MORE AXIOMS FOR THE SET-THEORETIC HIERARCHY

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Van Aken [1986] introduces an elegant reformulation of Montague-Scott set theory in which relative rank order (a binary relation holding directly between sets) is taken as basic rather than the notion of hierarchical level. Van Aken's approach can itself easily be modified to suit those of us with a Cantorian taste for ordinals who believe that an explicit account of the overall vertical structure of an iterative hierarchy should appear no later than an account of the relative position of objects within that vertical structure. The appropriate alteration of Van Aken's language is obvious enough: we simply replace "$x \lhd y$" ("x is lower in rank than y") by "$p(x) \in p(y)$" ("the rank of x precedes that of y" or "the ordinal assigned to x precedes the ordinal assigned to y"), taking the p-operator as primitive (whereas for Van Aken it is defined). We shall also allow ourselves the luxury of monadic second order logic – its respectability as a tool for set theorists having been established by Boolos [1984]. (See also Pollard [1986].) These modifications cause Van Aken's principles of rank comprehension and strict rank order to read as follows.

Axiom I $\forall X(\exists y \forall x(Xx \rightarrow p(x) \in p(y)) \rightarrow \exists y \forall x(x \in y \leftrightarrow Xx))$

Axiom II $\forall x, y(p(x) \in p(y) \leftrightarrow \exists z \in y\ \forall w \in x\ p(w) \in p(z))$

From these two axioms we can derive the principle of $\in$-induction. Furthermore, we can show that the ordering of ranks is determined by the ordering of the objects ranked and that $\in$ forms a partial ordering between our ranks:

Theorem 1 $\forall X(\forall x(\forall y \in x\ Xy \rightarrow Xx) \rightarrow \forall x\ Xx)$
Theorem 2 $\forall y \forall x \in y\ p(x) \in p(y)$
Theorem 3 $\forall x\ p(x) \notin p(x)$
Theorem 4 $\forall x, y, z((p(x) \in p(y) \land p(y) \in p(z)) \rightarrow p(x) \in p(z))$

Suppose we assume that ranks are preceded only by ranks (writing "$x \in \mathfrak{R}$" for "$\exists x'\ x = p(x')$"):

Axiom III $\forall x \in \mathfrak{R}\ \forall y \in x\ y \in \mathfrak{R}$

Then we can show that each rank is $\in$-transitive and we can derive a strong well foundedness principle for ranks:

Theorem 5 $\forall x \in \mathfrak{R}\ \forall y \in x\ y \subset x$
Theorem 6 $\forall X(\exists x \in \mathfrak{R}\ Xx \rightarrow \exists x \in \mathfrak{R}\ (Xx \wedge \forall y \in x\ \neg Xy))$

Suppose we adopt an extensionality axiom:

Axiom IV $\forall x, y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x=y)$

Then we can establish that each rank is $\in$-comparable to every other:

Theorem 7 $\forall x, y \in \mathfrak{R}(x \in y \vee y \in x \vee x=y)$

Theorems 3, 4, 6 and 7 say that $\in$ forms a well ordering between our ranks. So Axiom III implies that our ranks are internally well ordered by $\in$. Hence, by Theorem 5, our ranks are von Nuemann (or, as I prefer to say, Mirimanoff) ordinals. That is (letting "$x \in$ ORD" express that x is a Mirimanoff ordinal – i.e., that x is $\in$-transitive and internally well ordered by $\in$):

Theorem 8 $\forall x \in \mathfrak{R}\ x \in \text{ORD}$

We can also establish that p(x) is the first rank preceded by all the ranks of members of x; that each rank is its own rank; that every $\in$-transitive object preceded only by ranks is itself a rank; and that every Mirimanoff ordinal is a rank (our ranks being then all and only the Mirimanoff ordinals):

Theorem 9 $\forall x, y(x \in p(y) \rightarrow \exists z \in y\ p(z) \notin x)$
Theorem 10 $\forall x \in \mathfrak{R}\ \forall y((\forall z \in y\ p(z) \in x \wedge \forall z \in x\ \exists w \in y\ p(w) \notin z) \rightarrow x=p(y))$
Theorem 11 $\forall x \in \mathfrak{R}\ p(x)=x$
Theorem 12 $\forall x(\forall y \in x(y \subset x \wedge y \in \mathfrak{R}) \rightarrow x \in \mathfrak{R})$
Theorem 13 $\forall x \in \text{ORD}\ x \in \mathfrak{R}$

A nice feature of this reformulation of Van Aken's set theory is that its primitive non-logical vocabulary contains only "$\in$" and an operator expressing a notion which is central to mainstream set theory (namely, the notion of rank). In fact, our ranks turn out to be our old friends the Mirimanoff ordinals. Furthermore, the reformulation is motivated by a general philosophical outlook which promises to dismiss our doubts about the intuitiveness of the ZF axioms. My Cantorian taste for ordinals is a symptom of my Husserlian belief that iterative hierarchies are *pure structures* which happen to be conveniently described using set talk, but which nonetheless consist solely of contentless nodes whose properties are entirely relational (nodes which cannot be regarded as sets in anything approaching an ordinary sense). On this structuralist view, an iterative hierarchy is intended to provide a comprehensive inventory of mathematically interesting sub-structures. It is the already established respectability of the specialized study of a particular pure structure which justifies the inclusion of that structure in an iterative hierarchy. To justify our adoption of, say, Axiom I, we would show that it merely introduces into our hierarchy denatured versions of respectable objects of specialized study. (Specialized areas of mathematics are the foundations on which "set" theory rests – thus making the latter one of the *least* foundational parts of mathematics. That is, set theory is not foundational in the Aristotelian sense: it does not deal with the most familiar objects and the most evident truths of mathematics. Nonetheless, it certainly deserves to be called "fundamental" by virtue of its abstractness and universality.)

Of particular mathematical interest and respectability are well ordered structures. Since our ranks are well ordered by ∈, they are natural candidates to be representatives within our hierarchy of well orderings in general – that is, to be ordinals. Indeed, someone who takes the notion "ordinal" to be basic would say that a central reason for axiomatizing the notion of rank is to supply ourselves with a general theory of ordinals. Yet our ranks would be unfit for this task if they were all exceeded in complexity by a sub-structure of an initial segment of our hierarchy. (After all, ordinals are supposed to be abstract representatives of all available well orderings.) It is to insure that no such calamity befalls us that we adopt the following axiom (the use in which of notation for ordered pairs can be justified by our previous axioms together with Axiom VI below).

Axiom V $\forall X(\forall y \exists ! y' X < y, y' > \rightarrow \forall x \exists z \in \mathfrak{R} \forall w \in \mathfrak{R}(\exists y \in x$
$X<y, w> \rightarrow w \in z))$

This essentially says that if objects from an initial segment of the hierarchy are projected onto the sequence of ranks, the image which is thus formed will be bounded from above by some rank. With this axiom and the two following ones in hand we can prove that every well ordering representable by an object in our hierarchy is isomorphic to one of our ranks (or, more precisely, to the structure formed by the predecessors of one our ranks).

Our desire that the sub-structures of our hierarchy amount to a comprehensive inventory of mathematically interesting structures will lead us to assume that every rank is succeeded by another and that among our ranks there is (switching terminology) a limit ordinal. That is:

Axiom VI $\forall x \in \mathfrak{R}\ \exists y \in \mathfrak{R}\ x \in y$

Axiom VII $\exists x \in \mathfrak{R}(\exists y\ y \in x \wedge \forall y \in x\ \exists z \in x\ y \in z)$

This puts us in a position to derive all the axioms of second order ZF.

Two final remarks: (1) If the above theory has a model M, then so does the result of replacing Axiom III by its negation. (Modify M by replacing each p(x) with 1+p(x).) (2) The assumption expressed by Axiom III could easily have been incorporated into Axiom II (thus leaving us with one less axiom at the price of only a slight complication of another).

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