbf{B}}_j = \underline{\mathbf{B}}_i$ and $\mathbf{C}Y_jY_k = Y_i$

By the inductive step we have

$$\underline{\mathbf{B}}_j \vdash \mathbf{F}\alpha(\mathbf{F}\beta_k\beta_i)Y_j$$

so by (**F***e*), (**C**) and (d) we have (b).

We use the following notation.

$$\mathbf{F}_0 = \mathbf{I}$$
$$\mathbf{F}_{n+1}\alpha_1 \ldots \alpha_n\beta\gamma = \mathbf{F}_n\alpha_1 \ldots \alpha_n(\mathbf{F}\beta\gamma).$$

A term X has a *principal type scheme* (1) (P.T.S.)ω, relative to the types assigned to its variables, if all the T.S.s of X (and no extra ones) can be obtained from ω by substituting T.S.s for type variables.

(1) For a more detailed definition see HINDLEY and SELDIN [5].

We now prove a lemma which will lead to the proof of Theorem 1 of [1].

Lemma 1 For any T.S.β and any linear **BCK**-term X, whose free variables are $x_1,\ldots,x_k$, there exist T.S.s $\gamma_1,\ldots,\gamma_k$, $\alpha_1,\ldots,\alpha_n$, (with $n \geq 0$) such that $\gamma_1 x_1,\ldots,\gamma_k x_k \vdash \mathbf{F}_n\alpha_1 \ldots \alpha_n\beta X$.

Proof If X is x_i, let $k = 1$, $n = 0$, $\gamma_1 = \beta$. If X is **K, B** or **C** the T.S.s are given by **(K)**, **(B)** and **(C)**.

To prove Lemma 1 for general X, we proceed by induction on m the glength of X. The $m = 1$ and some $m = 3$ cases, we have dealt with above.

If $m = 2$, $X \equiv x_1 x_2$ and then, $\mathbf{F}\alpha\beta x_1,\ \alpha x_2 \vdash \beta X$.

If $m = 3$ and X is not a combinator, $X \equiv x_1\ x_2\ x_3$ or $x_1\ (x_2\ x_3)$. In the former case $\gamma_1 = \mathbf{F}_2\gamma_2\gamma_3\beta$ (γ_2, γ_3 arbitrary) in the latter $\gamma_1 \equiv \mathbf{F}\gamma\beta$ and $\gamma_2 \equiv \mathbf{F}\gamma_3\gamma$ for some γ. In both cases $n = 0$.

If $m > 3$ X must be composite and we can assume that our result holds for terms of glength less than m. There are several cases.

Case 1 $X \equiv x_i\ Y_1,\ldots,Y_p$, where as X is a linear **BCK** term, x_i, $Y_1,\ldots$ and Y_p have no variables in common. Also $Y_1,\ldots,Y_p$ have glengths $< m$ so we have for $1 \leq j \leq p$, by the induction hypothesis, for some T.S.s $\gamma_{j_1},\ldots,\gamma_{j_{q(j)}}$, $\alpha_{j_1},\ldots,\alpha_{j_{m(j)}}$ and arbitrary β_j:

$$\gamma_{j_1}x_{j_1},\ldots,\gamma_{j_{q(j)}} \vdash \mathbf{F}_{m_j}\alpha_{j_1}\ldots\alpha_{j_{m(j)}}\beta_j Y_j$$

where $x_{j_1},\ldots,x_{j_{q(j)}}$ are the variables in Y_j.

Letting $\mathbf{F}_{m_j}\alpha_{j_1}\ldots\alpha_{j_{m(j)}}\beta_j \equiv S_j$, we obtain Lemma 1 with $\gamma_i \equiv \mathbf{F}_p S_1 \ldots S_p\beta$ and $n = 0$.

Case 2 $X \equiv \mathbf{B}Z$. Then $X\ x_{k+1}x_{k+2} = Z(x_{k+1}x_{k+2})$ by (B).

$Z(x_{k+1}x_{k+2})$ has glength $m - 1$, so by the inductive hypothesis for some T.S.s, $\gamma_1,\ldots,\gamma_{k+2}$, $\alpha_3,\ldots,\alpha_n$, $\gamma_1 x_1,\ldots,\gamma_{k+1}x_{k+1}$, $\gamma_{k+2}x_{k+2} \vdash \mathbf{F}_{n-2}\alpha_3\ldots\alpha_n\beta(Z\ (x_{k+1}x_{k+2}))$.

As there is no duplication or cancellation of terms in the reduction from $Xx_{k+1}x_{k+2}$ tot $Z(x_{k+1}x_{k+2})$ it follows by the Subject Expansion Theorem (Curry and Feys [3] §9C) that the above also holds for $Xx_{k+1}x_{k+2}$.

The lemma then follows by **(F***i***)** with $\gamma_{k+1} \equiv \alpha_1$ and $\gamma_{k+2} \equiv \alpha_2$.

Case 3 $X \equiv \mathbf{B}UZ$. Then $Xx_{k+1} = U(Zx_{k+1})$ by **(B)**.

$U(Zx_{k+1})$ has glength $m - 2$, so by the inductive hypothesis for certain

T.S.s $\gamma_1,...,\gamma_{k+1},\alpha_2,...,\alpha_n$, $\gamma_1 x_1,...,\gamma_{K+1}x_{k+1} \vdash \mathbf{F}_{n-1}\alpha_2,...,\alpha_n(U(Zx_{k+1}))$.

The lemma then follows by the Subject Expansion Theorem and (**Fi**) as above.

Case 4 $X \equiv \mathbf{B}U_1,...,U_p$ where $p \geq 3$.

In this case $X = U_1(U_2, U_3)...,U_p$, which has glength $m - 3$. We therefore have $\gamma_1 x_1,...,\gamma_k \vdash \mathbf{F}_n\alpha_1,...,\alpha_n\beta(U_1(U_2,U_3),...,U_p)$ and the Lemma holds for X as above.

Case 5 $X \equiv \mathbf{K}Z$. Then $Xx_{k+1} = Z$ and Z has glength $m - 3$. We therefore have by the inductive hypothesis, adding $\gamma_{k+1}x_{k+1}$,

$$\gamma_1 x_1,...,\gamma_{k+1}x_{k+1} \vdash \mathbf{F}_{n-1}\alpha_2,...,\alpha_n\beta Z.$$

As x_{k+1} has T.S. γ_{k+1}, we can replace Z in the above by $\mathbf{K}Zx_{k+1}$ by the Subject Expansion Theorem. The lemma for X then follows by (**Fi**).

Case 6 $X \equiv \mathbf{K}U_1...U_p$ where $p \geq 2$.

In this case $X = U_1U_3...U_p$, which has glength less than m. Also U_2 has glength less than m so if the variables of U_2 are $u_{i_1},...,x_{i_q}$ and those of $U_3U_3...U_p$ are $x_{i_{q+1}},...,x_{i_k}$ where $(i_1,...,i_q,...,i_k)$ is a permutation of $(1,2,...,k)$ we have by the inductive hypothesis for T.S.s $\gamma_{i_1},...,\gamma_{i_k}$, $\delta_1,...,\delta_{r+1},\alpha_1,...,\alpha_n$:

$$\gamma_{i_1}x_{i_1},...,\gamma_{i_q}x_{i_q} \vdash \mathbf{F}_r\delta_1...\delta_{r+1}U_2$$
$$\gamma_{i_{q+1}}x_{i_{q+1}},...,\gamma_{i_k}x_{i_k} \vdash \mathbf{F}_n\alpha_1...\alpha_n\beta(U_1U_3...U_p)$$

It then follows by the Subject Expansion Theorem that

$$\gamma_{i_1}x_{i_1},...,\gamma_{i_k}x_{i_k} \vdash \mathbf{F}_n\alpha_1...\alpha_n\beta X$$

so that the lemma holds.

The other cases $X = \mathbf{C}Z$, $\mathbf{C}UZ$ and $\mathbf{C}U_1...U_p (p \geq 3)$ are similar to cases 2, 3 and 4, hence the lemma holds in all cases.

Theorem 1 All **BCK**-combinators have a principal type scheme.

The Subsystem of **BCK** *logic*

The subsystem of **BCK** logic that we require, for a correct version of (I), cannot allow the proof of $A \supset A$ for arbitrary A and so cannot have axiom schemes **B**, **C** and **K** in their full generality. The subsystem will therefore, have these corresponding axioms with instead of A, B and C, the statement variables p, q and r.

Axioms

$\underline{\mathbf{B}}$ $p \supset q \cdot \supset \cdot (r \supset p) \supset (r \supset q)$
$\underline{\mathbf{C}}$ $p \supset (q \supset r) \cdot \supset \cdot q \supset (p \supset r)$
$\underline{\mathbf{K}}$ $p \supset (q \supset p)$.

Modus ponens is replaced by condensed detachment, a rule first proposed and used by C.A. Meredith (see Kalman [7] or D. Meredith [8] for historical details), which can be defined using the notion of "most general unification" defined below.

If $A(p_i, ..., p_n)$ is a statement form that has $p_i, ..., p_n$ as its distinct propositional variables and $B(q_1, ..., q_m)$ is a statement form that has $q_1, ..., q_m$ as its distinct variables, then a *most general unification*∎ of these statement forms consists of two sequences of statement forms $A_t, ..., A_n$ and $B_1, ..., B_m$ such that:

(1) $A(A_1, ..., A_n)$ is identical to $B(B_1, ..., B_m)$
(2) the total length (in variables and $\supset$ s) of $A_1, ..., A_n, B_1, ...,$ and B_m is minimal.
(3) Given (1) and (2) the number of different variables in $A(A_1, ..., A_n)$ is maximal.

Robinson's Unification Theorem (page 33 of [9]) shows that if sequences can be found such that (1) holds then a most general unification exists.

The required rule of inference for our system:

Condensed Detachment If $A(p_1, ..., p_n)$ and $B(q_1, ..., q_m) \supset C(q_k, ..., q_t)$ are theorems of our logic and $q_k, ..., q_t$)are those of the variables $q_1, ..., q_m$ that appear in $C(q_k, ..., q_t)$ then $C(B_k, ..., B_t$ is a theorem if $A_1, ..., A_n$ and $B_1, ..., B_m$ give a most general unification of $A(p_1, ..., p_n)$ and $B(q_1, ..., q_m)$ and no variables of $C(q_k, ..., q_t)$ other than $q_k, ..., q_t$ appear in $A_k, ..., A_t$.

We call this logic "Condensed **BCK**-logic". The result below was first conjectured by C.A. Meredith.

(2) This Theorem was proved independently by Curry in [2] the algorithm used in it was, because of the relation between statements of the implicational logic and the combinators, foreshadowed by Robinson's Unification Algorithm.

Theorem 2 If $A \supset B$ and C are theorems of condensed **BCK** logic, condensed detachment can be applied to these theorems, and results in the proof of a substitution instance of B.

Proof For eacht axiom of our logic, there is a corresponding combinator and this combinator has a P.T.S. which can be obtained by replacing each $A \supset B$ in the axiom by an appropriate $\mathbf{F}\alpha\beta$.

The combinator **B** for example corresponds to the axiom:

$$p \supset q \,.\supset.\, (r \supset p) \supset (r \supset q)$$

and its P.T.S. is $\mathbf{F}(\mathbf{F}ab)(\mathbf{F}(\mathbf{F}ca)(\mathbf{F}cb))$.

The application of condensed detachment to theorems of the logic corresponding to combinators X and Y then corresponds exactly to the operation of finding the P.T.S. of the combinator (XY). (See Curry [2] or Hindley [4]). The fact that this operation can always be performed is proved in Theorem 1 and it follows that condensed detachment can always be applied.

This Theorem will also apply to other logics, for example **BCI** logic, which has Axiom **K** replaced by:

Axiom I $\quad p \supset p.$

Theorem 2 will in fact hold for any logic with axioms corresponding to combinators which do not allow duplications.

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